CS 161
Section 4

[CA Name]

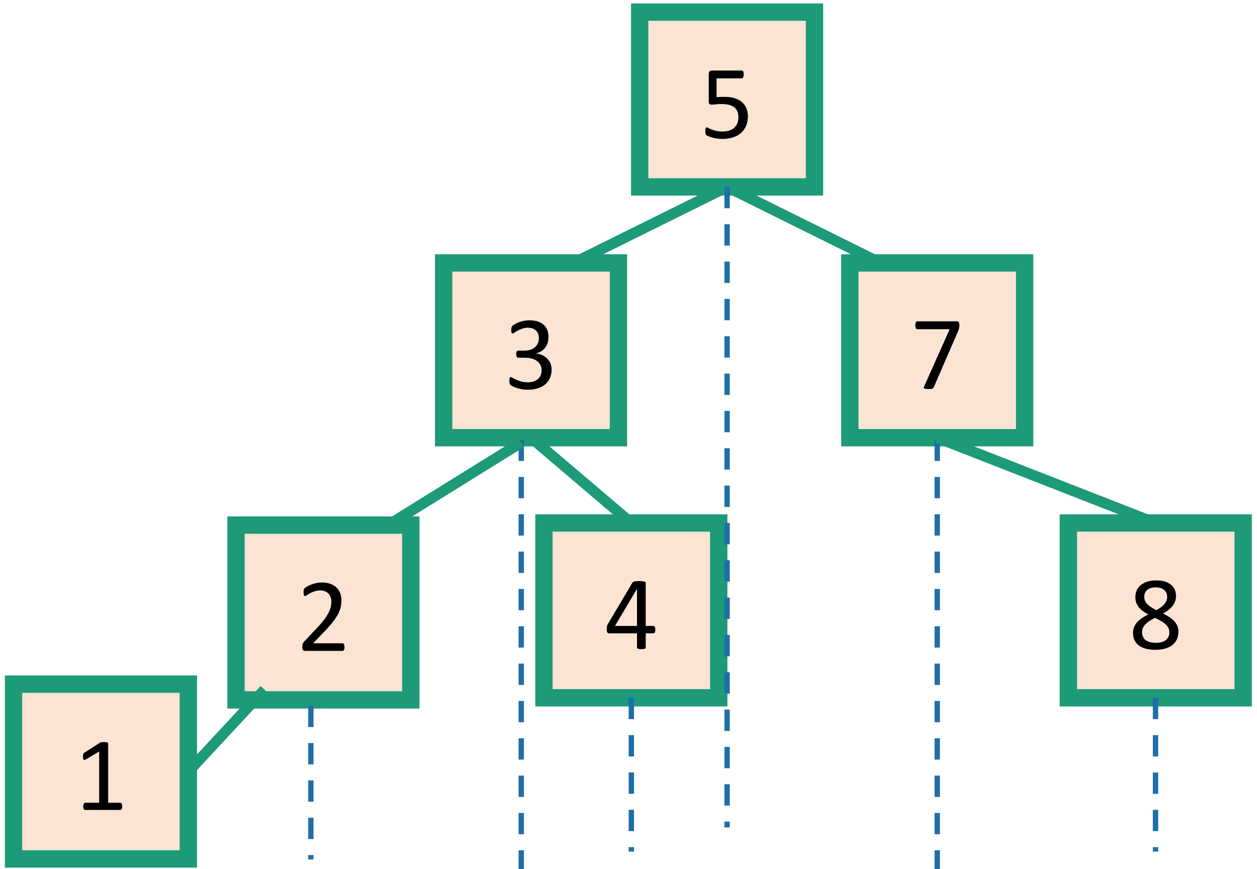
Agenda

- Recap
  - Trees
  - Hashing
- Handout

Trees (BSTs and Red-Black)

Binary Search Trees

- A BST is a binary tree so that:
  - Every LEFT descendant of a node has key less than that node.
  - Every RIGHT descendant of a node has key larger than that node.
- Example of building a binary search tree:

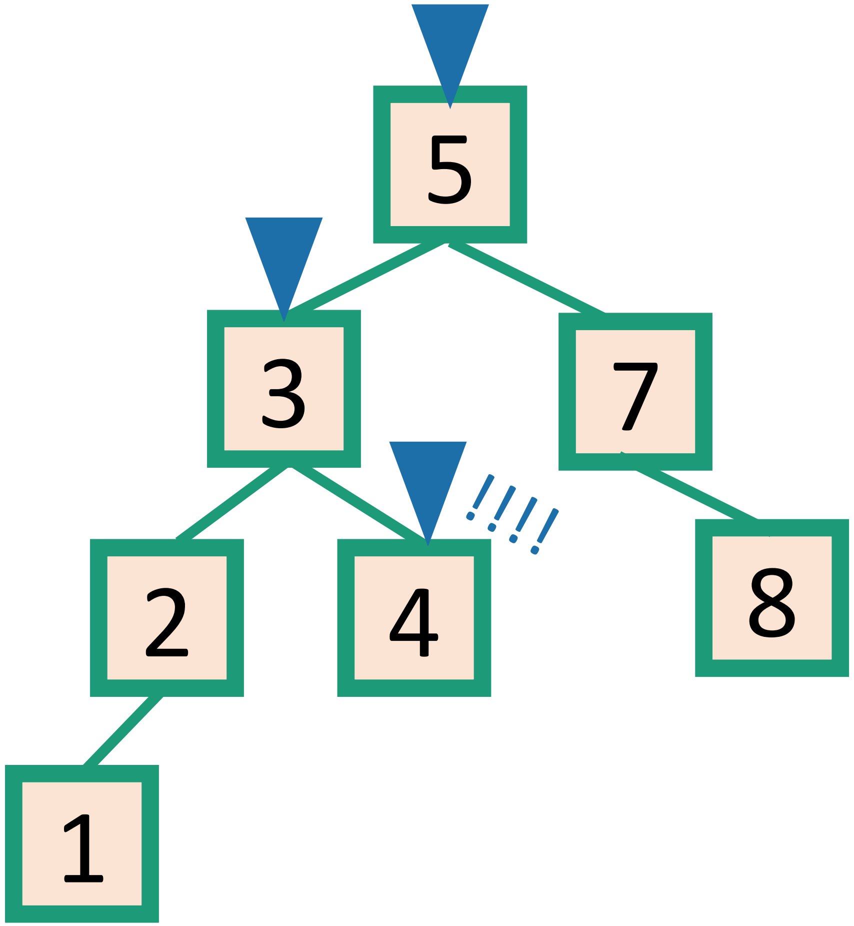


Q: Is this the only binary search tree I could possibly build with these values?

A: **No.** I made choices about which nodes to choose when. Any choices would have been fine.

SEARCH in a Binary Search Tree

definition by example

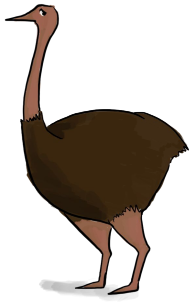


EXAMPLE: Search for 4.

EXAMPLE: Search for 4.5

- It turns out it will be convenient to **return 4** in this case
- (that is, **return** the last node before we went off the tree)

Write pseudocode (or actual code) to implement this!

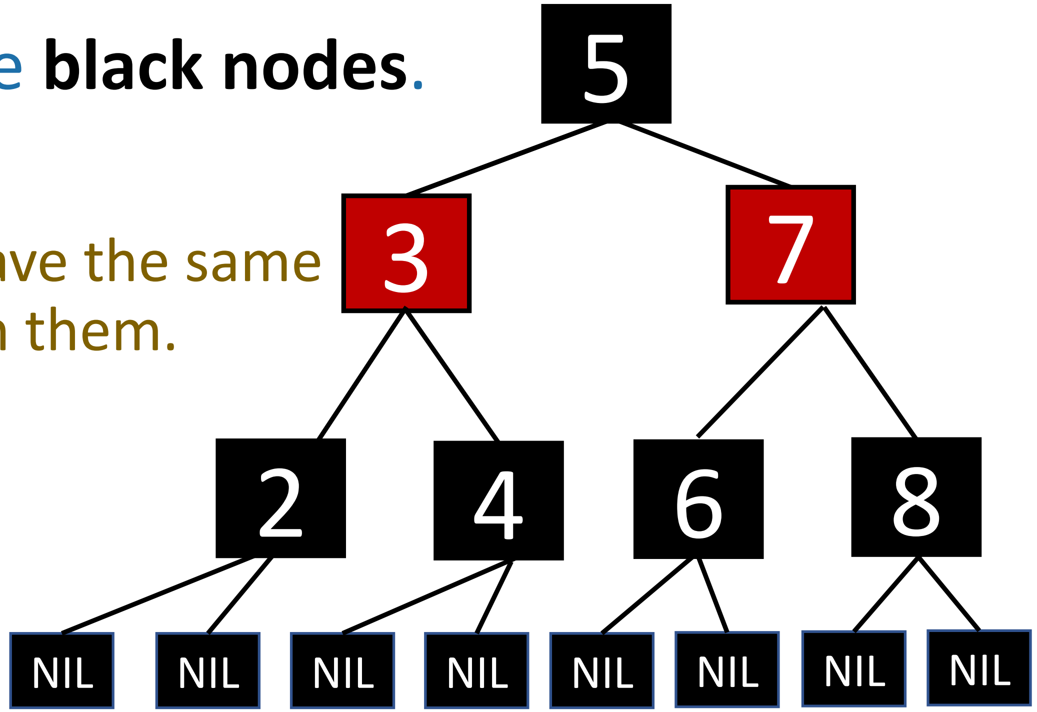


Ollie the over-achieving ostrich

Red-Black Trees

obey the following rules (which are a proxy for balance)

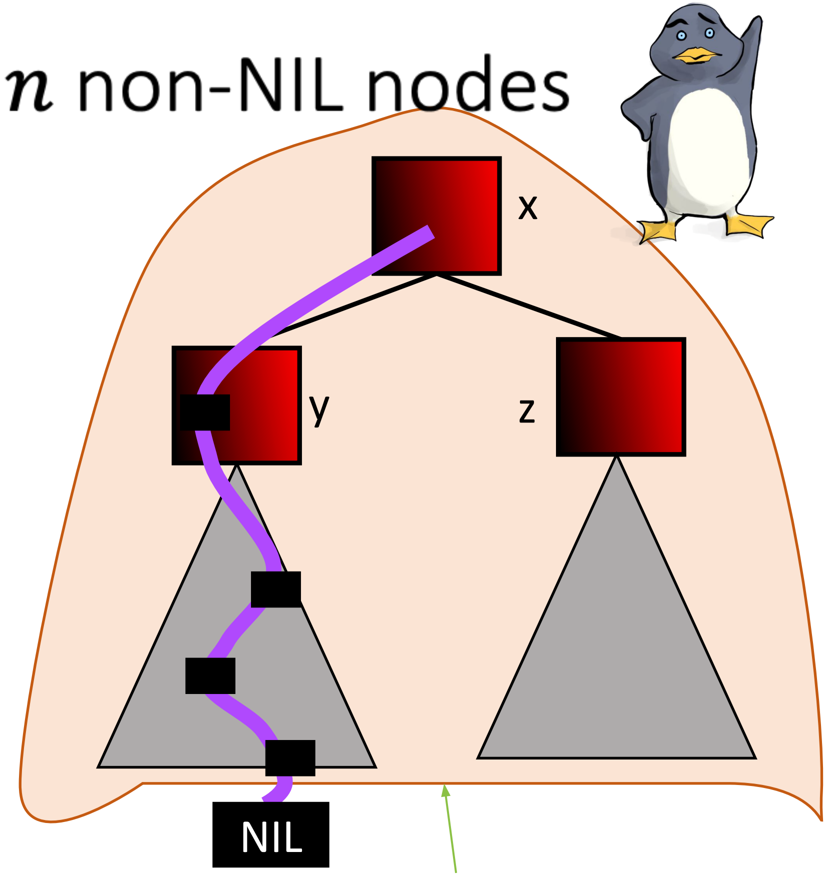
- Every node is colored **red** or **black.**
- The root node is a **black node**.
- NIL children count as **black nodes**.
- Children of a **red node** are **black nodes**.
- For all nodes x:
  - all paths from x to NIL's have the same number of **black nodes** on them.



The NIL children are treated as black nodes.

The height of a RB-tree with n non-NIL nodes is at most $2\log(n+1)$

- Define b(x) to be the number of black nodes in any path from x to NIL.
  - (excluding x, including NIL).
- Claim:
  - There are at least $2^{b(x)} - 1$ non-NIL nodes in the subtree underneath x. (Including x).
- [Proof by induction – see lecture notes]



Claim: at least $2^{b(x)} - 1$ nodes in this WHOLE subtree (of any color).

Then:

$$n \geq 2^{b(root)} - 1$$ using the Claim

$$\geq 2^{height/2} - 1$$ b(root) >= height/2 because of RBTree rules.

Rearranging:

$$n + 1 \geq 2^{height/2} \Rightarrow height \leq 2\log(n+1)$$

Running time comparison (Red-Black Trees)

	Sorted Arrays	Linked Lists	Red-Black Binary Search Trees
Search	O(log(n))	O(n)	O(log(n))
Delete	O(n)	Search +O(1)	O(log(n))
Insert	O(n)	O(1)	O(log(n))
Extract-min	O(1)	O(n)	O(log(n))

Hashing

This is a **hash table** (with chaining)

For demonstration purposes only!
This is a terrible hash function! Don't use this!

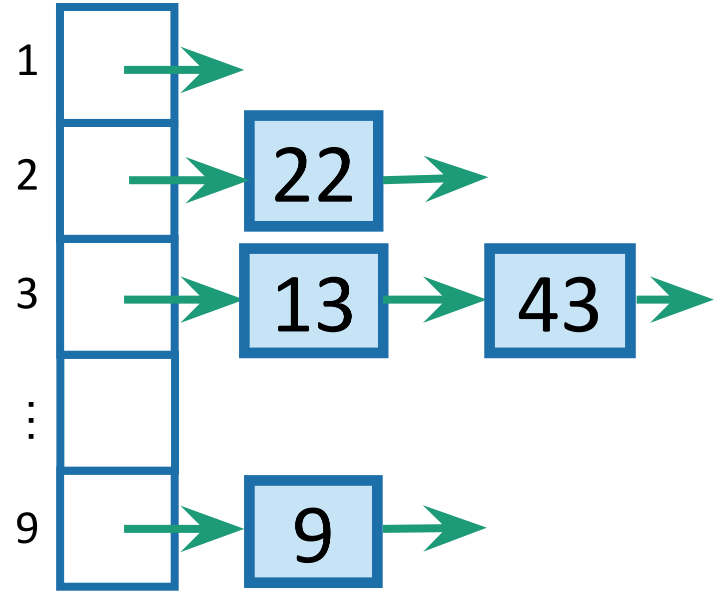
- Array of n buckets.
- Each bucket stores a linked list.
  - We can insert into a linked list in time O(1)
  - To find something in the linked list takes time O(length(list)).
- $h:U \to \{1,\ldots,n\}$ can be any function:
  - but for concreteness let's stick with h(x) = least significant digit of x.

INSERT:

13 22 43 9

SEARCH 43:

Scan through all the elements in bucket h(43) = 3.



n buckets (say n=9)

The game

What does **random** mean here? Uniformly random?

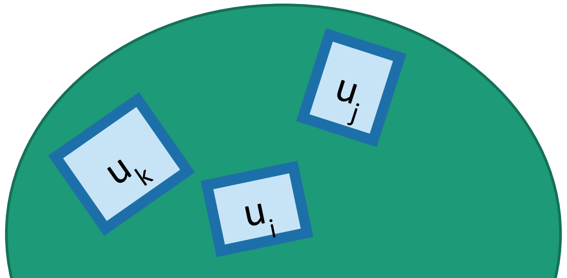
Plucky the pedantic penguin

2. You, the algorithm, chooses a **random** hash function $h: U \to \{1, \ldots, n\}$.

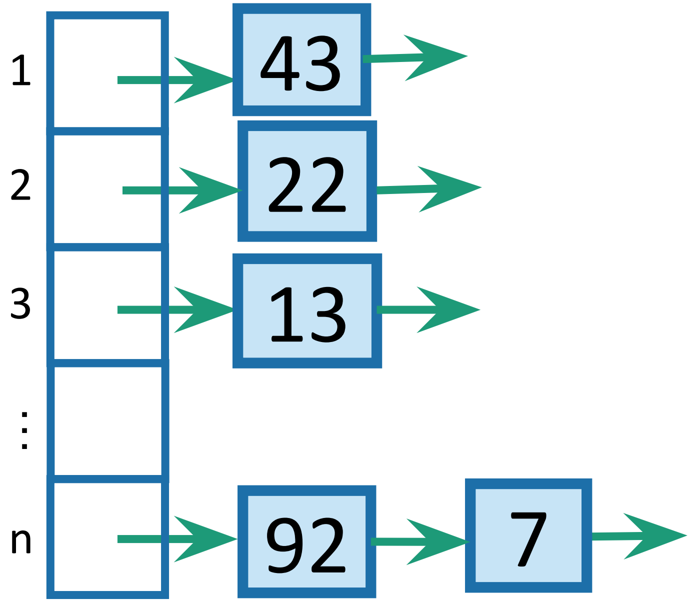
1. An adversary chooses any n items $u_1, u_2, \ldots, u_n \in U$, and any sequence of INSERT/DELETE/SEARCH operations on those items.

13 | 22 | 43 | 92 | 7

INSERT 13, INSERT 22, INSERT 43, INSERT 92, INSERT 7, SEARCH 43, DELETE 92, SEARCH 7, INSERT 92



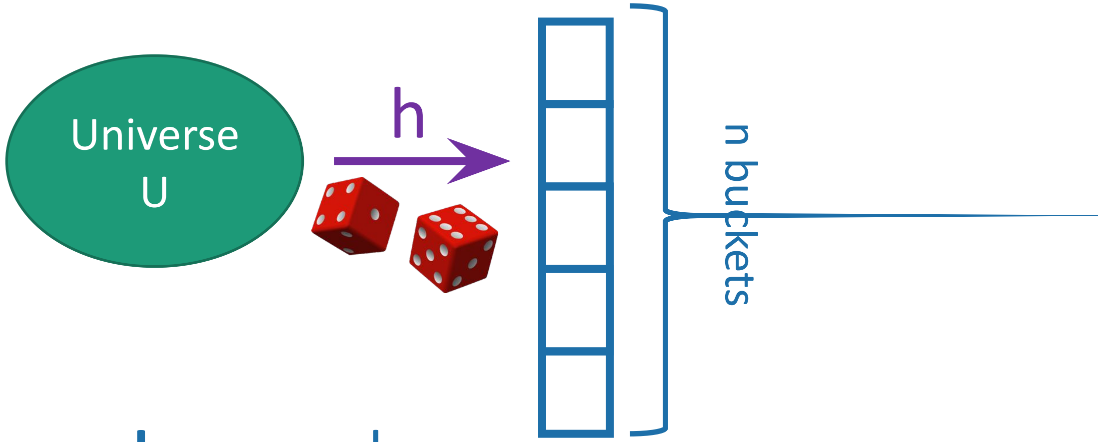
3. **HASH IT OUT** #hashpuns



What if we have lots of collisions?

Solution: Randomness

Example



- Say that h *is* uniformly random.
  - That means that **h(1)** is a **uniformly random** number between 1 and n.
  - **h(2)** is also a **uniformly random** number between 1 and n, independent of h(1).
  - **h(3)** is also a **uniformly random** number between 1 and n, independent of h(1), h(2).
  - …
  - **h(n)** is also a **uniformly random** number between 1 and n, independent of h(1), h(2), …, h(n-1).

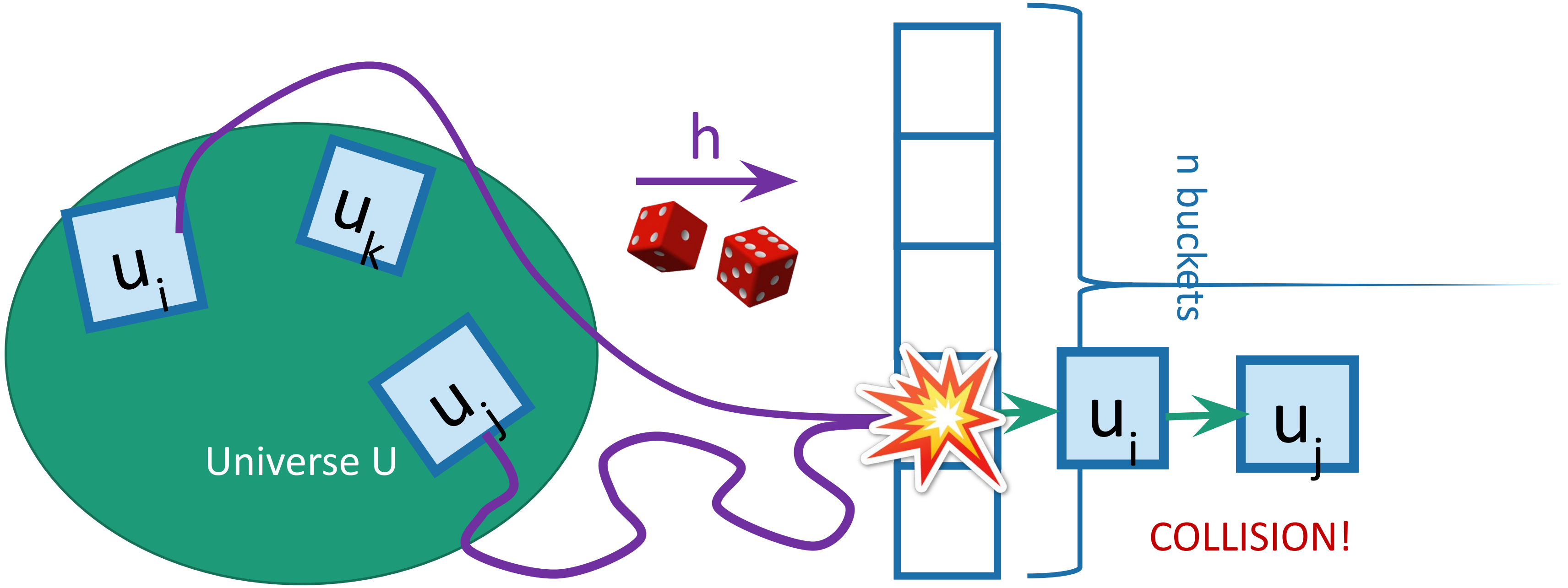
h is uniformly random

Expected number of items in u_i's bucket?

- $E[\ ] = \sum_{j=1}^{n} P\{\, h(u_i) = h(u_j) \}$
- $\quad = 1 + \sum_{j \neq i} P\{\, h(u_i) = h(u_j) \}$
- $\quad = 1 + \sum_{j \neq i} 1/n$
- $\quad = 1 + \frac{n-1}{n} \leq 2.$

That's what we wanted.

All that we needed was that this is 1/n



Universal hash family

Let's stare at this definition

- H is a ***universal hash family*** if:
  - When h is chosen uniformly at random from H,

$$\text{for all } u_i, u_j \in U \quad \text{with } u_i \neq u_j,$$
$$P_{h\in H}\{\, h(u_i) = h(u_j)\} \leq \frac{1}{n}$$

Sounds good? But How to pick the Hash functions?

Solution: Universal Hash Family

Why Do We Need Hash Family?

- Remember that after we select the hash function, we need to store the hash function for search/delete.
    - What will happen if we did not remember the hash function?

- The set of all hash functions contains n^M functions.
  So to store the index of a function, we need $\log(n^M)=M\log(n)$ bits ☹.

- A smaller universal hash family it's easier to remember which function from the family we're using.

An example of small universal hash family

- Here's one:
  - Pick a prime $p \geq M$.
  - Define

$$f_{a,b}(x) = ax + b \quad mod\, p$$

$$h_{a,b}(x) = f_{a,b}(x) \quad mod\, n$$

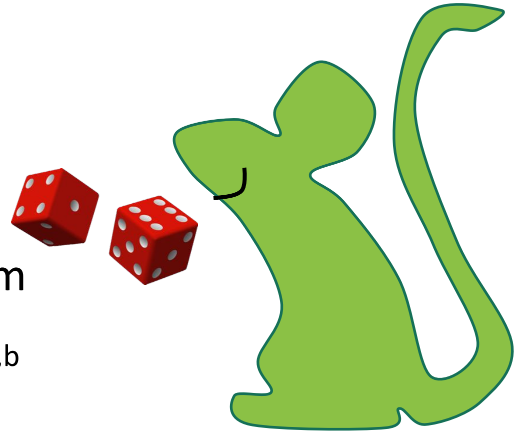
  - Claim:

$$H = \{\, h_{a,b}(x) \;:\; a \in \{1, \dots, p-1\}, b \in \{0, \dots, p-1\} \,\}$$

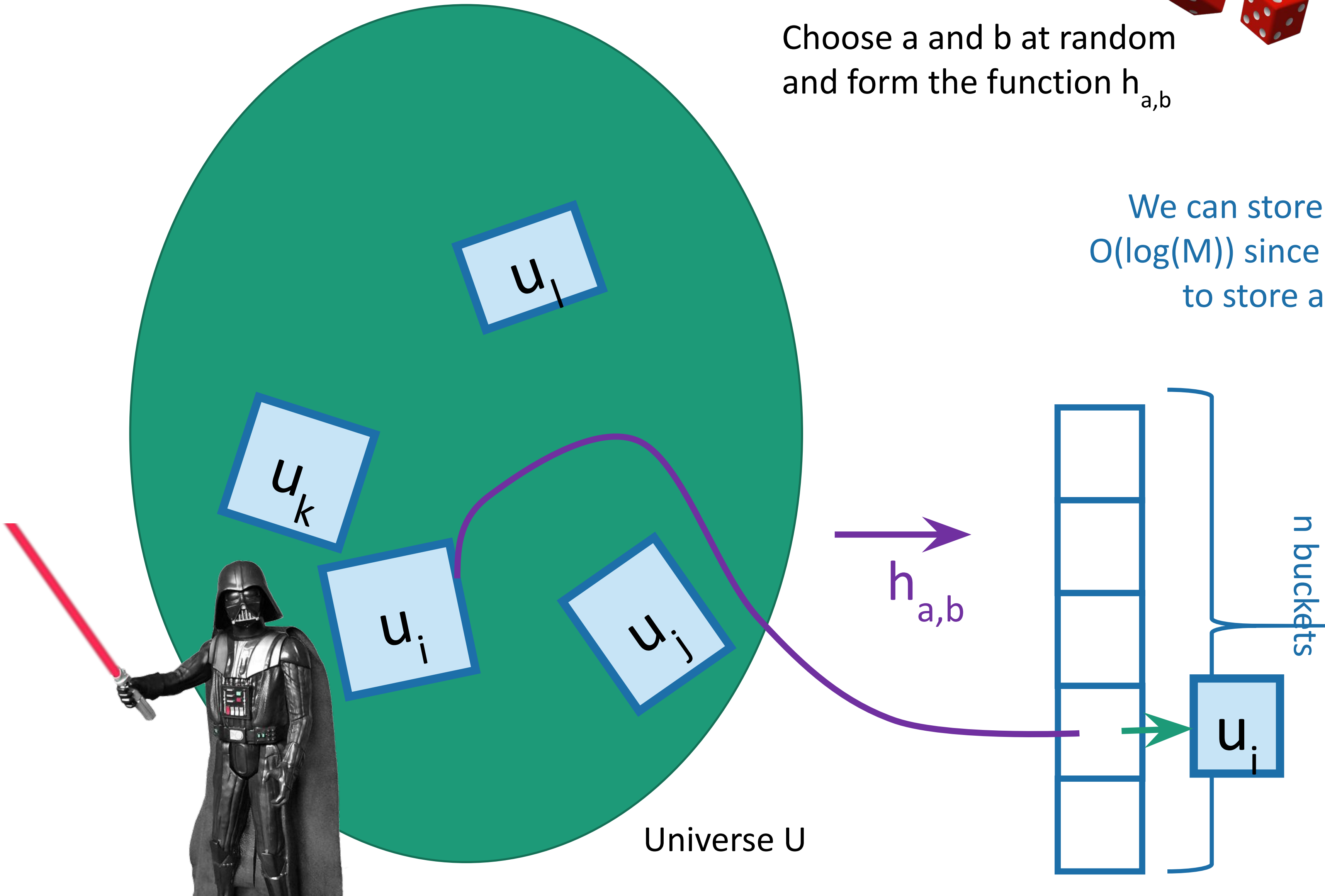
is a universal hash family.

So the whole scheme will be

Choose a and b at random and form the function $h_{a,b}$



We can store h in space O(log(M)) since we just need to store a and b.



Probably these buckets will be pretty balanced.

Conclusion:

- We can build a hash table that supports INSERT/DELETE/SEARCH in O(1) expected time,
  - if we know that only n items are every going to show up, where n is waaaayyyyyy less than the size M of the universe.

- The space to implement this hash table is

$$O(n \log(M)) \text{ bits.}$$

  - O(n) buckets
  - O(n) items with log(M) bits per item
  - O(log(M)) to store the hash fn.
- M is waaayyyyyy bigger than n, but log(M) probably isn't.

Thank you!