Induction + log review

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When to Use?

- Need to prove a claim $P(n)$ holds $\forall n \in \mathbb{N}$ - Hard

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- Need to prove a claim $P(n)$ holds $\forall n \in \mathbb{N}$ - Hard
- Easier to generalise if $P(n)$ holds for some n

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$$\begin{aligned} 1 + 2 + \cdots + k &= \frac{k(k+1)}{2} \\ \implies 1 + 2 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) = (k+1)\left(\frac{k}{2} + 1\right) \\ &= \frac{(k+1)(k+2)}{2} = \frac{(k+1)((k+1)+1)}{2} \end{aligned}$$

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$$\begin{aligned} 2k+1 &= 2e_k \\ \implies 2k+1+2 &= 2e_k+2 \\ \implies 2(k+1)+1 &= 2(e_k+1) \end{aligned}$$

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3. **Conclusion**: Wait what?

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- Say the number of vertices $= n$

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    2. Remove this leaf $\rightarrow$ exactly 1 edge removed!
    3. Now a tree with k vertices, with $\leq k-1$ edges!

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- Easier to generalise if $P(n)$ holds for all $n \leq k$
  - ie, assuming it holds for just one n isn't sufficient

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How to decide base case?
If $n \geq 16$, then I could use one of my 4¢ coins to reduce to simpler problem!

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1. **Base Case**: Show that $P(n)$ holds for $12 \leq n \leq 15$.
   - $12 = 4 + 4 + 4$
   - $13 = 4 + 4 + 5$
   - $14 = 4 + 5 + 5$
   - $15 = 5 + 5 + 5$

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3. **Inductive Step**: If $k \geq 15, k + 1 = 4 + (k - 3)$.
   Is the assumption that $k \geq 15$ needed? Does the inductive hypothesis work for $k - 3$ if $k < 15$?

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Logarithms to the base e or often denoted by the symbol 'ln'. $\log_e 2 = \ln 2$.
In this course, unless explicitly mentioned, the base is 2. $\log 8 = 3$.

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Let's start with a few simple properties:

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1. $a^1 = a \implies \log_a a = 1$
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3. $a^{-1} = 1/a \implies \log_a(1/a) = -1$

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Remember from exponents,

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(Let $a^x = m, a^y = n$)

$$\implies \log_a mn = \log_a m + \log_a n$$

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Why?

Properties

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Why?

Use the fact that $a^{-y} = 1/a^y$ to convince yourself of this following a similar proof as the product case.

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$$\begin{aligned} (a^x)^n &= a^{xn} \\ \implies \log_a (a^x)^n &= \log_a a^{xn} \\ &= xn = n \log_a a^x \end{aligned}$$

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Let's start by assuming $\log_a m = x$ and $\log_a b = y$.
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Let's start by assuming $\log_a m = x$ and $\log_a b = y$.
Then, $m = a^x, b = a^y \implies m = a^{(x/y)y} = (a^y)^{x/y}$.
So, $m = b^{x/y} \implies \log_b m = \log_a m / \log_a b$

Properties

8. $\log_b m = 1/\log_m b$
   Try to show this yourself, perhaps using the previous property?

Properties

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As an exercise, let's try to use a bunch of previous properties to show this...

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Why?Property 7!

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$$\log_a c \log_b a = \log_a c^{\log_b a}$$

Why?Property 6!

$$\implies \log_a c^{\log_b a} = \log_b c \implies c^{\log_b a} = a^{\log_b c}$$

Summary of Properties

1. $a^1 = a \implies \log_a a = 1$
2. $a^0 = 1 \implies \log_a 1 = 0$
3. $a^{-1} = 1/a \implies \log_a(1/a) = -1$
4. $\log mn = \log m + \log n$
5. $\log(m/n) = \log m - \log n$
6. $\log m^n = n \log m$
7. $\log_b m = \log_a m / \log_a b$
8. $\log_b m = 1/\log_m b$
9. $a^{\log_b c} = c^{\log_b a}$