CS 161 Section

W1: Asymptotic analysis, proof by induction.

Spring 2023

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PSEUDOCODE

What is good pseudocode?

PSEUDOCODE

☐ Clear description of the steps

Setup: You have light bulbs of different sizes placed in order of their size, and have 1 socket. How do you match the sock to the light bulb?

find_matching_light_bulb (LightBulbs, Socket)
 	
search the light bulbs and return the matching lightbulb with the socket
	find_matching_light_bulb (lightBulbs, socket)
	
	for lightBulb in lightBulbs:
	if lightBulb and socket match:
	return lightBulb
	return no matching light bulb found

✗	✓

```
find_matching_light_bulb (LightBulbs, Socket)

    search the light bulbs and return the
    matching lightbulb with the socket
```

```
find_matching_light_bulb (lightBulbs, socket)

    for lightBulb in lightBulbs:
        if lightBulb and socket match:
        return lightBulb
    return no matching light bulb found
```

PSEUDOCODE

☐ Use algorithms covered in lectures without their pseudocode

```
find_matching_light_bulb
(LightBulbs, Socket)

    run binary_search to find
the matching light bulb.
```

✗

```
find_matching_light_bulb (lightBulbs, socket)

    ind=modified_binary_search(lightBulbs,
                                   socket)
    if lightBulbs[ind] matches with socket:
        return lightBulbs[ind]
    else:
        return no matching light bulb found
```

❖ *modified_binary_search is binary_search, but we change it [in this specific way].*

✓

PSEUDOCODE

- ☐ Simplify pseudocode as much as possible

```
find_matching_light_bulb (lightBulbs, socket)
    if the 1st light bulb matches the socket:
        return the first light bulb
    if the 2nd light bulb matches the socket:
        return the 2nd light bulb
    if the 3rd light bulb matches the socket:
        return the 3rd light bulb
                    ...
    if the nth light bulb matches the socket:
        return the nth light bulb
    return no matching light bulb found
```

✗

```
find_matching_light_bulb (lightBulbs, socket)

      for lightBulb in lightBulbs:
             if lightBulb and socket match:
      return lightBulb
      return no matching light bulb found
```

✓

PSEUDOCODE

- ☐ Clear description of the steps
- ☐ Use algorithms covered in lectures without their pseudocode
- ☐ Simplify pseudocode as much as possible

BIG-O NOTATION

BIG-O NOTATION

Let T(n) & f(n) be functions defined on the positive integers.

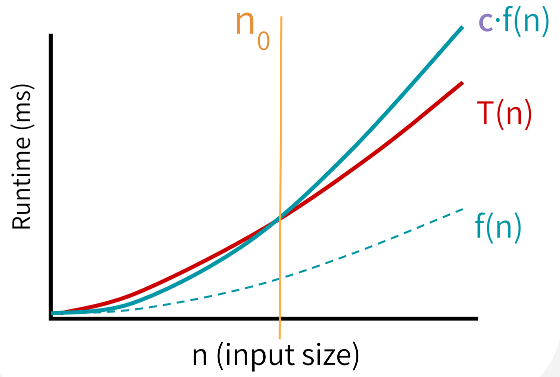
(In this class, we'll typically write T(n) to denote the worst case runtime of an algorithm)

What do we mean when we say "T(n) is O(f(n))"?

In English

T(n) = O(f(n)) if and only if T(n) is *eventually* **upper bounded** by a constant multiple of f(n)

In Pictures



In Math

T(n) = O(f(n)) if and only if there exists positive **constants** **c** and **n_0** such that *for all $n \geq n_0$*

$$T(n) \leq c \cdot f(n)$$

BIG-O NOTATION

Let T(n) & f(n) be functions defined on the positive integers.

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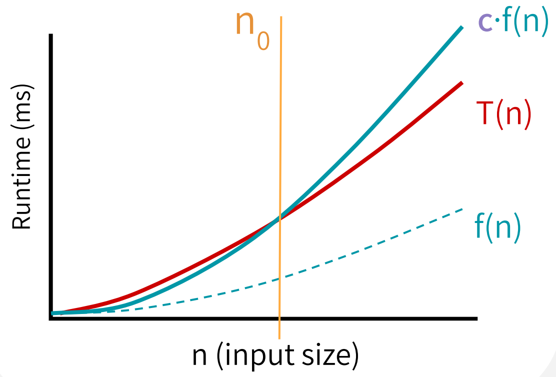


In *Math*

$$T(n) = O(f(n))$$

"if and only if" $\rightarrow$ $\Leftrightarrow$

"for all"

$$\exists\ c, n_0 > 0 \text{ s.t. } \forall\ n \geq n_0,$$

"there exists"

"such that"

$$T(n) \leq c \cdot f(n)$$

PROVING BIG-O BOUNDS

If you're ever asked to formally prove that T(n) is O(f(n)), use the *MATH* definition:

$$T(n) = O(f(n))$$
$$\Leftrightarrow$$
$$\exists\ c, n_0 > 0 \text{ s.t. } \forall\ n \geq n_0,$$
$$\mathbf{T(n) \leq c \cdot f(n)}$$

must be constants! i.e. c & n_0 cannot depend on n!

- To **prove** T(n) = O(f(n)), you need to announce your c & n_0 up front!
  - Play around with the expressions to find appropriate choices of c & n_0 (positive constants)
  - Then you can write the proof! Here how to structure the start of the proof:

"Let c = __ and n_0 = __. We will show that T(n) ≤ c·f(n) for all n ≥ n_0."

PROVING BIG-O BOUNDS: EXAMPLE

$$T(n) = O(f(n))$$

$$\Leftrightarrow$$

$$\exists\ c, n_0 > 0 \text{ s.t. } \forall\ n \geq n_0,$$

$$T(n) \leq c \cdot f(n)$$

Prove that $3n^2 + 5n = O(n^2)$.

My thinking: I want to find a c & n_0 such that for all $n \geq n_0$:

$$3n^2 + 5n \leq c \bullet n^2$$

I can rearrange this inequality just to see things a bit more clearly:

$$5n \leq (c - 3) \bullet n^2$$

Now let's cancel out the n:

$$5 \leq (c - 3)\, n$$

Let's choose:

$c = 4$

$n_0 = 5$

(other choices work too! e.g. $c = 10$, $n_0 = 10$)

PROVING BIG-O BOUNDS: EXAMPLE

$$T(n) = O(f(n))$$
$$\Leftrightarrow$$
$$\exists\ c, n_0 > 0 \text{ s.t. } \forall\ n \geq n_0,$$
$$T(n) \leq c \cdot f(n)$$

Prove that $3n^2 + 5n = O(n^2)$.

Let $c = 4$ and $n_0 = 5$. We will now show that $3n^2 + 5n \leq c \cdot n^2$ for all $n \geq n_0$.
We know that for any $n \geq n_0$, we have:

$$5 \leq n$$
$$5n \leq n^2$$
$$3n^2 + 5n \leq 4n^2$$

Using our choice of c and n_0, we have successfully shown that $3n^2 + 5n \leq c \cdot n^2$ for all $n \geq n_0$. From the definition of Big-O, this proves that $3n^2 + 5n = O(n^2)$. ■

DISPROVING BIG-O BOUNDS

If you’re ever asked to formally disprove that T(n) is O(f(n)), use **proof by contradiction!**

This means you need to show that NO POSSIBLE CHOICE of c & n_0 exists such that the Big-O definition holds

DISPROVING BIG-O BOUNDS

If you're ever asked to formally disprove that T(n) is O(f(n)), use **proof by contradiction!**

For sake of contradiction, assume that T(n) is O(f(n)). In other words, assume there does indeed exist a choice of c & n_0 s.t. $\forall$ $n \geq n_0$, $\mathbf{T(n) \leq c \cdot f(n)}$

pretend you have a friend that comes up and says "I have a c & n_0 that will prove T(n) = O(f(n))!!!", and you say "ok fine, let's assume your c & n_0 does prove T(n) = O(f(n))"

$\downarrow$

Treating c & n_0 as "variables", derive a contradiction!

although you are skeptical, you'll entertain your friend by saying: "let's see what happens. [some math work... and then...] AHA! regardless of what your constants c & n_0, trusting you has led me to something *impossible!!!*"

$\downarrow$

Conclude that the original assumption must be false, so **T(n) is *not* O(f(n))**.

you have triumphantly proven your silly (or lying) friend wrong.

DISPROVING BIG-O: EXAMPLE

$T(n) = O(f(n))$

$\Leftrightarrow$

$\exists\ c, n_0 > 0$ s.t. $\forall\ n \geq n_0$,

$T(n) \leq c \cdot f(n)$

Prove that $3n^2 + 5n$ is *not* $O(n)$.

For sake of contradiction, assume that $3n^2 + 5n$ is $O(n)$. This means that there exists positive constants c & n_0 such that $3n^2 + 5n \leq c \cdot n$ for all $n \geq n_0$.

Then, we would have the following:

$$3n^2 + 5n \leq c \cdot n$$
$$3n + 5 \leq c$$
$$n \leq (c - 5)/3$$

However, since $(c - 5)/3$ is a constant, we've arrived at a contradiction since n cannot be bounded above by a constant for all $n \geq n_0$. For instance, consider $n = n_0 + c$: we see that $n \geq n_0$, but $n > (c - 5)/3$. Thus, our original assumption was incorrect, which means that $3n^2 + 5n$ is not $O(n)$. ■

BIG-O EXAMPLES

lower order terms don't matter!

$$\log_2 n + 15 = O(\log_2 n)$$

remember, big-O is upper bound!

$$3^n = O(4^n)$$

Polynomials

Say $p(n) = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n + a_0$ is a polynomial of degree $k \geq 1$.

Then:

i. $p(n) = O(n^k)$
ii. $p(n)$ is **not** $O(n^{k-1})$

constant multipliers & lower order terms don't matter

$$6n^3 + n \log_2 n = O(n^3)$$

$$25 = O(1)$$

[any constant] = O(1)

AN ASIDE: O(n log n) vs. O(n^2)?

log(n) grows very slowly! (Much more slowly than n)

ALL LOGARITHMS IN THIS COURSE ARE BASE 2

log(2) = 1
log(4) = 2
...
log(64) = 6
log (128) = 7
...
log(4096) = 12
...
log(**# particles in the universe**) < 280

Logs are slow!
In fact,
log n = O(n^d)
for any d > 0

n log n grows much more slowly than n^2

Punchline: A running time of O(n log n) is a LOT better than O(n^2)

BIG-Ω NOTATION

Let $T(n)$ & $f(n)$ be functions defined on the positive integers.

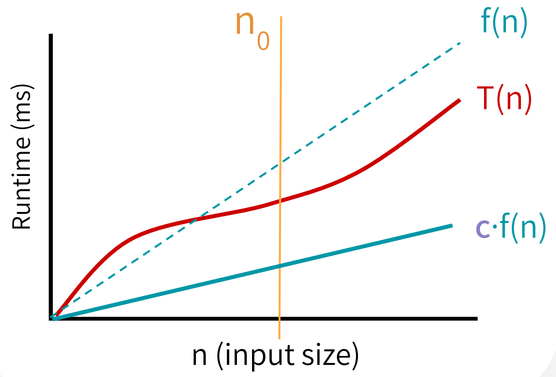
(In this class, we'll typically write $T(n)$ to denote the worst case runtime of an algorithm)

What do we mean when we say "$T(n)$ is $\Omega(f(n))$"?

In English

$T(n) = \Omega(f(n))$ if and only if $T(n)$ is eventually **lower bounded** by a constant multiple of $f(n)$

In Pictures



In *Math*

$$T(n) = \Omega(f(n))$$

$$\Leftrightarrow$$

$$\exists\ c, n_0 > 0 \text{ s.t. } \forall\ n \geq n_0,$$

$$T(n) \underset{\uparrow}{\geq} c \cdot f(n)$$

inequality switched directions!

BIG-ϴ NOTATION

We say **"T(n) is ϴ(f(n))"** if and only if both

$$\mathbf{T(n) = O(f(n))}$$

and

$$\mathbf{T(n) = \Omega(f(n))}$$

$$T(n) = \Theta(f(n))$$

$$\Leftrightarrow$$

$$\exists\ c_1, c_2, n_0 > 0 \text{ s.t. } \forall\ n \ge n_0,$$

$$c_1 \cdot f(n) \le T(n) \le c_2 \cdot f(n)$$

ASYMPTOTIC NOTATION CHEAT SHEET

BOUND	DEFINITION (HOW TO PROVE)	WHAT IT REPRESENTS
$T(n) = O(f(n))$	$\exists\ c > 0,\ \exists\ n_0 > 0$ s.t. $\forall\ n \geq n_0,\ T(n) \leq c \cdot f(n)$	upper bound
$T(n) = \Omega(f(n))$	$\exists\ c > 0,\ \exists\ n_0 > 0$ s.t. $\forall\ n \geq n_0,\ T(n) \geq c \cdot f(n)$	lower bound
$T(n) = \Theta(f(n))$	$T(n) = O(f(n))$ and $T(n) = \Omega(f(n))$	tight bound

KARATSUBA INTEGER MULTIPLICATION

Three subproblems instead of four!

CHOOSING SUBPROBLEMS WISELY

$$[x_1 x_2 \ldots x_{n-1} x_n] \times [y_1 y_2 \ldots y_{n-1} y_n]$$

$$= (a \times 10^{n/2} + b) \times (c \times 10^{n/2} + d)$$

$$= (\boxed{a \times c}) 10^n + (\boxed{a \times d} + \boxed{b \times c}) 10^{n/2} + (\boxed{b \times d})$$

The subproblems we choose to solve just need to provide these quantities:

$\boxed{ac}$ $\boxed{ad + bc}$ $\boxed{bd}$

Originally, we assembled these quantities by computing FOUR things: ac, ad, bc, and bd.

KARATSUBA'S TRICK

$$\text{end result} = (\boxed{ac})10^n + (\boxed{ad + bc})10^{n/2} + (\boxed{bd})$$

ac & **bd** can be recursively computed as usual

ad + bc is equivalent to $\mathbf{(a+b)(c+d) - ac - bd}$

$$= (ac + ad + bc + bd) - ac - bd$$
$$= ad + bc$$

So, instead of computing **ad** & **bc** as two separate subproblems, let's just compute **(a+b)(c+d)** instead!

OUR THREE SUBPROBLEMS

These *three* subproblems give us everything we need to compute our desired quantities:

1	ac
2	bd
3	(a+b)(c+d)

(a+b) and (c+d) are both going to be n/2-digit numbers!

⇩

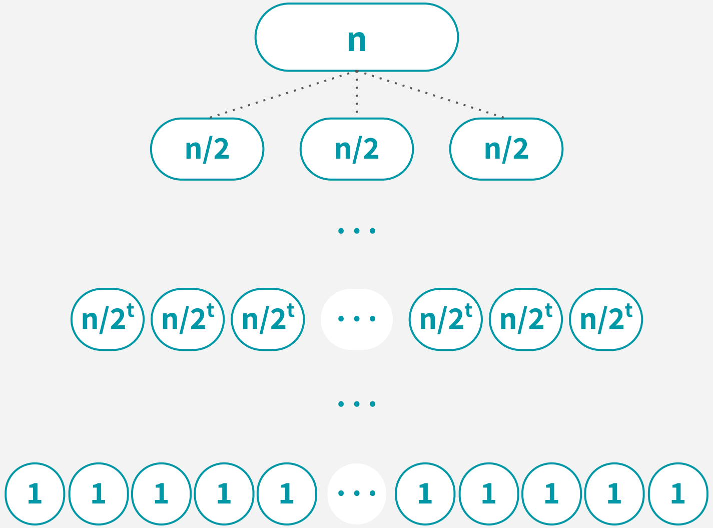
This means we still have half-sized subproblems!

Assemble our overall product by combining these three subproblems:

$$\overset{1}{(\ ac\ )}10^n + \overset{3 - 1 - 2}{(\ ad + bc\ )}10^{n/2} + \overset{2}{(\ bd\ )}$$

WHAT'S THE RUNTIME?

Karatsuba Multiplication Recursion Tree



Level 0: 1 problem of size n

Level 1: 3^1 problems of size n/2

Level t: 3^t problems of size $n/2^t$

Level $\log_2 n$: $\underline{\mathbf{n^{1.6}}}$ problems of size 1

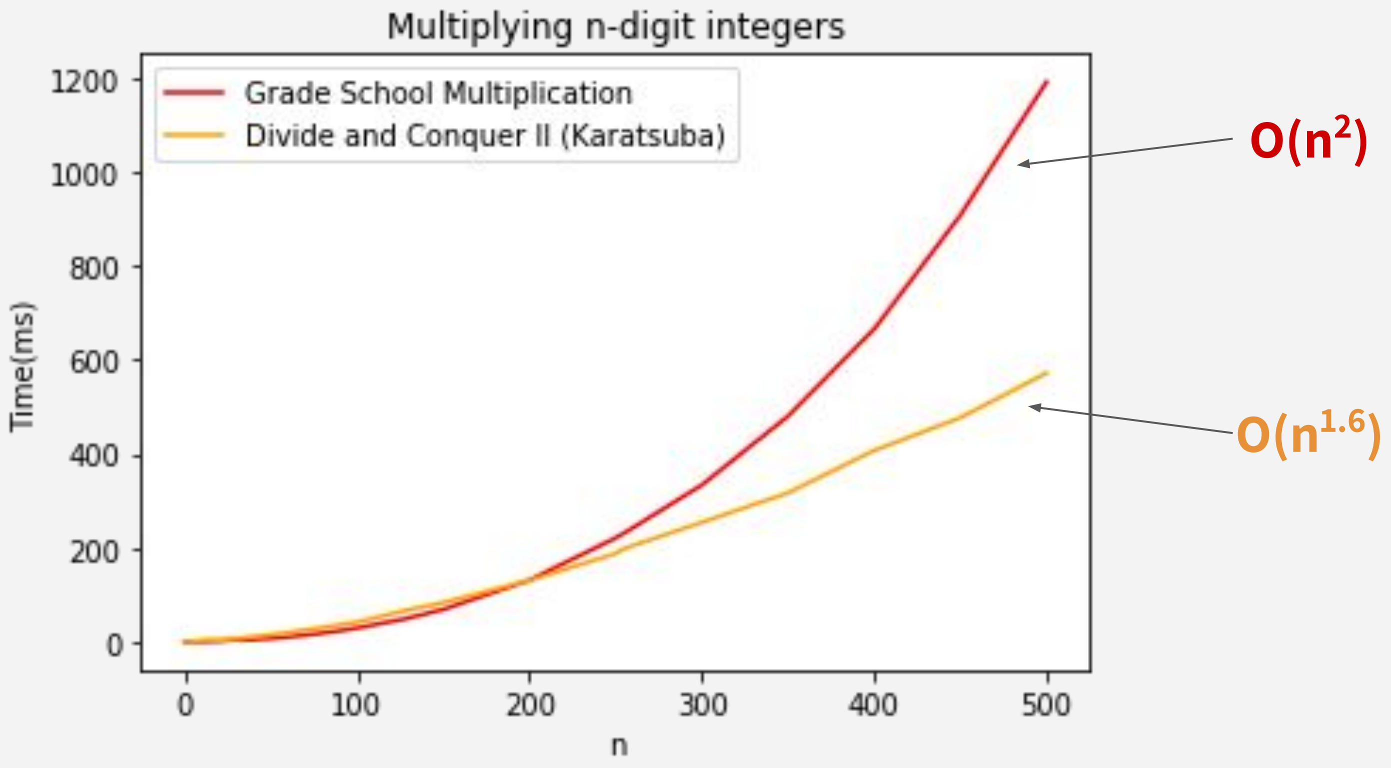
$\log_2 n$ levels
(you need to cut n in half $\log_2 n$ times to get to size 1)

# of problems on last level (size 1)
$= 3^{\log_2 n} = n^{\log_2 3}$

$\approx \mathbf{n^{1.6}}$

Thus, the runtime is $O(n^{1.6})$!

IT WORKS IN PRACTICE TOO!



REVIEW OF INDUCTION

How to write proof by induction.

4 INGREDIENTS OF INDUCTION

INDUCTIVE HYPOTHESIS (IH)

This is a statement that's basically what you're trying to prove, except it's written in terms of some variable (e.g. **i**). We need to set up the inductive hypothesis clearly, and our goal in the next three steps is to prove that the IH holds for a whole *range* of values for **i**.

BASE CASE

First establish that the inductive hypothesis holds for some base case value(s) of **i**.

INDUCTIVE STEP *(strong/complete induction version)*

Next, assume that the IH holds when **i** takes on any value *between [base case value(s)] and some number* **k**. Now prove that the IH holds as well when **i** takes on the value **k**.

CONCLUSION

By induction, conclude that the IH holds across the range of **i** you're dealing with.

PROVE CORRECTNESS w/ INDUCTION

ITERATIVE ALGORITHMS

1. **Inductive hypothesis**: some state/condition will always hold throughout your algorithm by any iteration **i**
2. **Base case**: show IH holds for iteration 0 (i.e. start of algorithm)
3. **Inductive step**: Assume IH holds for k ⇒ prove k+1
4. **Conclusion**: IH holds for i = # total iterations ⇒ yay!

RECURSIVE ALGORITHMS

1. **Inductive hypothesis**: your algorithm is correct for sizes *up to* **i**
2. **Base case**: IH holds for i < small const.
3. **Inductive step**:
    - assume IH holds for k ⇒ prove k+1, *OR*
    - assume IH holds for {1,2,...,k-1} ⇒ prove k (*it's not important that I chose k instead of k+1, using k is can just be syntactically cleaner!)
4. **Conclusion**: IH holds for i = n ⇒ yay!

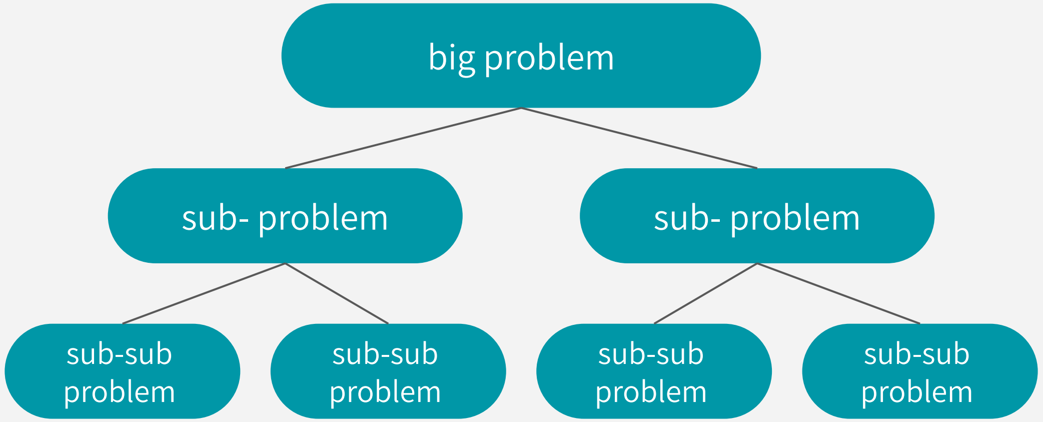
Example: MERGESORT

Algorithm, Proof of Correctness, Runtime

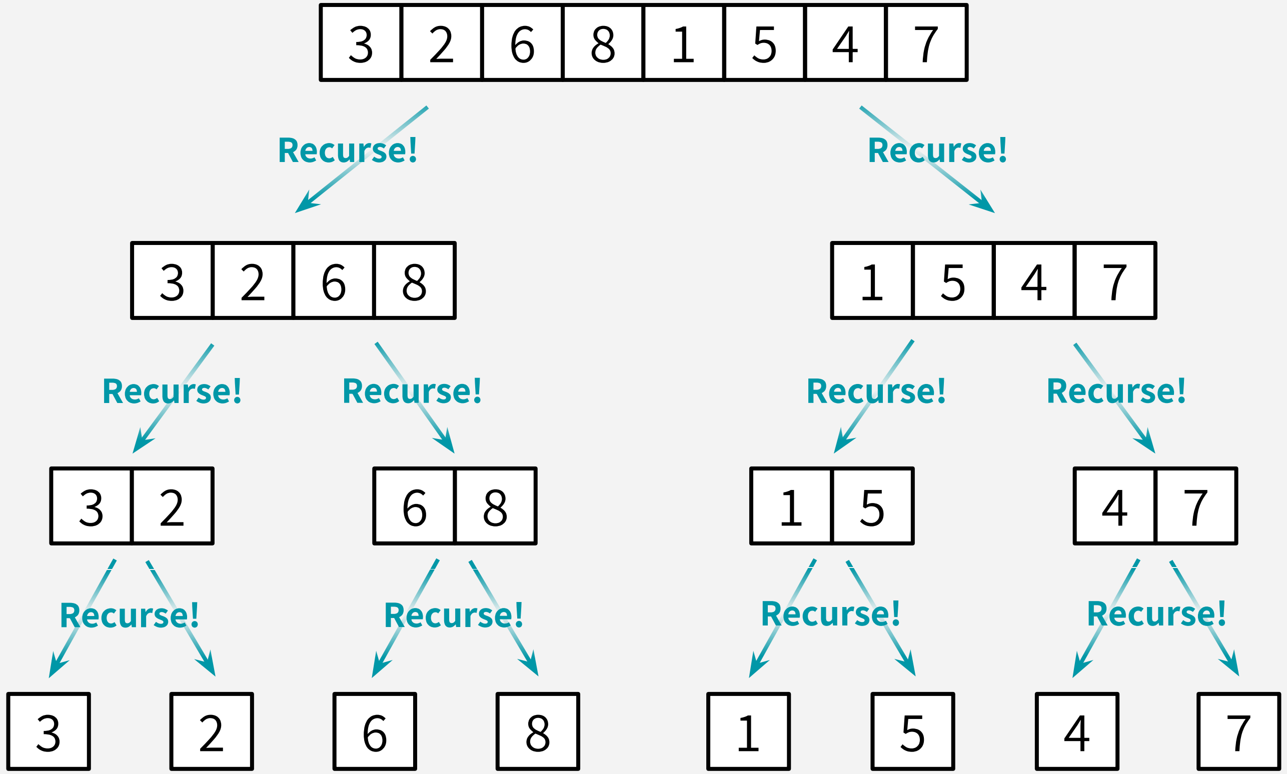
MERGESORT

FROM MONDAY!

- **DIVIDE-AND-CONQUER: an algorithm design paradigm**
  1. break up a problem into smaller subproblems
  2. solve those subproblems *recursively*
  3. combine the results of those subproblems to get the overall answer

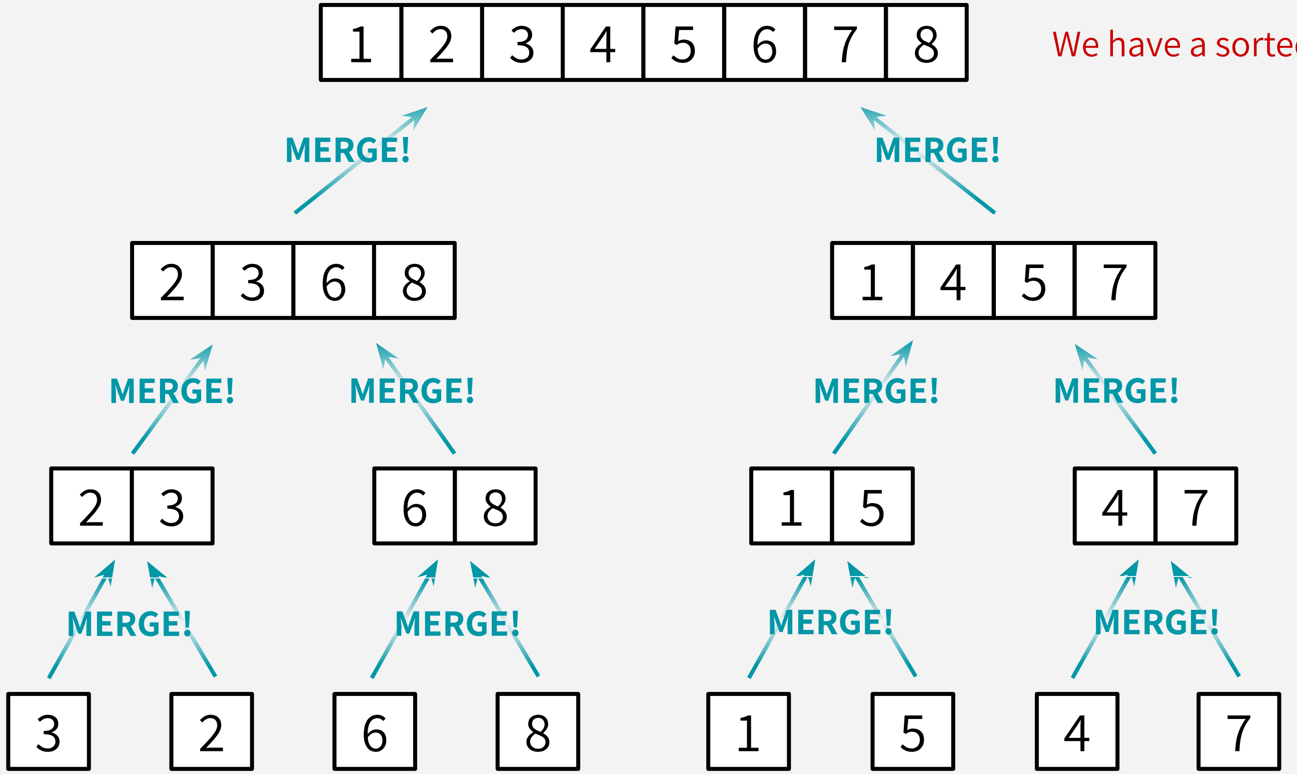


MERGESORT: RECURSIVE CALLS



This is where we hit our base case!

MERGESORT: MERGE STEPS



We have a sorted sequence!

MERGESORT: PSEUDOCODE

Intuition: Divide and Conquer. If you sort your left and right halves, it's easier to "Merge" them into a sorted list.

```
MERGESORT(A):
    n = len(A)
    if n <= 1:
        return A
    L = MERGESORT(A[0:n/2])
    R = MERGESORT(A[n/2:n])
    return MERGE(L,R)
```

```
MERGE(L,R):
    result = length n array
    i = 0, j = 0
    for k in [0,...,n-1]:
        if L[i] < R[j]:
            result[k] = L[i]
            i += 1
        else:
            result[k] = R[j]
            j += 1
    return result
```

MERGESORT: DOES IT WORK?

THIS IS A JOB FOR: ***PROOF BY INDUCTION!***

(This time, we perform induction on the *length of input list*, rather than # of iterations)

MERGESORT: INDUCTION PROOF

INDUCTIVE HYPOTHESIS (IH)

In every recursive call on an array of length *at most* **i**, MERGESORT returns a sorted array.

BASE CASE

The IH holds for i = 1: A 1-element array is always sorted.

INDUCTIVE STEP *(strong/complete induction)*

Let k be an integer, where 1 < k ≤ n. Assume that the IH holds for i < k, so MERGESORT correctly returns a sorted array when called on arrays of length less than k. We want to show that the IH holds for i = k, i.e. that MERGESORT returns a sorted array when called on an array of length k.

[INSERT INDUCTION PROOF TO PROVE THE MERGE SUBROUTINE IS CORRECT WHEN GIVEN TWO SORTED ARRAYS]

Since the two “child” recursive calls are executed on arrays of length k/2 (which is strictly less than k), then our inductive hypothesis tells us that MERGESORT will correctly sort the left and right halves of our length-k array. Then, since the MERGE subroutine is correct when given two sorted arrays, we know that MERGESORT will ultimately return a fully sorted array of length k.

Try out this inner proof on your own!

CONCLUSION

By induction, we conclude that the IH holds for all 1 ≤ i ≤ n. In particular, it holds for i = n, so in the top recursive call, MERGESORT returns a sorted array.

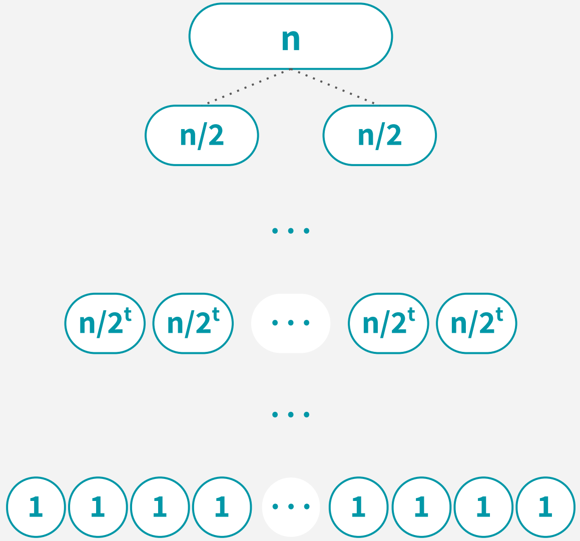
MERGESORT: IS IT FAST?

```
MERGESORT(A):
    n = len(A)
    if n <= 1:
        return A
    L = MERGESORT(A[0:n/2])
    R = MERGESORT(A[n/2:n])
    return MERGE(L,R)
```

CLAIM: MergeSort runs in time **O(n log n)**

MERGESORT RECURSION TREE

If a subproblem is of size **n**, then the work done in that subproblem is **O(n)**.
⇒ **Work ≤ c · n** (c is a constant)



Level	# of Problems	Size of each Problem	Work done per Problem ≤	Total work on this level
0	1	n	$c \cdot n$	**O(n)**
1	2^1	n/2	$c \cdot (n/2)$	$2^1 \cdot c \cdot (n/2) =$ **O(n)**
. . .				
t	2^t	$n/2^t$	$c \cdot (n/2^t)$	$2^t \cdot c \cdot (n/2^t) =$ **O(n)**
. . .				
$\log_2 n$	$2^{\log_2 n} = n$	1	$c \cdot (1)$	$n \cdot c \cdot (1) =$ **O(n)**

We have ($\log_2 n + 1$) levels, each level has O(n) work total ⇒ **O(n log n)** work overall!