Big-O and Summations Review Session

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Agenda

1. Big-O Motivation
2. Big-O Definitions/Examples
3. Summation Relations
4. Summation Examples

Please ask questions as they come up!

Why Asymptotics?

$O, \Omega, \Theta, o, \omega$

1. Concise description of runtime/space complexity
2. Abstracts away implementation details
3. Answers the guiding question:

   How does my runtime grow as the input size grows?

What Big-O is, and what it isn’t

- Lower bound
- Describes short term behavior
- Worst-case analysis

✗

- Upper bound
- Characterize long term behavior
- Growth rate

✓

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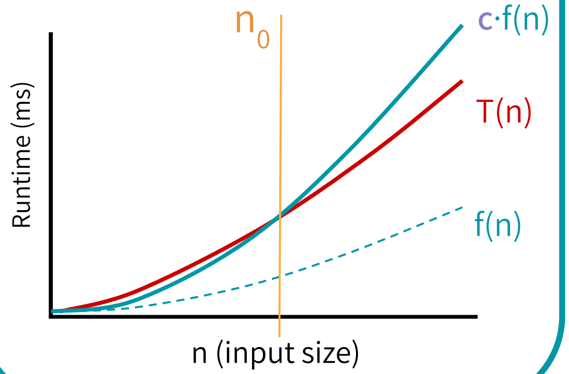
Big-O Notation

What do we mean when we say "T(n) is O(f(n))"?

In English

T(n) = O(f(n)) if and only if T(n) is *eventually* **upper bounded** by a constant multiple of f(n)

In Pictures



In Math

T(n) = O(f(n)) if and only if there exists positive **constants** **c** and **n_0** such that *for all $n \geq n_0$*

$$T(n) \leq c \cdot f(n)$$

Big-O Notation

What do we mean when we say “T(n) is O(f(n))”?

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$T(n) = O(f(n))$ if and only if $T(n)$ is *eventually* **upper bounded** by a constant multiple of $f(n)$

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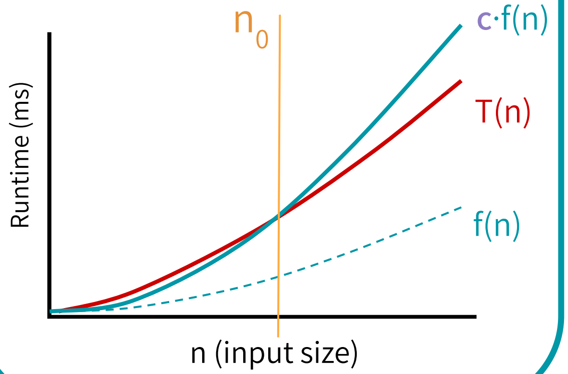


In *Math*

$$T(n) = O(f(n))$$

$$\iff$$

$$\exists\ c, n_0 > 0 \ \text{s.t.}\ \forall\ n \ge n_0,$$

$$T(n) \le c \cdot f(n)$$

“if and only if”

“for all”

“such that”

“there exists”

Limit sufficient Big-O Condition

What do we mean when we say “T(n) is O(f(n))”?

$$\lim_{n \to \infty} \frac{T(n)}{f(n)} \neq \infty$$

$$\Rightarrow$$

T(n) = O(f(n))

(provided this limit exists!)

Prove that n = O(n²).

Proof: Let $c=1$, $n_0=1$. $\forall\ n \geq n_0$, we have

$$n \geq 1$$

$$n^2 \geq n$$

$$n \leq n^2 \qquad \square$$

Alternatively, we have

$$\lim_{n\to\infty} \frac{n}{n^2}$$
$$= \lim_{n\to\infty} \frac{1}{n}$$
$$= 0,$$

which is not ∞.

“Racetrack Principle”

Prove that log n = O(n).

Proof: Define f(x) = log n, g(n) = n, and let c = 1, n_0=2.

Noting that $f(2) = \log_2 2 = 1 \leq g(2) = 2,$ and $f^{'}(n) = \frac{1}{\ln 2 \cdot n} \leq g^{'}(n) = 1$ for all $n \geq 2,$ it follows that $\log n \leq n,$ which is what we wanted to show.

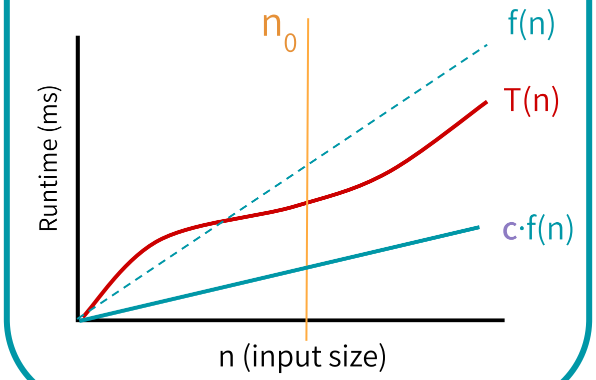
Big-Ω Notation

What do we mean when we say "T(n) is Ω(f(n))"?

In English

T(n) = Ω(f(n)) if and only if T(n) is eventually **lower bounded** by a constant multiple of f(n)

In Pictures



In *Math*

$$T(n) = \Omega(f(n))$$

$$\Leftrightarrow$$

$$\exists\ c, n_0 > 0 \text{ s.t. } \forall\ n \geq n_0,$$

$$T(n) \geq c \cdot f(n)$$

inequality switched directions!

Limit sufficient Big-Ω Condition

What do we mean when we say “T(n) is Ω(f(n))”?

$$\lim_{n \to \infty} \frac{T(n)}{f(n)} \neq 0$$

$$\Rightarrow$$

$$T(n) = \Omega(f(n))$$

(provided this limit exists!)

Big-Θ Notation

We say **“T(n) is Θ(f(n))”** if and only if both

$T(n) = O(f(n))$ *and* **$T(n) = \Omega(f(n))$**

$$T(n) = \Theta(f(n))$$

$$\Leftrightarrow$$

$$\exists\ c_1, c_2, n_0 > 0 \text{ s.t. } \forall\ n \geq n_0,$$

$$c_1 \cdot f(n) \leq T(n) \leq c_2 \cdot f(n)$$

Limit sufficient Big-Θ Condition

What do we mean when we say "T(n) is Θ(f(n))"?

$$\lim_{n\to\infty} \frac{T(n)}{f(n)} \neq \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{T(n)}{f(n)} \neq 0$$

or in other words $\lim_{n\to\infty} \frac{T(n)}{f(n)} = c$, for some constant c

$$\Rightarrow$$

T(n) = Θ(f(n))

(provided this limit exists!)

$f(n) = 3^n$, $g(n) = n^3$. Is $f(n) = O, \Omega$, or $\Theta(g(n))$?

Claim: $f(n) = \Omega(g(n))$. Proof: By induction on n. Let $c=1$, $n_0=4$.

Let $P(n)$ be the statement that $3^n \geq n^3$.

Base case ($n = n_0 = 4$): $3^4 = 81 \geq 4^3 = 64$.

Inductive step: Assume $P(j)$: $3^j \geq j^3$ holds for some $4 \leq j \leq k$. Showing $P(k+1)$, we have $3^{k+1} = 3 \cdot 3^k \geq 3k^3 = k^3 + 2k^3 \geq k^3 + 8k^2 = k^3 + 3k^2 + 5k^2 \geq$ $k^3 + 3k^2 + 20k = k^3 + 3k^2 + 3k + 17k \geq k^3 + 3k^2 + 3k + 68 =$ $k^3 + 3k^2 + 3k + 1 + 67 = (k+1)^3 + 67 \geq (k+1)^3$. Induction complete!

$f(n) = 3^n$, $g(n) = n^3$. Is $f(n) = O, \Omega,$ or $\Theta(g(n))$?

We've narrowed it down to $f(n) = \Omega(g(n))$, or $f(n) = \Theta(g(n))$.

We now claim: $f(n) \neq O(g(n))$, meaning $f(n) \neq \Theta(g(n))$. Proof:

Assume for the sake of contradiction that $f(n) = O(g(n))$. Then there exists constants $c, n_0 > 0$ such that for all $n \geq n_0$, $3^n \leq c \cdot n^3$.

Logging both sides: $n \leq \log_3(cn^3) \Rightarrow n \leq \log_3 c + 3\log_3 n$.
Taking the limit of the ratio, we get $\lim_{n\to\infty} \frac{n}{\log_3 c + \log_3 n} = \lim_{n\to\infty} \frac{1}{3 \cdot \frac{1}{n \cdot \ln 3}}$
$= \lim_{n\to\infty} \frac{n \cdot \ln 3}{3} = \infty$, which is a contradiction!

Aside: little-o and little-ω notation

little-o

In *Math*

“for all”

$$T(n) = o(f(n))$$

$$\Leftrightarrow$$

$$\forall\ c\ ,\ \exists\ n_0 > 0 \text{ s.t. } \forall\ n \geq n_0\ ,$$

$$\mathbf{T(n) < c \cdot f(n)}$$

strict!

little-ω

In *Math*

$$T(n) = \omega(f(n))$$

$$\Leftrightarrow$$

$$\forall\ c\ ,\ \exists\ n_0 > 0 \text{ s.t. } \forall\ n \geq n_0\ ,$$

$$\mathbf{T(n) > c \cdot f(n)}$$

inequality switched directions!

Limit sufficient little-o, little-ω Conditions

$$\lim_{n \to \infty} \frac{T(n)}{f(n)} = 0$$

$$\Rightarrow$$

T(n) = o(f(n))

(provided this limit exists!)

$$\lim_{n \to \infty} \frac{T(n)}{f(n)} = \infty$$

$$\Rightarrow$$

T(n) = ω(f(n))

(provided this limit exists!)

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Summation Formulas

Geometric series $$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$ $|r| < 1$

Geometric sum $$\sum_{k=0}^{n-1} ar^k = a\frac{1-r^n}{1-r}$$ $r \neq 1$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

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Summation Practice!

What is $\sum_{k=1}^{\infty}(\frac{5}{8})^k$?

$$\sum_{k=1}^{\infty}(\frac{5}{8})^k = -1 + \sum_{k=0}^{\infty}(\frac{5}{8})^k = -1 + \frac{1}{1-\frac{5}{8}} = -1 + \frac{1}{\frac{3}{8}} = -1 + \frac{8}{3} = \frac{5}{3}.$$

Summation Practice!

What is $\sum_{k=0}^{9} 3 \cdot 1.2^k$?

$$\sum_{k=0}^{9} 3 \cdot 1.2^k = \sum_{k=0}^{9} 3 \cdot (\frac{6}{5})^k = 3 \cdot \frac{1-(\frac{6}{5})^{10}}{1-\frac{6}{5}} = -15(1-1.2^{10}) \approx 77.88$$

$$f(n) = \sum_{i=1}^{n} i \log i, g(n) = n^2 \log n. \text{ Is } f(n) = O, \Omega, \text{ or } \Theta(n)?$$

To prove Big-O, begin by inspecting the summation:

$$\sum_{i=1}^{n} i \log i = 1 \log 1 + 2 \log 2 + 3 \log 3 + ... + n \log n$$

$$\begin{aligned}\sum_{i=1}^{n} i \log i &\leq 1 \log n + 2 \log n + 3 \log n + ... + n \log n \\ &= \log n(1 + 2 + 3 + ... + n) \\ &= \frac{n(n+1)}{2} \log n\end{aligned}$$

$$f(n) = \sum_{i=1}^{n} i \log i, g(n) = n^2 \log n. \text{ Is } f(n) = O, \Omega, \text{ or } \Theta(n)?$$

$$\frac{n(n+1)}{2} \log n \leq cn^2 \log n$$

$$\frac{n(n+1)}{2} \leq cn^2$$

$$\frac{1}{2}n^2 + \frac{1}{2}n \leq cn^2$$

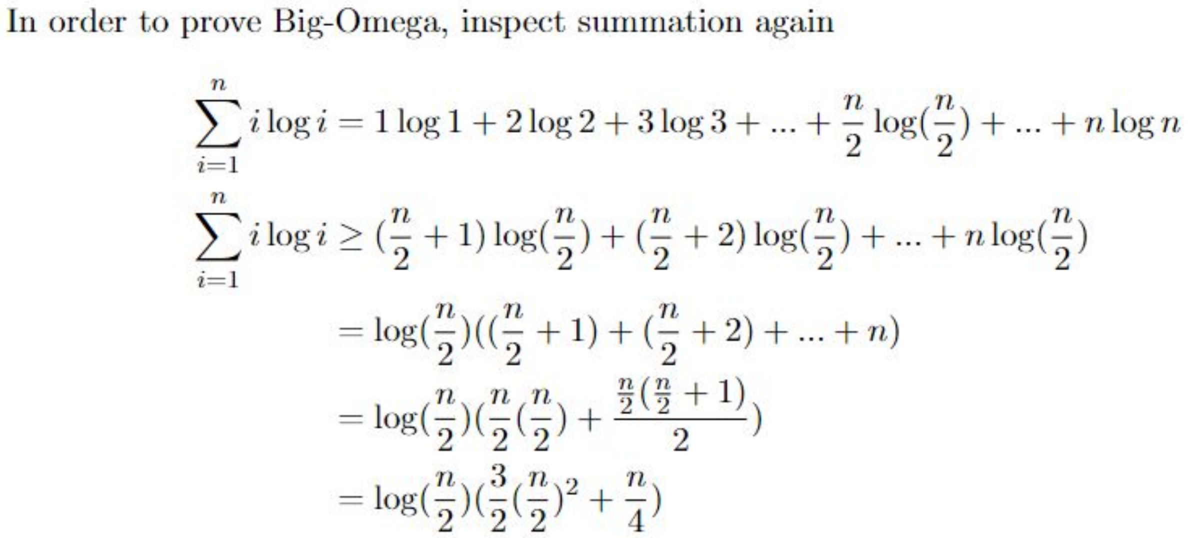
$$n^2(c - \frac{1}{2}) \geq \frac{1}{2}n$$

$$n(c - \frac{1}{2}) \geq \frac{1}{2}$$

So we can pick $c = 1, n_0 = 1$.

$$f(n) = \sum_{i=1}^{n} i\log i, g(n) = n^2 \log n. \text{ Is } f(n) = O, \Omega, \text{ or } \Theta(n)?$$

This is also Big-Ω, so overall f(n) = Ө(g(n))! (Similar analysis to big-O)



In order to prove Big-Omega, inspect summation again

$$\sum_{i=1}^{n} i\log i = 1\log 1 + 2\log 2 + 3\log 3 + ... + \frac{n}{2}\log(\frac{n}{2}) + ... + n\log n$$

$$\sum_{i=1}^{n} i\log i \geq (\frac{n}{2}+1)\log(\frac{n}{2}) + (\frac{n}{2}+2)\log(\frac{n}{2}) + ... + n\log(\frac{n}{2})$$

$$= \log(\frac{n}{2})((\frac{n}{2}+1) + (\frac{n}{2}+2) + ... + n)$$

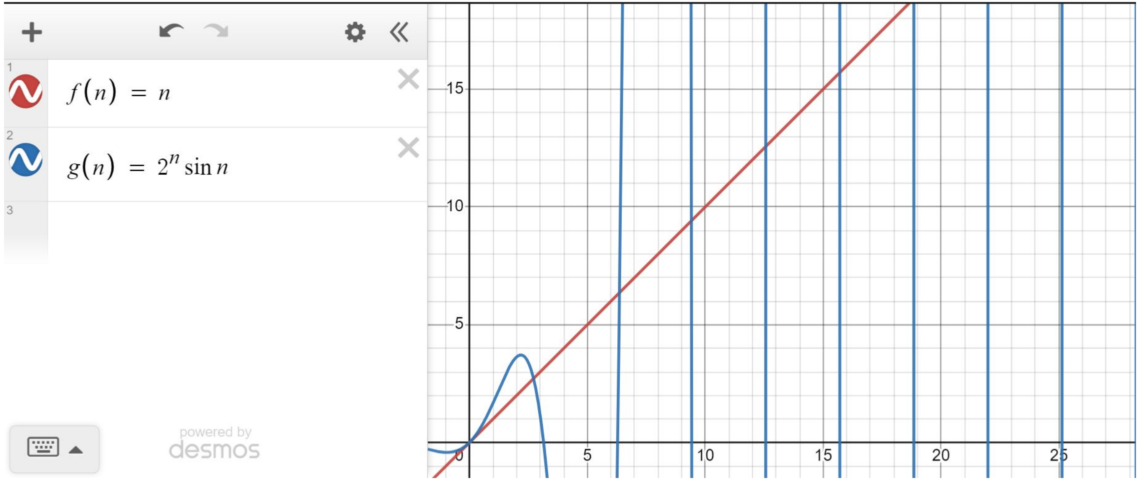
$$= \log(\frac{n}{2})(\frac{n}{2}(\frac{n}{2}) + \frac{\frac{n}{2}(\frac{n}{2}+1)}{2})$$

$$= \log(\frac{n}{2})(\frac{3}{2}(\frac{n}{2})^2 + \frac{n}{4})$$

(Fill in the rest!)

Give an example of f, g such that f ≠O(g) and g ≠O(f).

$$f(n) = n, g(n) = 2^n \sin n$$



$f(n) = n$

$g(n) = 2^n \sin n$

Takeaways

We have many "tools" to prove asymptotic bounds:

- Algebraic manipulation
- Limit analysis
- Racetrack principle
- Induction

To ***disprove*** an asymptotic bound, use proof by contradiction!

Questions?