CS 161 Section 5

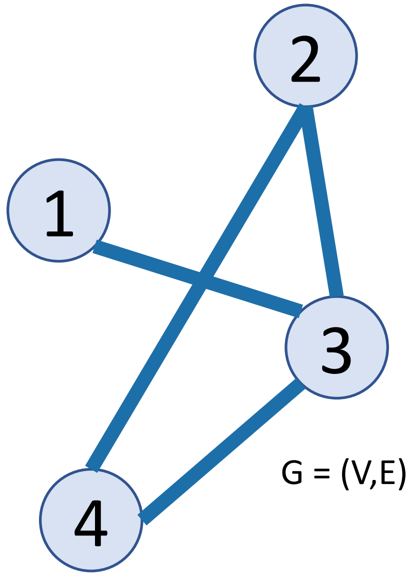
CA: [name of CA]

Lecture Recap

Undirected Graphs

- Have vertices and edges
  - V is the set of vertices
  - E is the set of edges
  - Formally, a graph is G = (V,E)

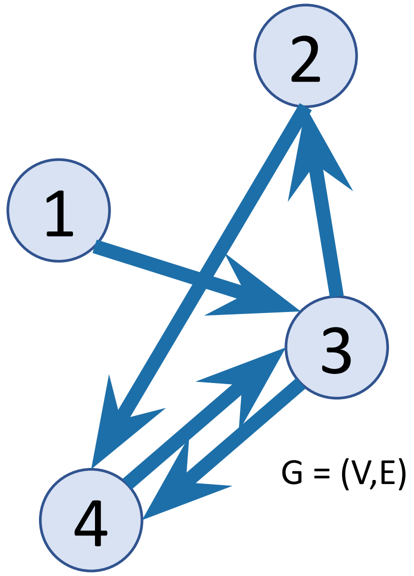
- Example
  - V = {1,2,3,4}
  - E = { {1,3}, {2,4}, {3,4}, {2,3} }



- The **degree** of vertex 4 is 2.
  - There are 2 edges coming out.
- Vertex 4's **neighbors** are 2 and 3

Directed Graphs

- Have vertices and edges
  - V is the set of vertices
  - E is the set of **DIRECTED** edges
  - Formally, a graph is G = (V,E)
- Example
  - V = {1,2,3,4}
  - E = { (1,3), (2,4), (3,4), (4,3), (3,2) }



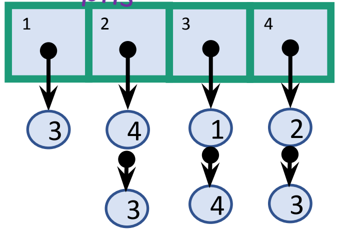
- The **in-degree** of vertex 4 is 2.
- The **out-degree** of vertex 4 is 1.
- Vertex 4's **incoming neighbors** are 2,3
- Vertex 4's **outgoing neighbor** is 3.

Trade-offs

n = # of vertices
m = # of edges

Say there are n vertices and m edges.

Generally better for **sparse** graphs

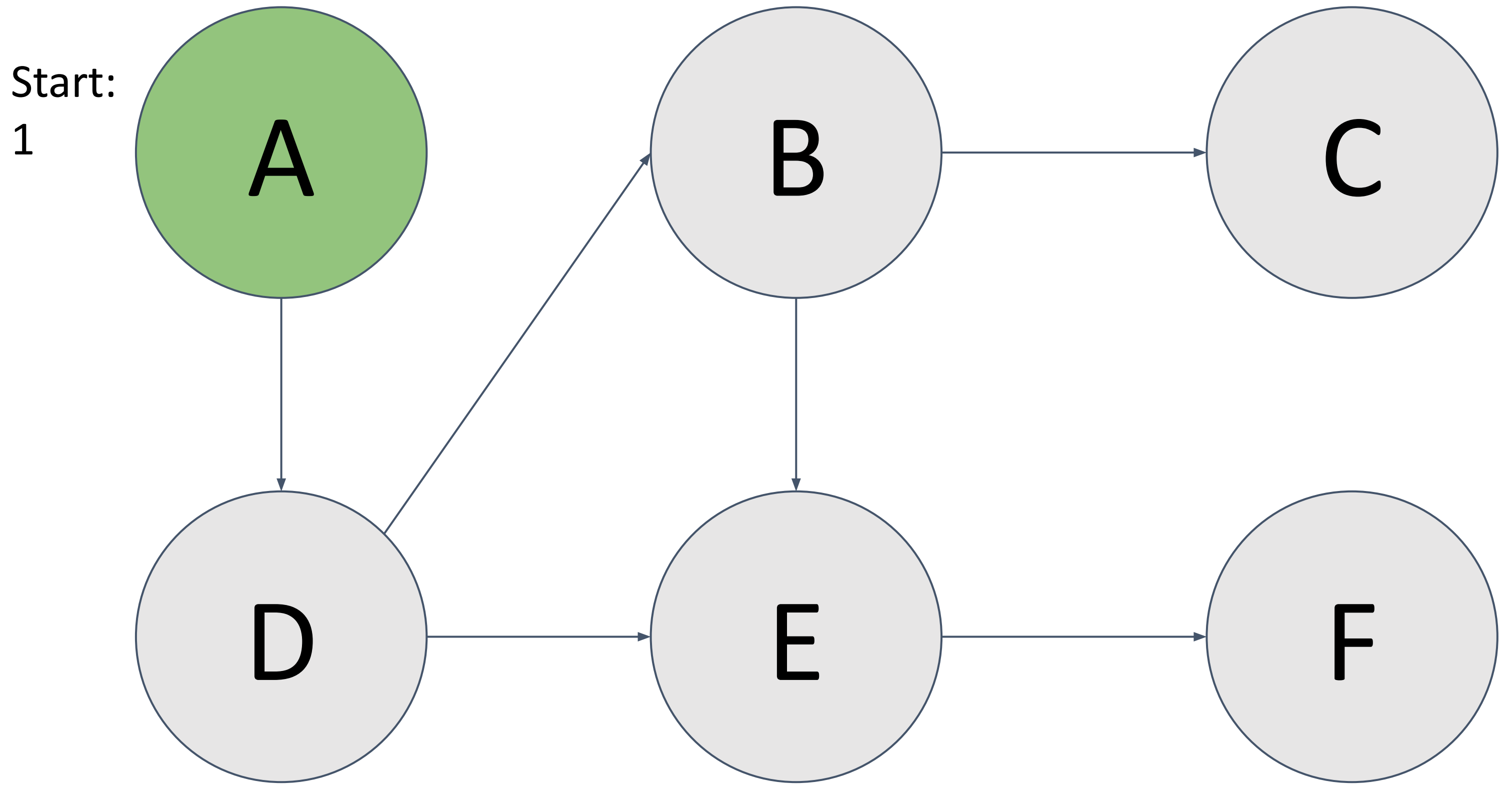
	$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$	
Edge membership Is e = {v,w} in E?	O(1)	O(deg(v)) or O(deg(w))
Neighbor query Give me v's neighbors.	O(n)	O(deg(v))
Space requirements	$O(n^2)$	O(n + m)

We'll assume this representation for the rest of the class

Depth-First Search: Pseudocode with start and end times

```
DFS(w, currentTime):
    w.startTime = currentTime
    currentTime ++
    Mark w as in progress.
    for v in w.neighbors:
        if v is unvisited:
            currentTime
              = DFS(v, currentTime)
            currentTime ++
    w.finishTime = currentTime
    Mark w as all done
    return currentTime
```

Example DFS (we sort edges in alphabetical order)



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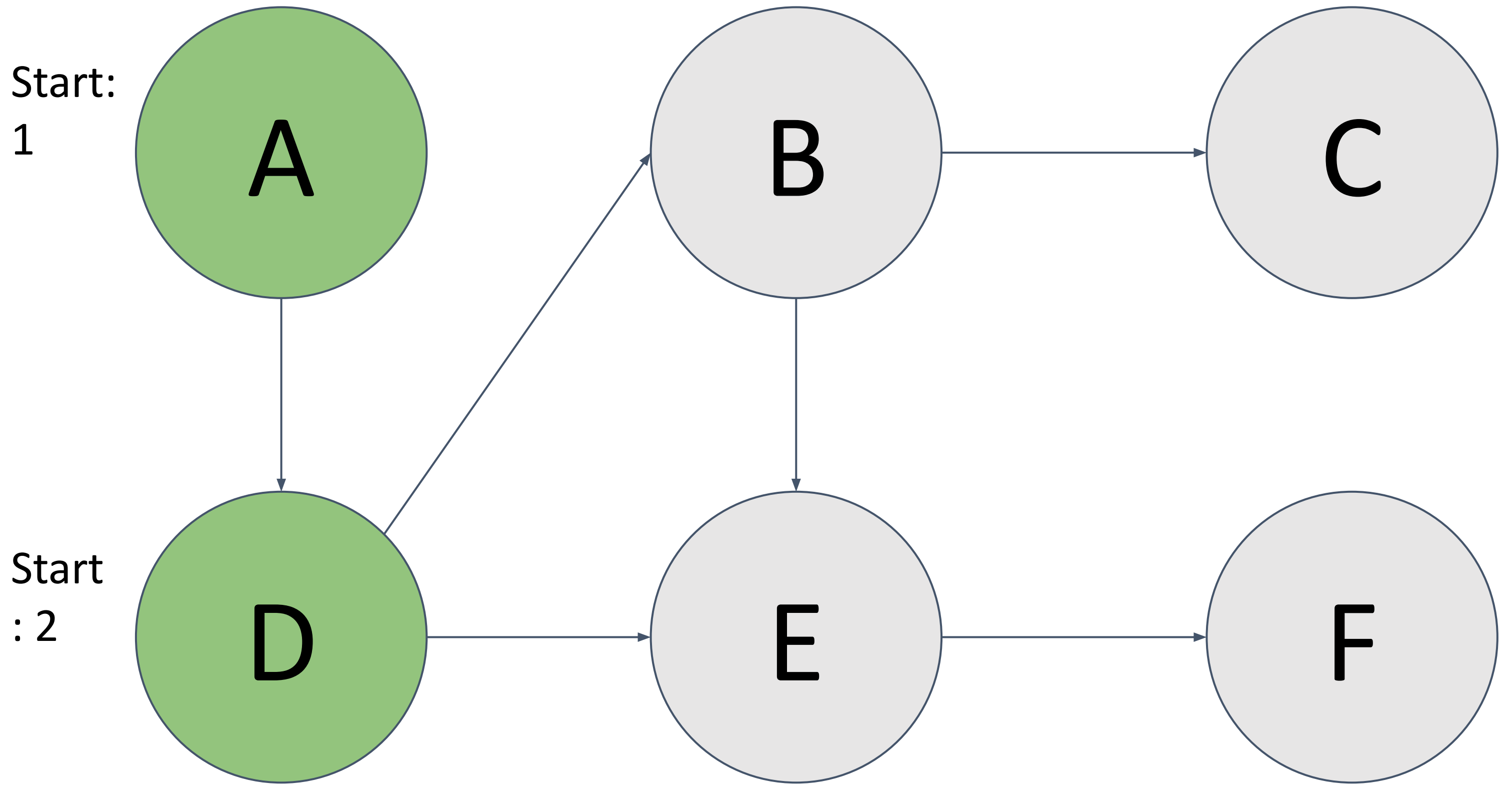

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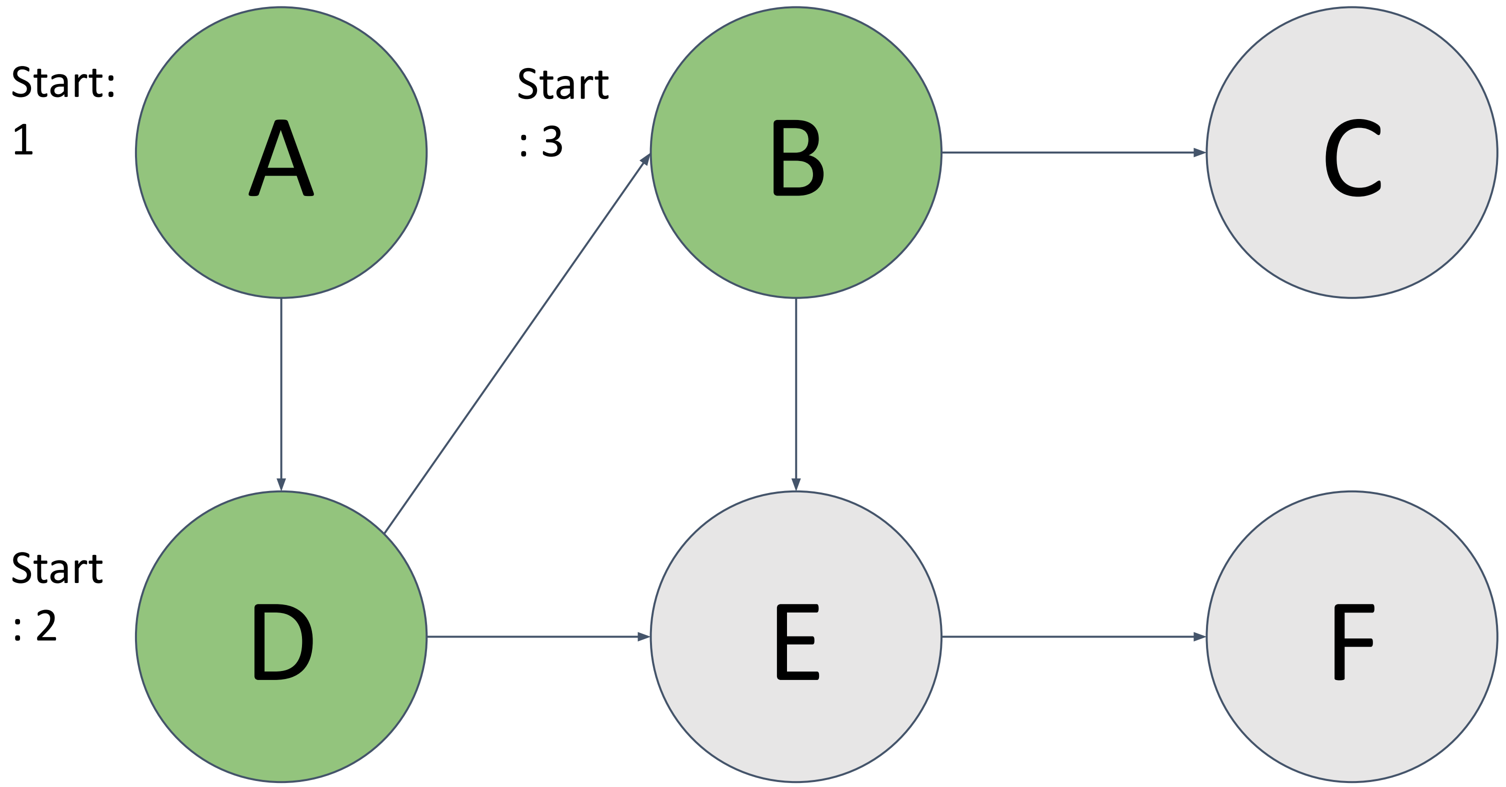

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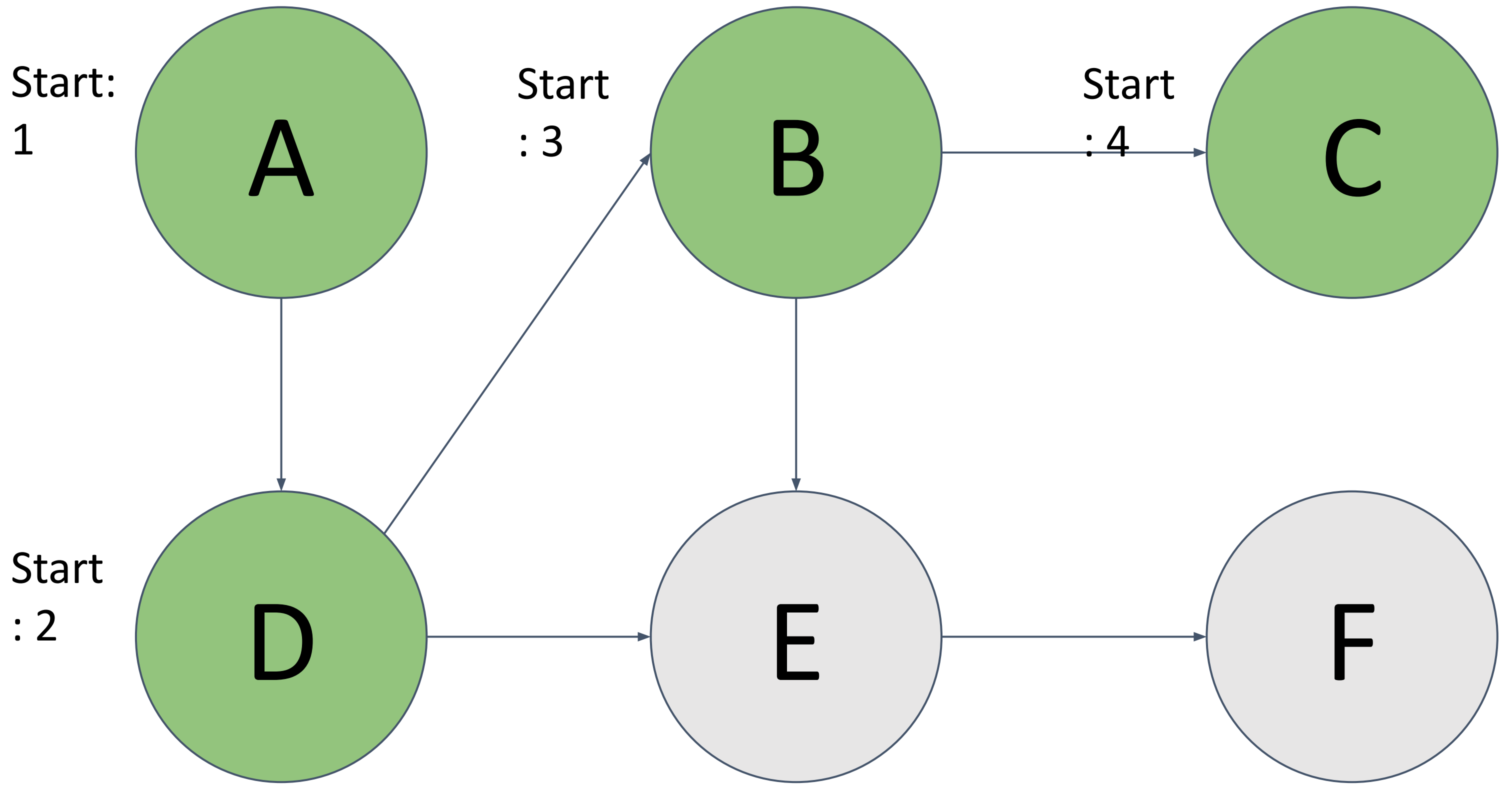

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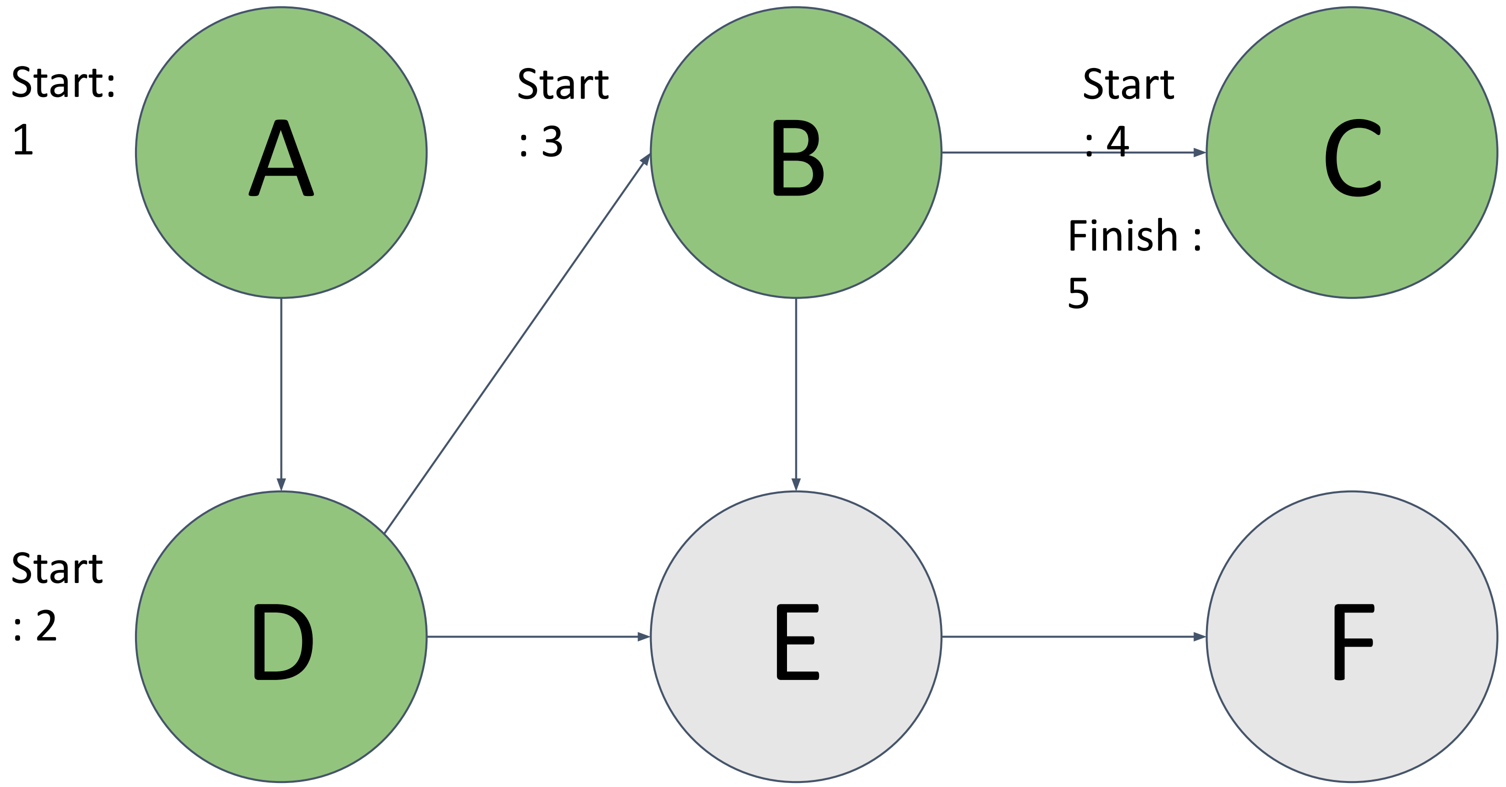

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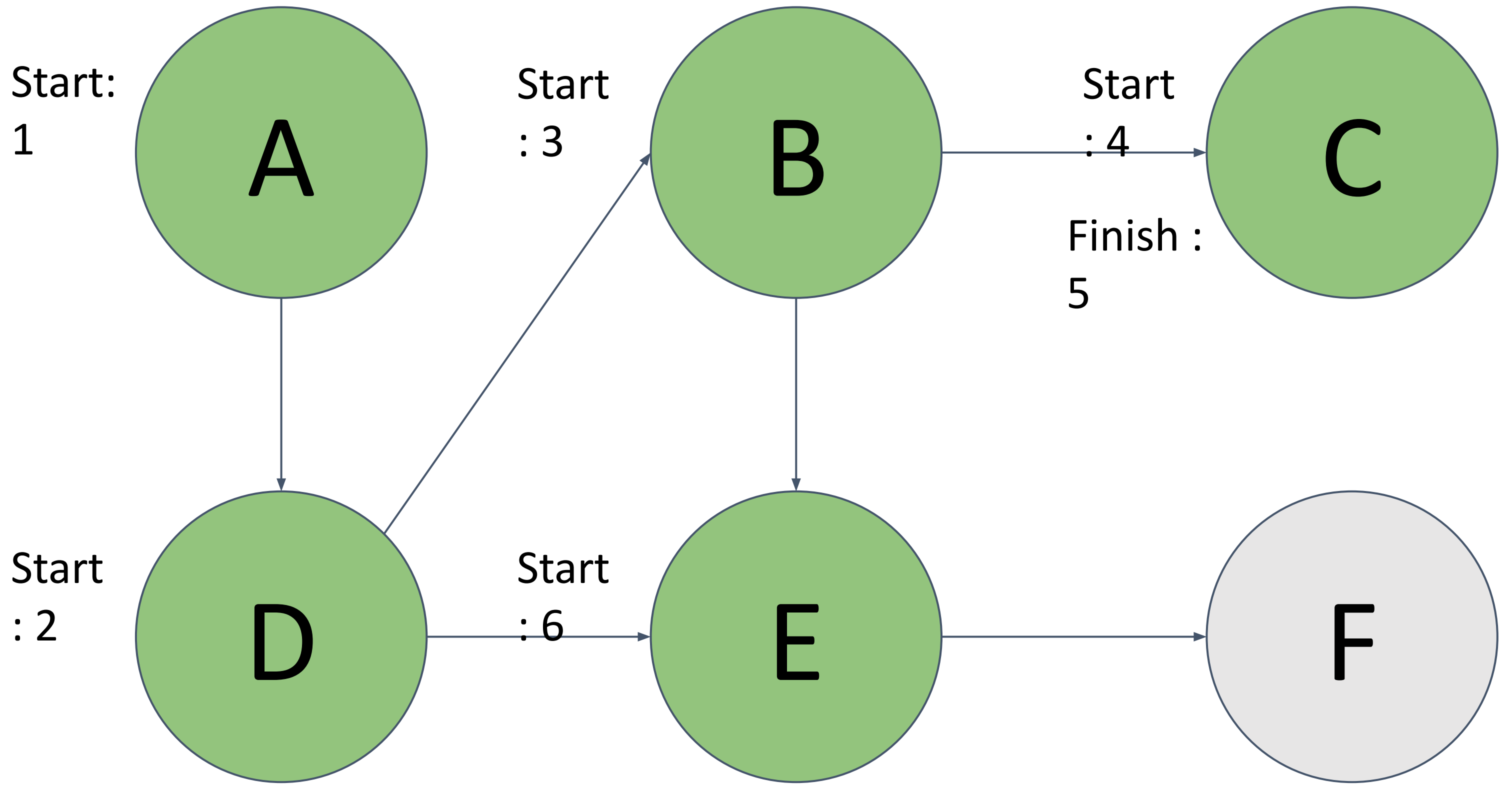

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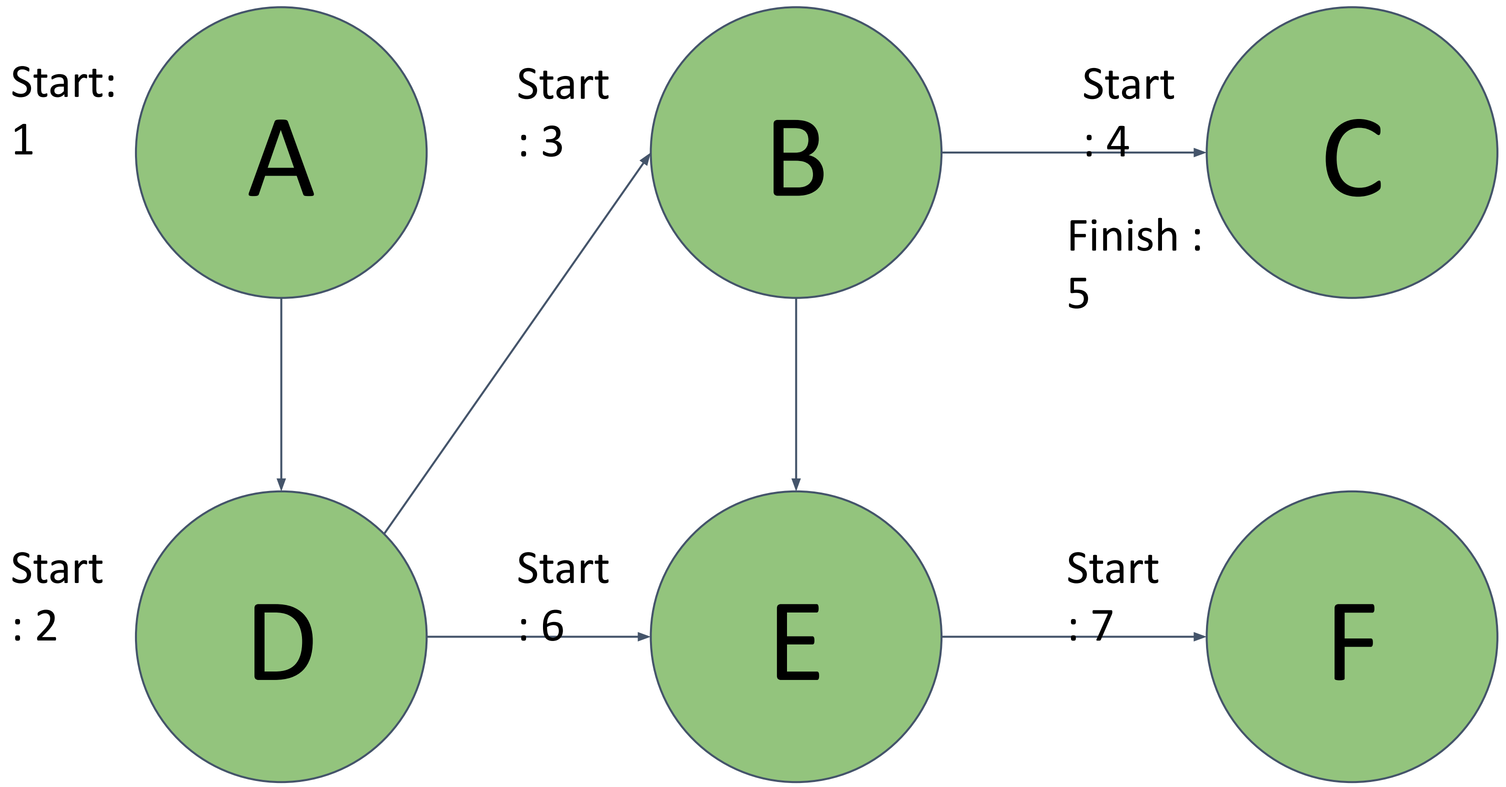

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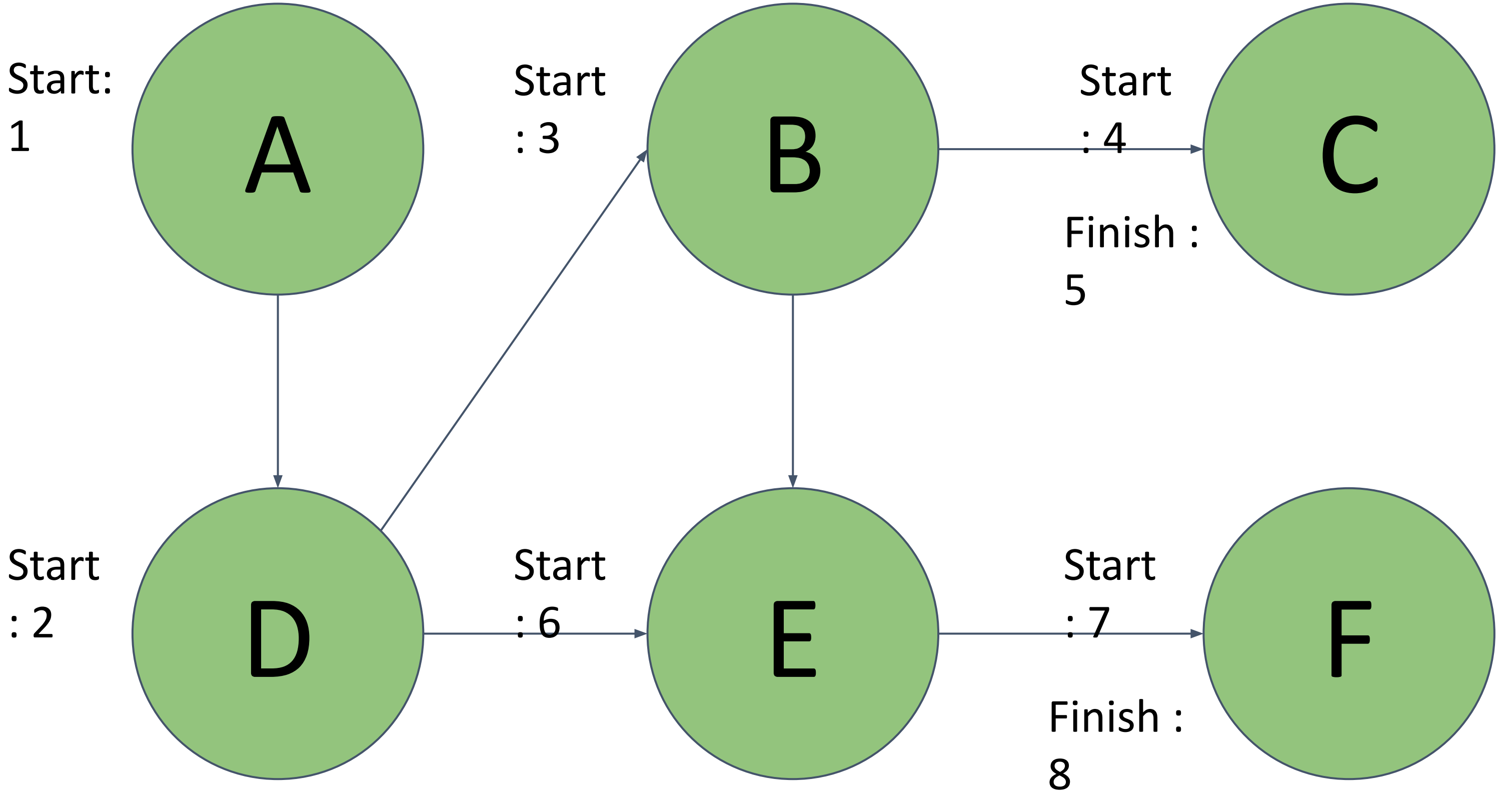

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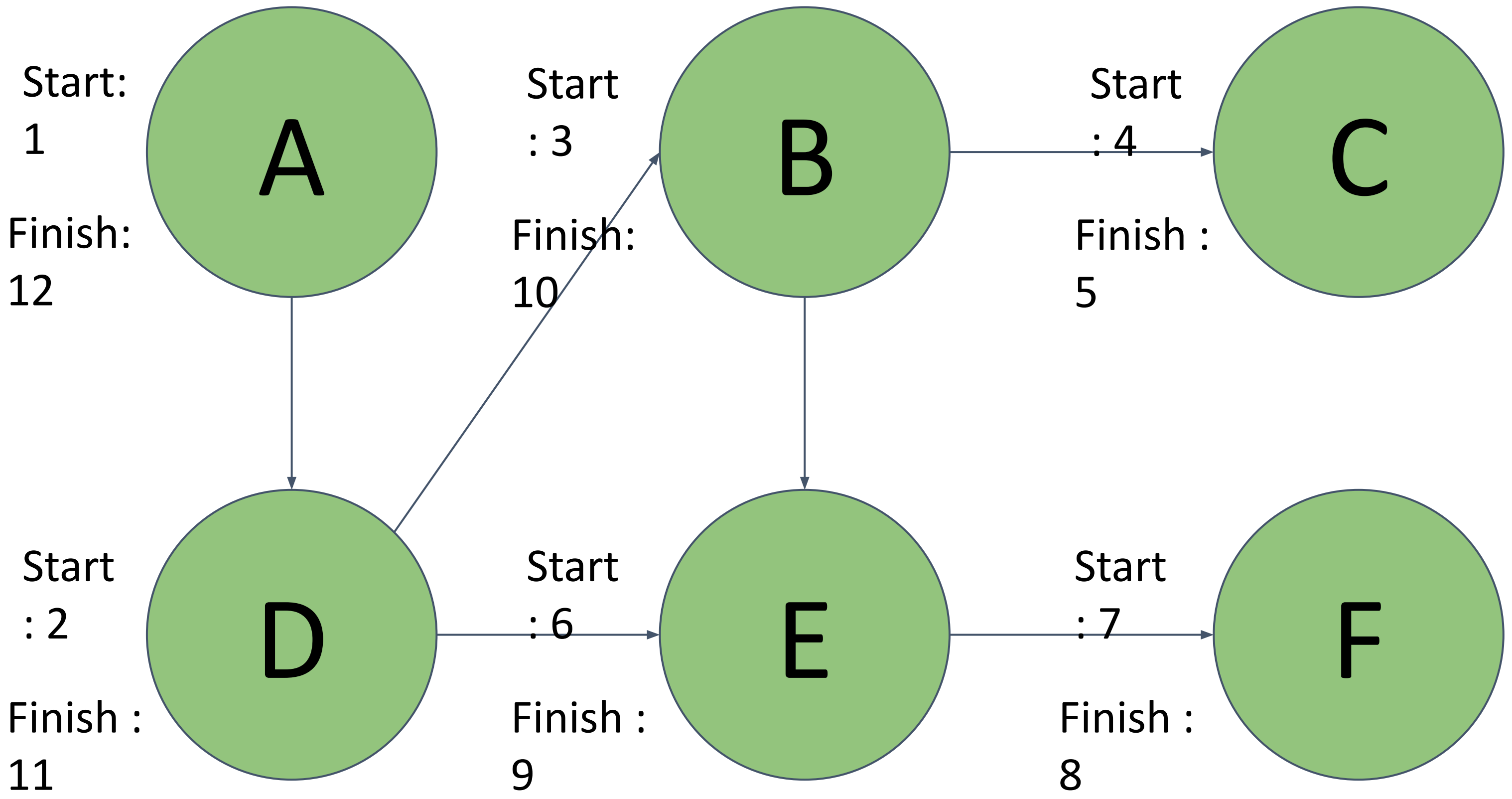


Relation in DFS tree and start/finish time

- u is v’s ancestor if start(u) < start(v) < finish(v) < finish(u)
- u is v’s descendant if start(v) < start(u) < finish(u) < finish(v)
- u is v’s cousin if start(v) < finish(v) < start(u) < finish(u)
- Notice that startTime and finishTime never interleave (start(v)<start(u)<finish(v)<finish(u)) will never happen.

Breadth-First Search: Pseudocode

- BFS(G, v)
- Input: graph G = (V, E); node v in V, edge e in E
- Output:
  - Array visited: V -> {True, False}
  - Layer i of vertices. Initialize L_0 with start vertex v.

Breadth-First Search: Pseudocode

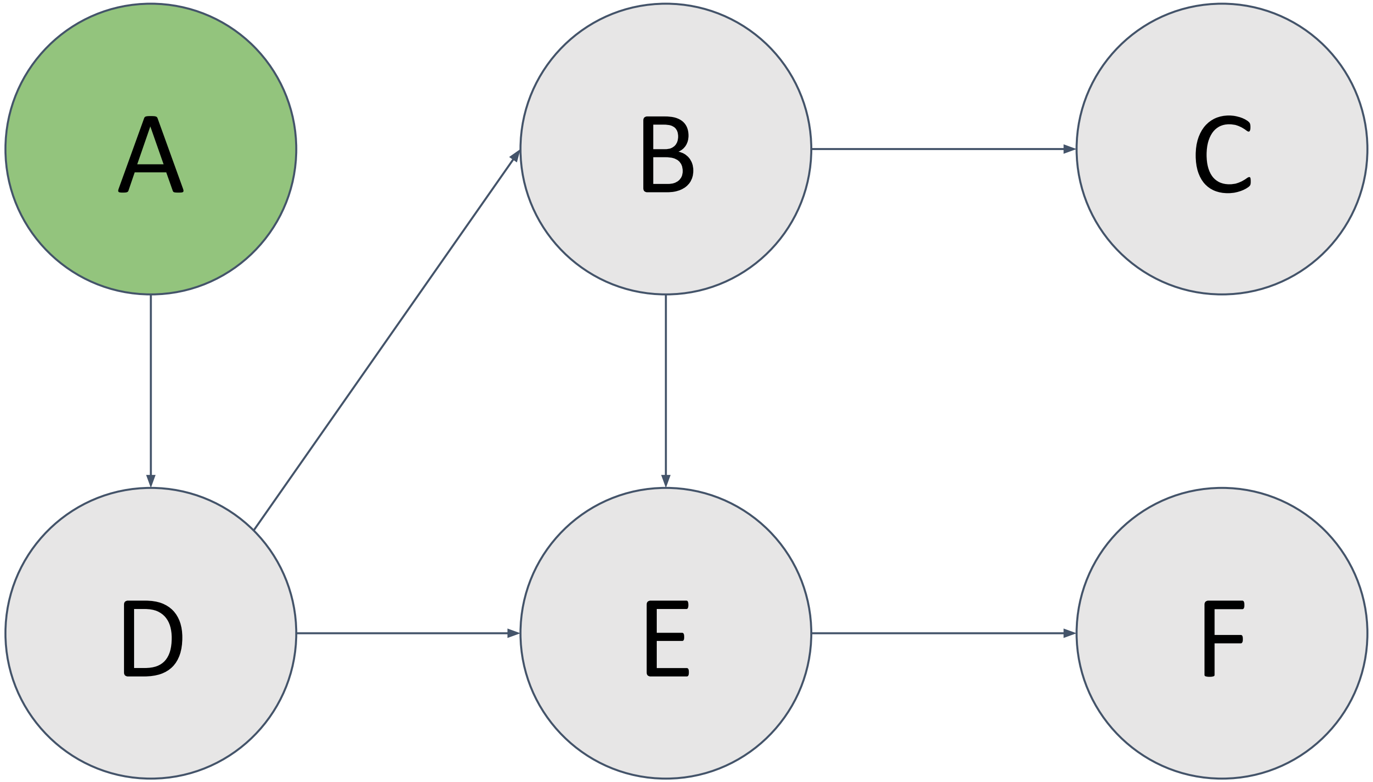
- BFS(G, v)
- Input: graph G = (V, E); node v in V, edge e in E
- Output:
    - Array visited: V -> {True, False}
    - Layer of vertices to visit.
        - Initialize L_0 with start vertex v. Mark v as visited
    - While L_i is non-empty: # keep looping as we have unexplored vertices
        - Visit neighbours of elements in L_i # vertices currently being explored

Breadth-First Search: Pseudocode

```
● BFS(G, v)
● Input: graph G = (V, E); node v in V, edge e in E
● Output:
    ○ Array visited: V -> {True, False}
    ○ Layer of vertices to visit.
        ■ Initialize $L_0$ with start vertex v. Mark v as visited
    ○ While $L_i$ is non-empty:     # keep looping as we have unexplored vertices
        ■ for u in $L_i$:       # vertices currently being explored
            ● for v neighbour of u:
                ○ if not visited v: # only work on vertices we have not explored
                    ■ mark v visited and add to $L_{i+1}$
        ■ i += 1
```

Can we use queues instead of lists?

Example BFS (we sort edges in alphabetical order)



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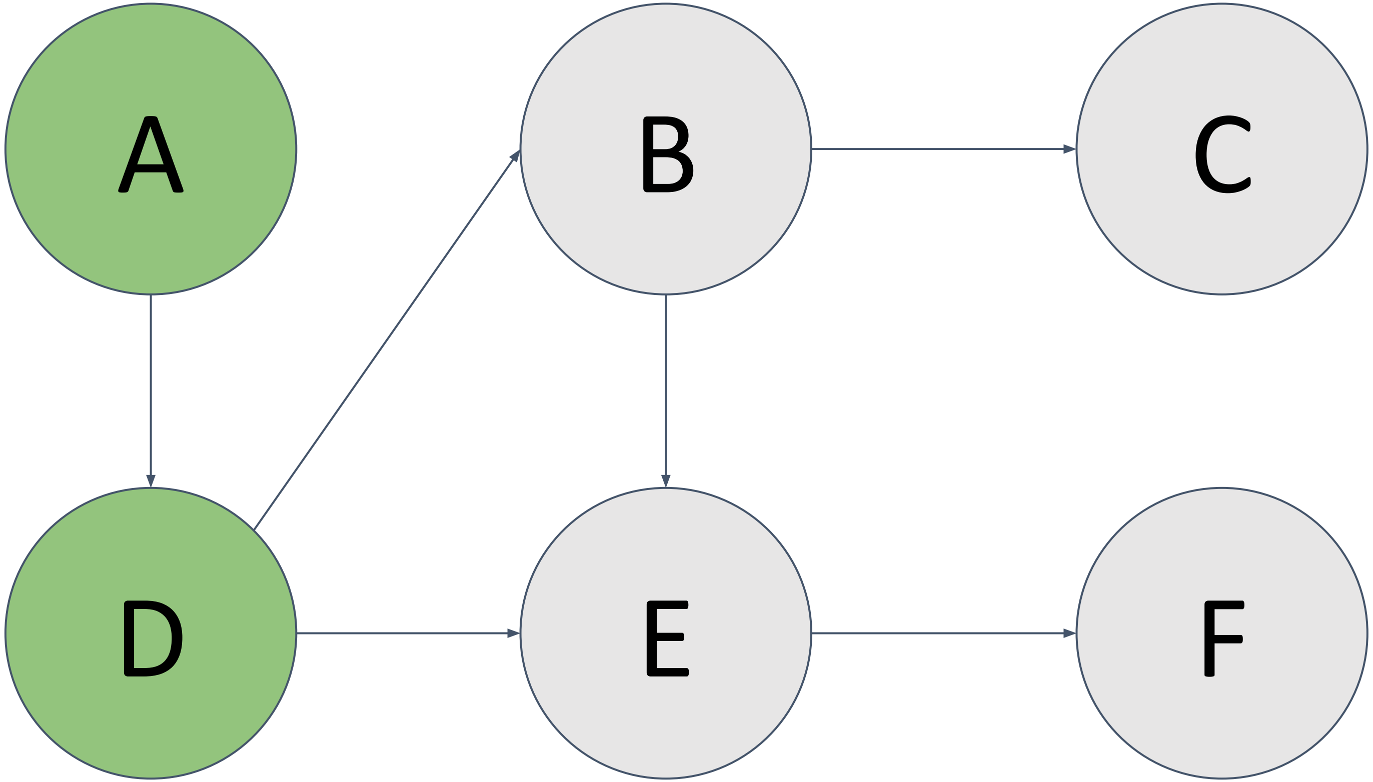

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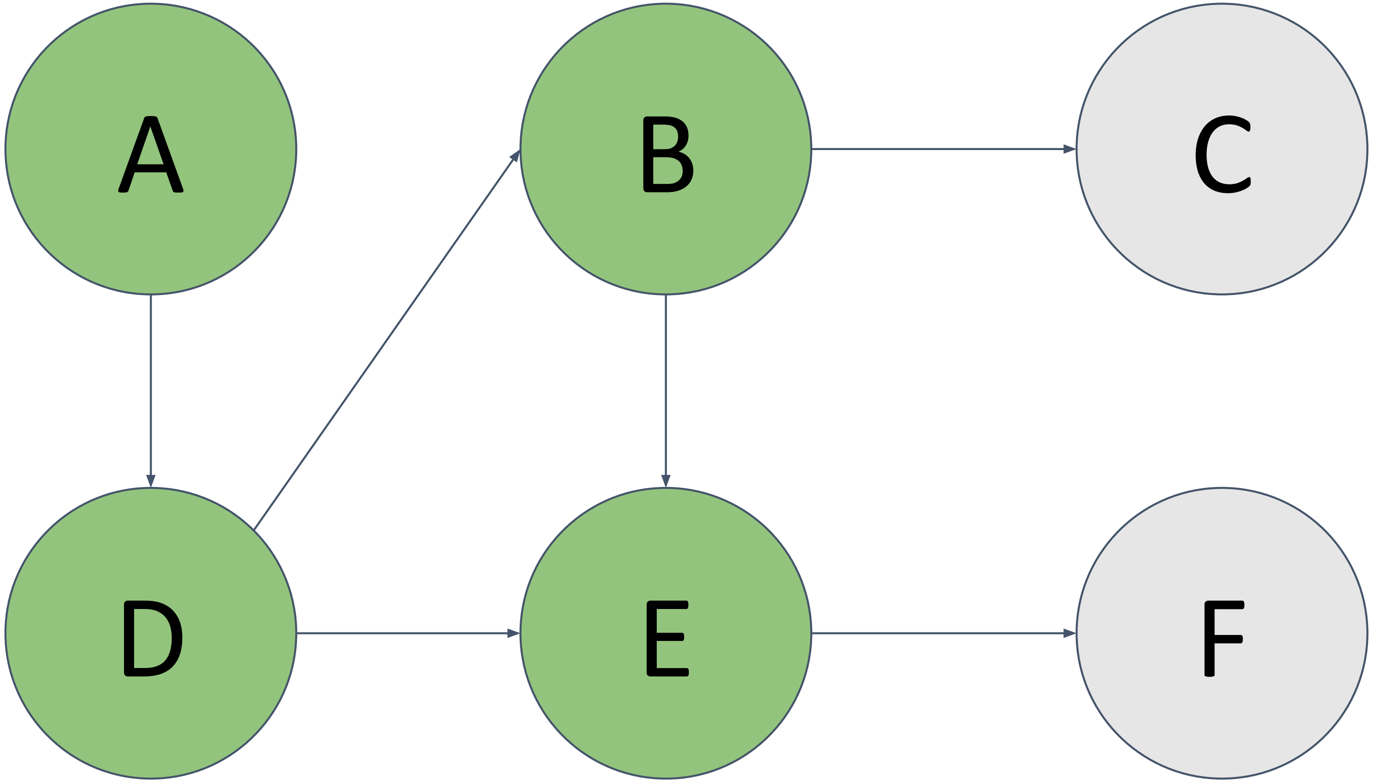

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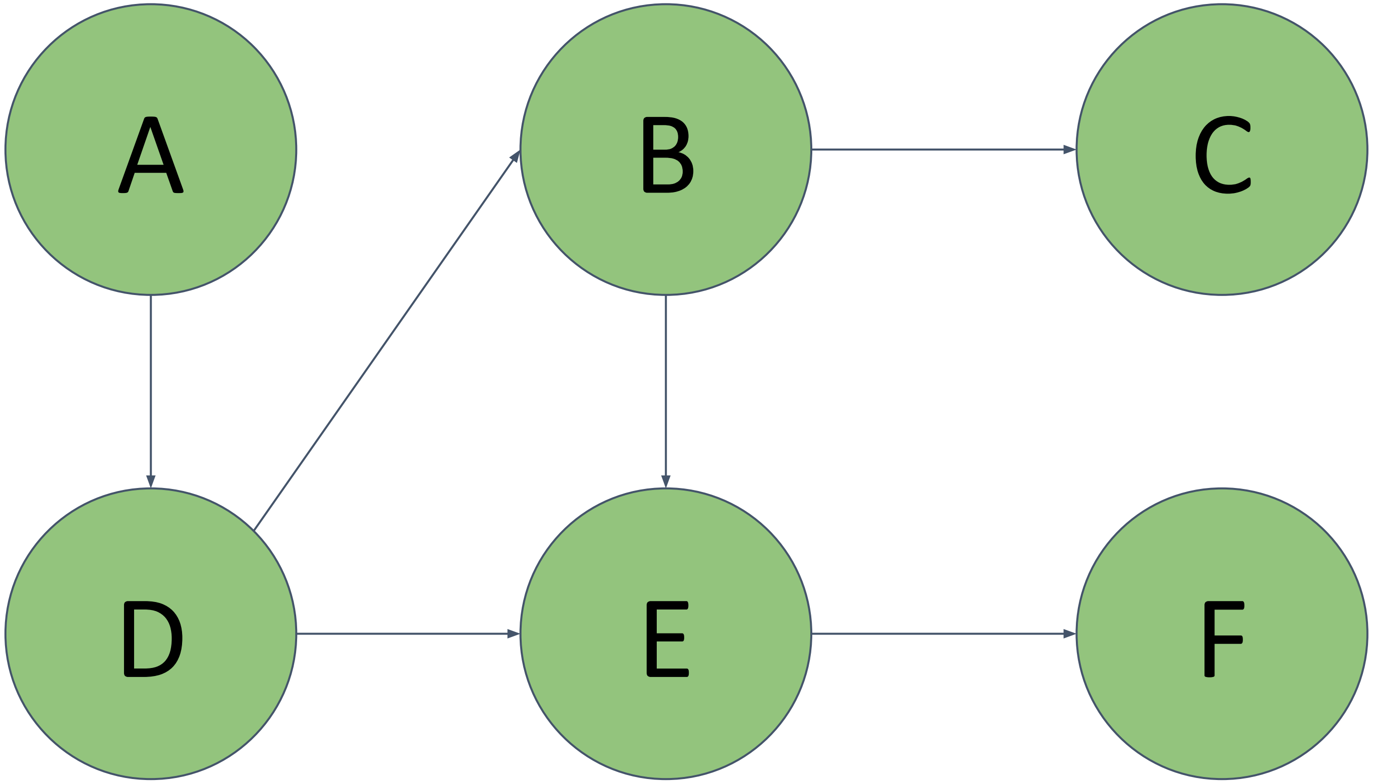
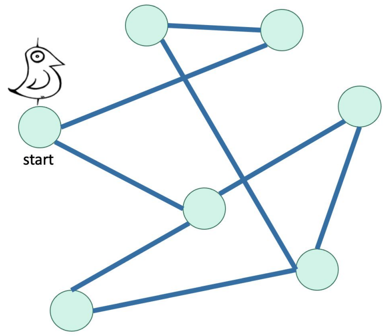
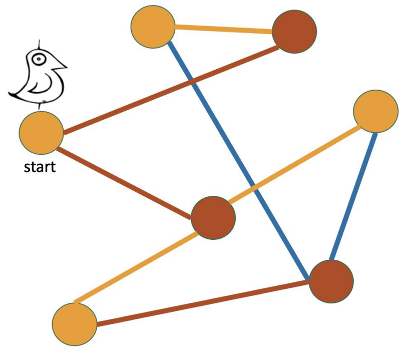


What can we do with BFS?

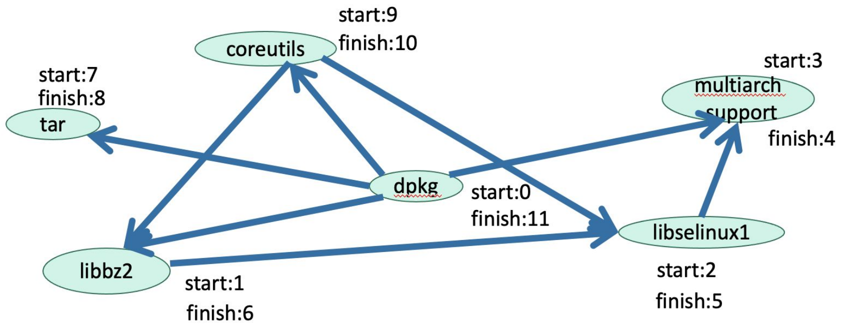
- Find all vertices reachable from the start vertex
- Find the shortest paths from a vertex to all the others
- Check if a graph is bipartite graph.
    - Color the levels of the BFS tree in alternating colors.
    - If you never color two connected nodes the same color, then it is bipartite.
    - Otherwise, it's not.

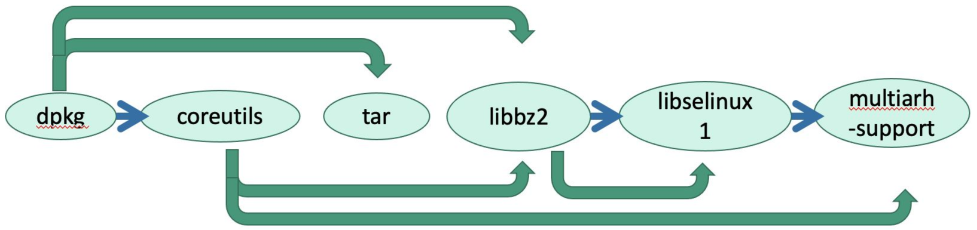




What can we do with DFS?

- Find all vertices reachable from the start vertex
- Topological sort for directed graph without any cycle (DAG)
    - In reverse order of finishing time in DFS!
- Both algorithms run in O(|V| + |E|).





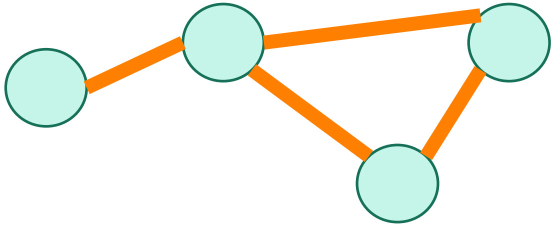
Strongly Connected Components

Please do not confuse them!

Strongly *vs* Connected Components

***Un*directed graphs**

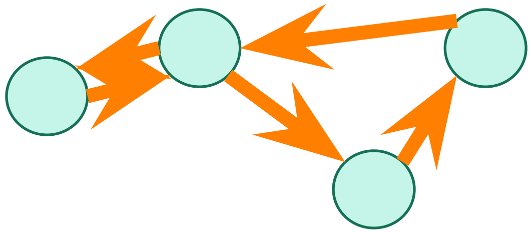
Connected Components (CC)



Find with BFS/DFS

Directed graphs

Strongly Connected Components (**S**CC)



Find with DFS (today)

[Actual algorithm!]

Algorithm

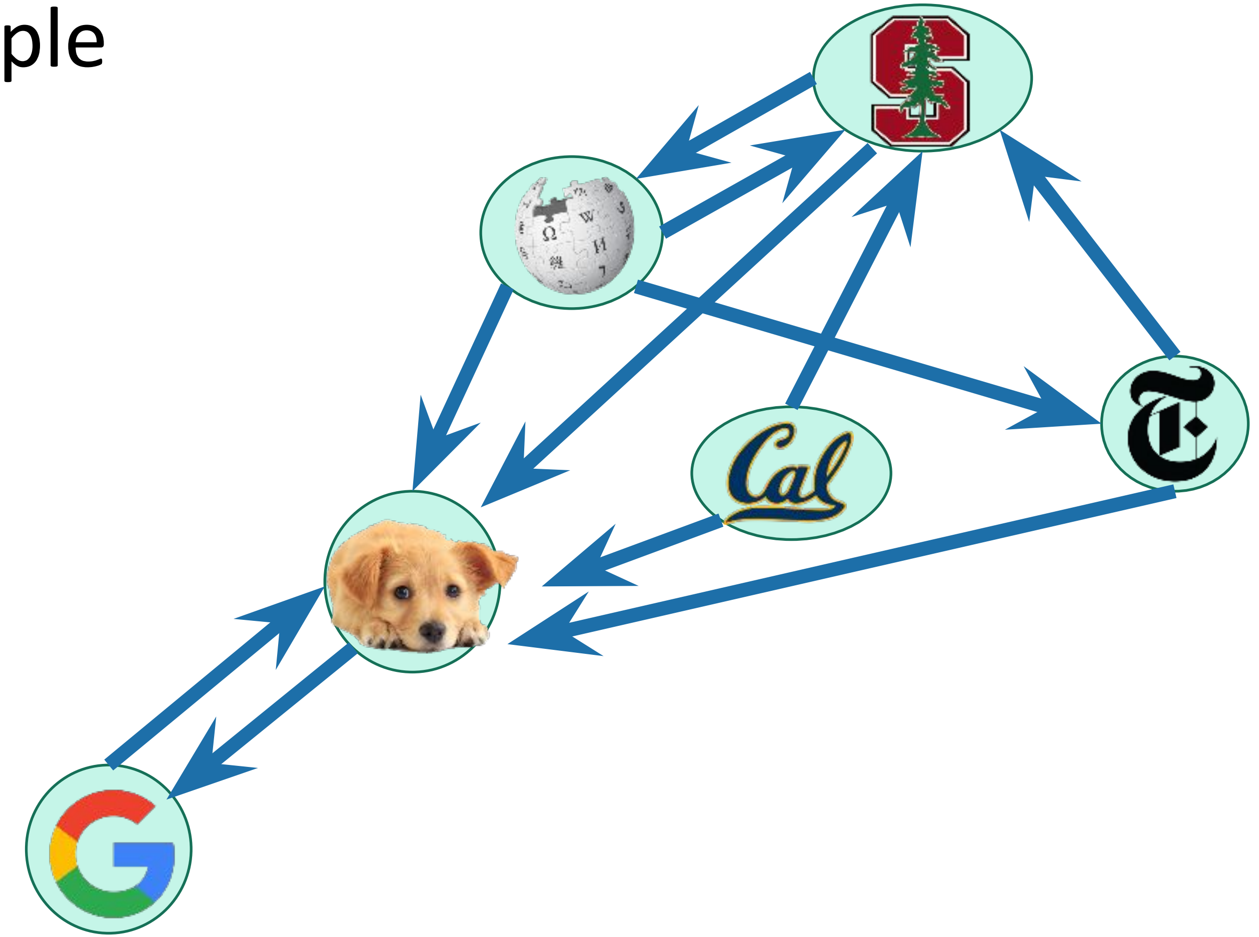
Running time: O(n + m)

- Run DFS.
  - Choose starting vertices in any order.
  - Order vertices by reverse finishing times.
- Reverse all the edges in the graph.
- Do DFS again to create **a new DFS forest**.
  - (Using the reverse-finish-time order from the first DFS run.)
- The SCCs are the different trees in the **new DFS forest.**

(This is basically "PretendTopoSort")

(This doesn't change SCCs)

Example



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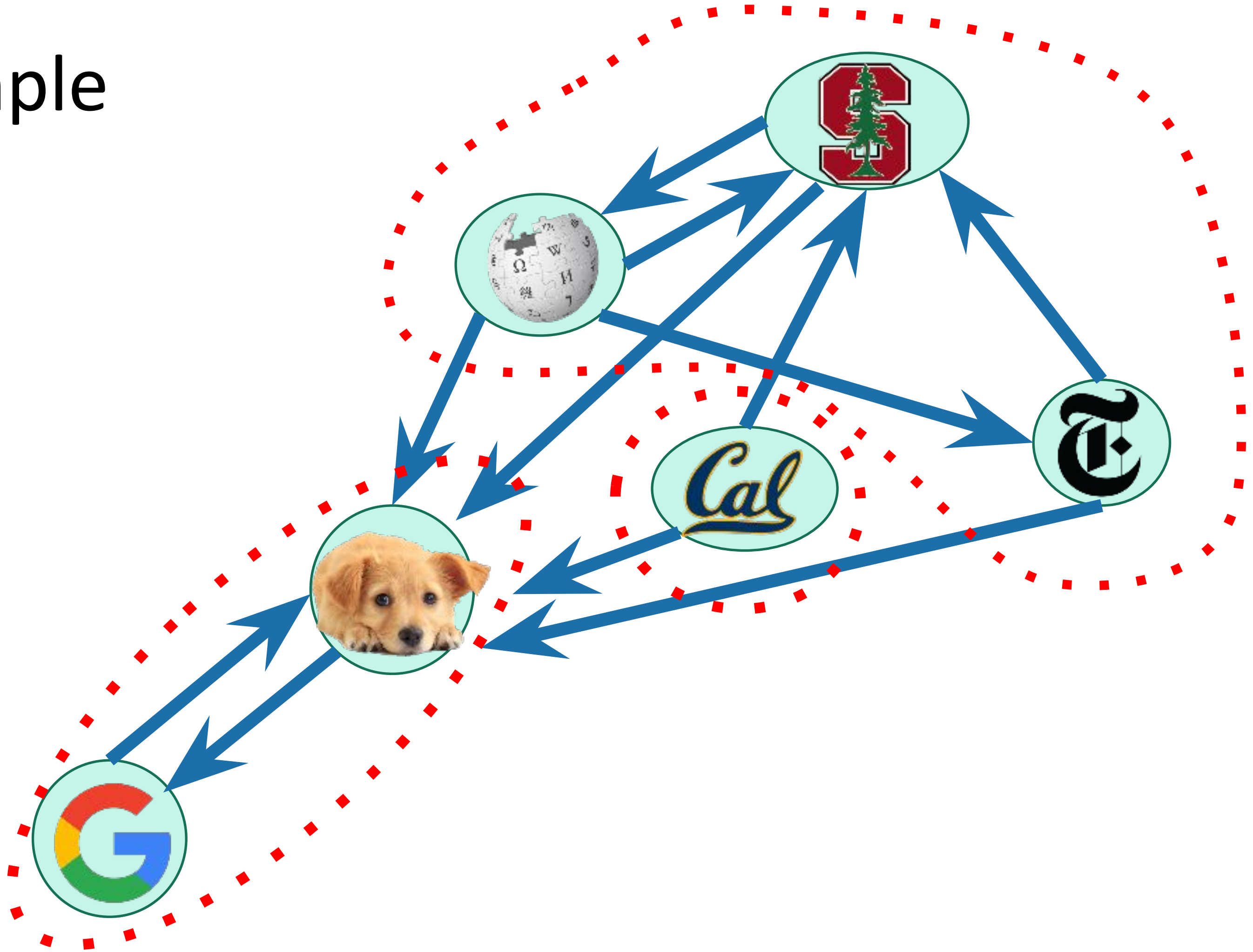

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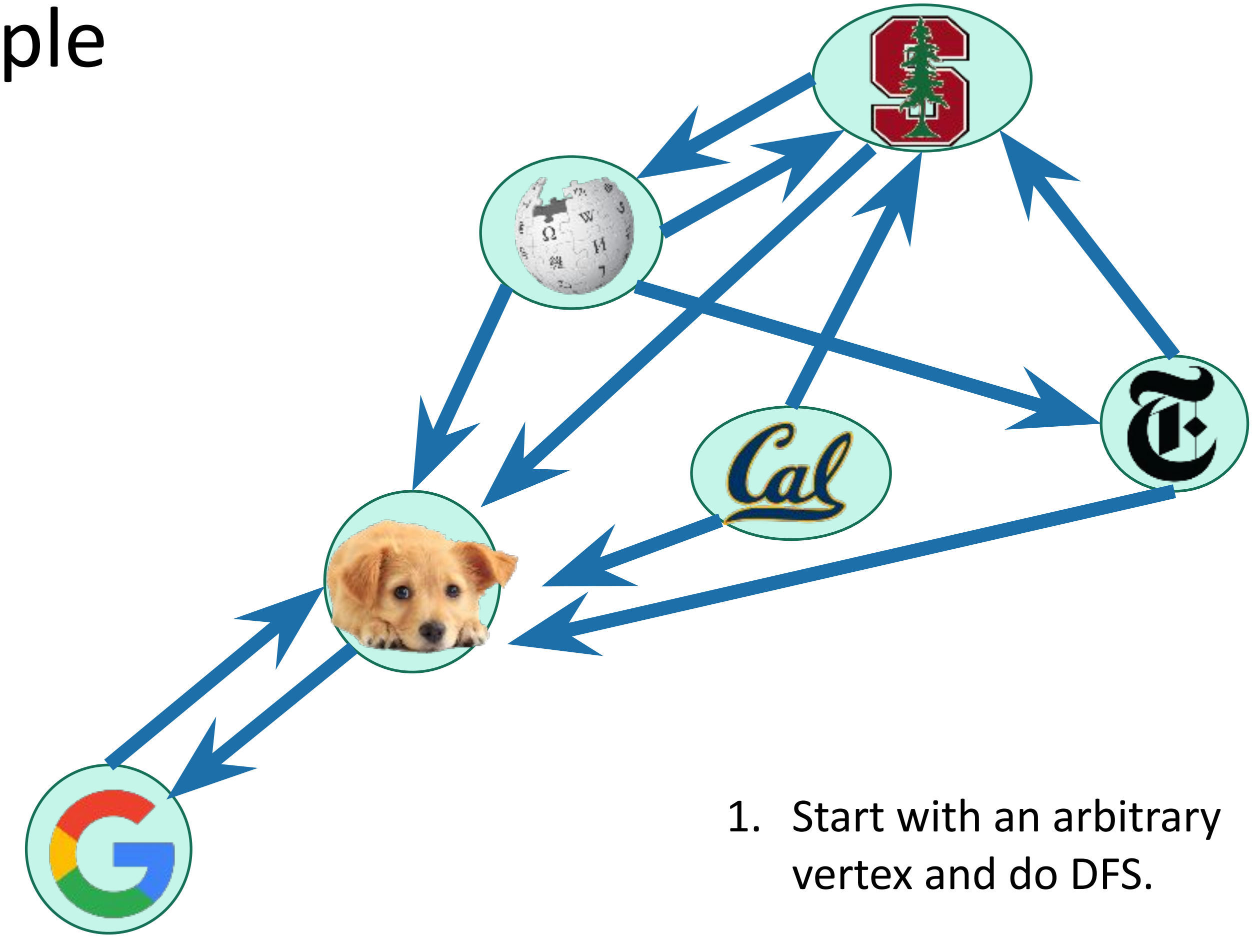
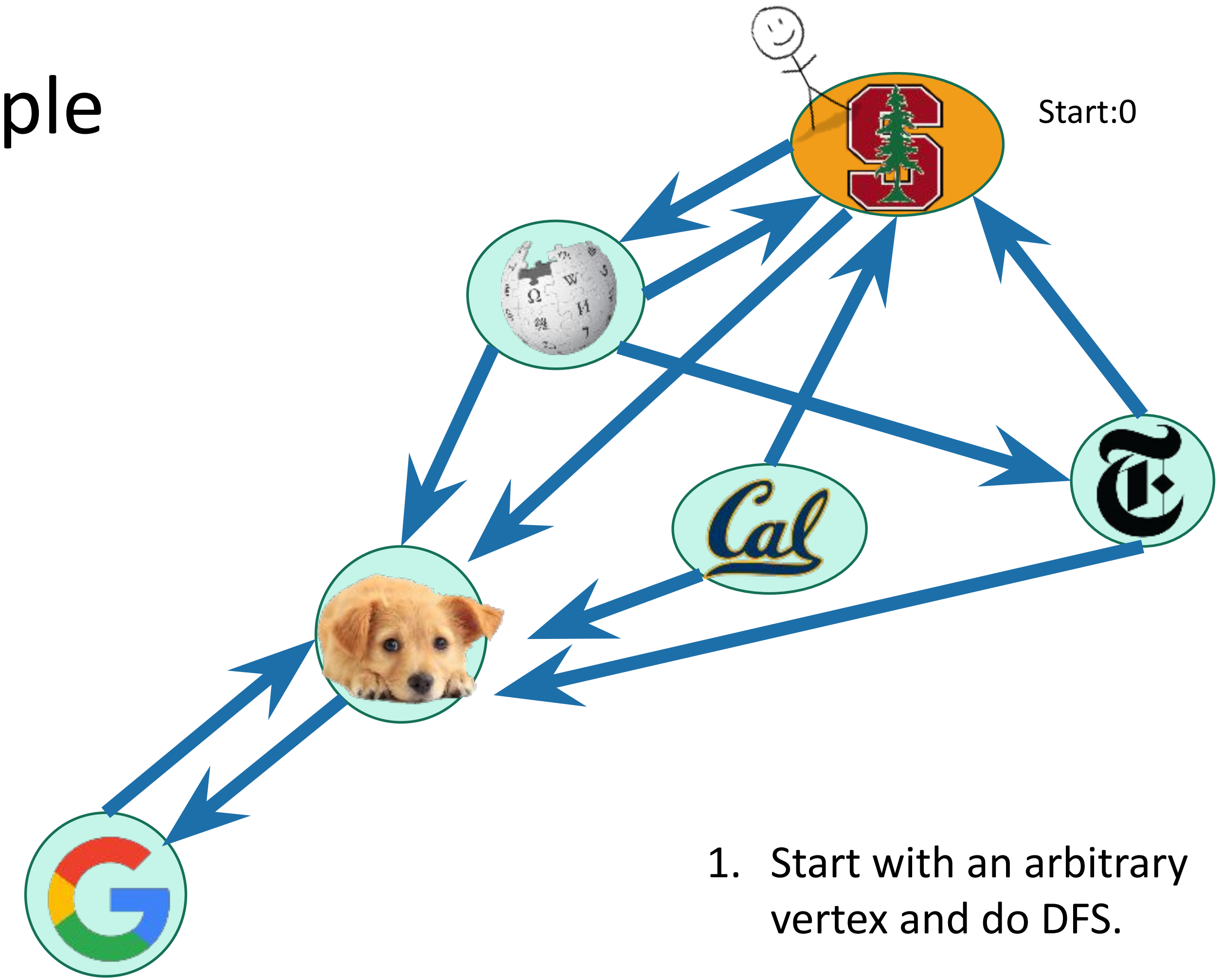


1. Start with an arbitrary vertex and do DFS.

Example



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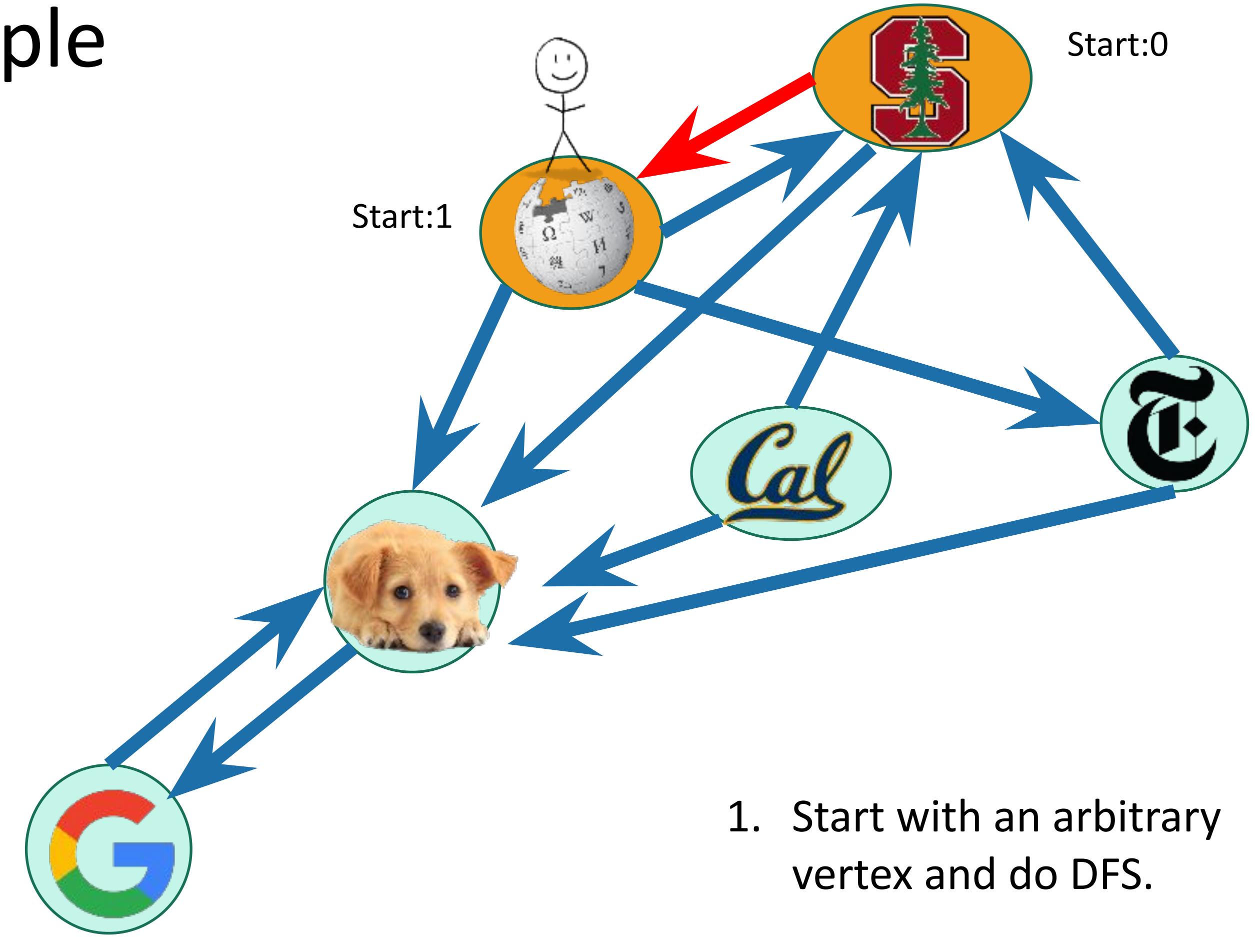

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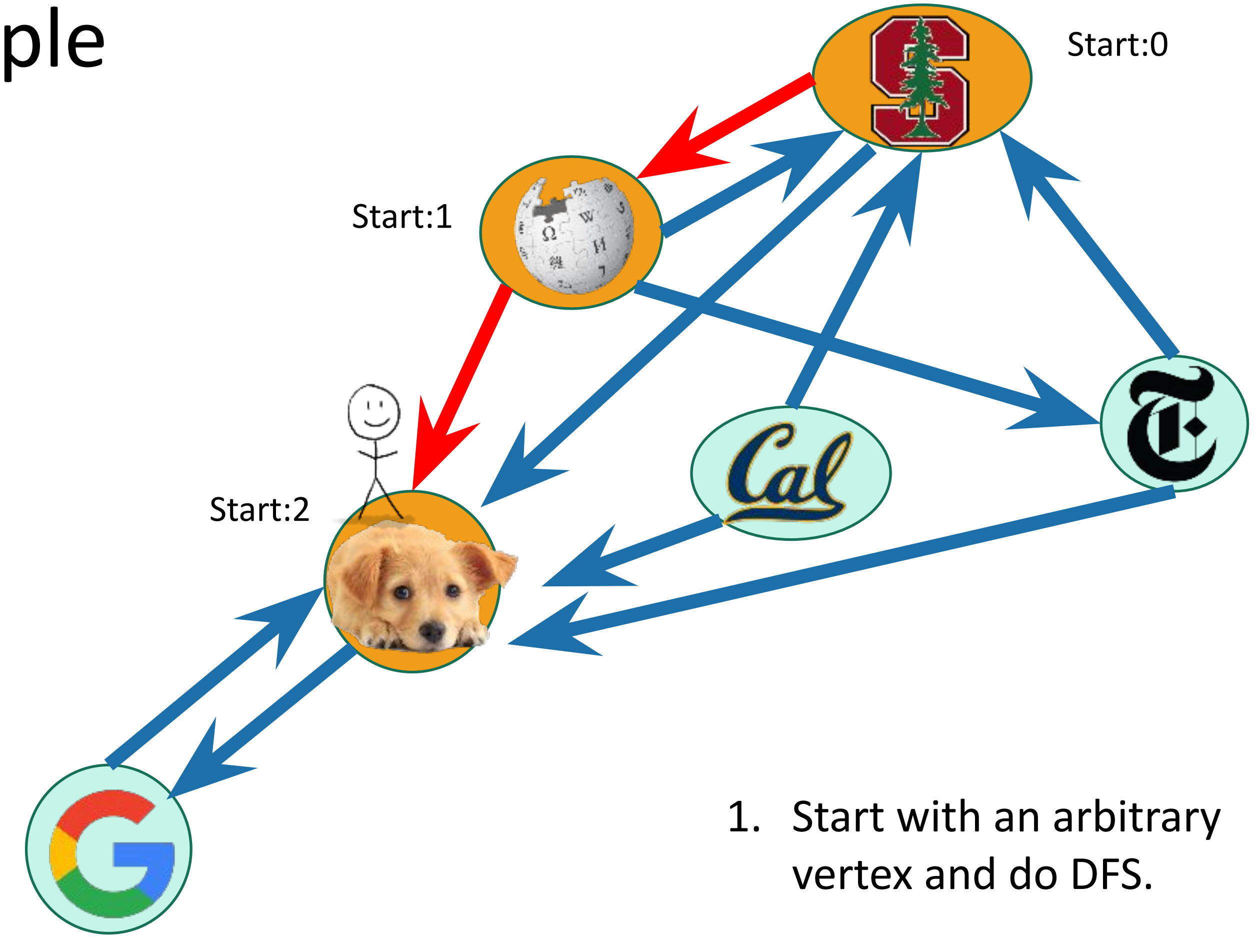

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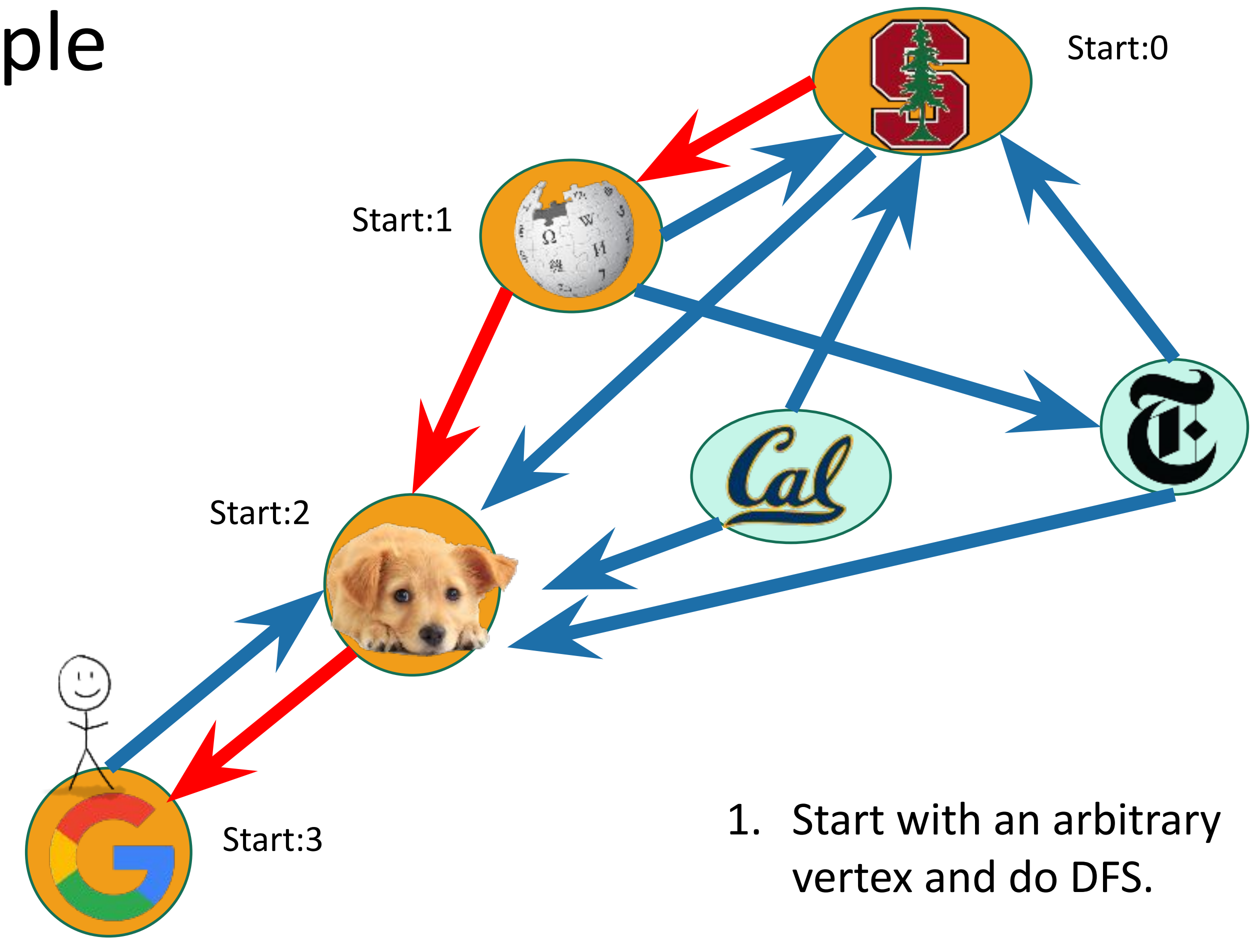

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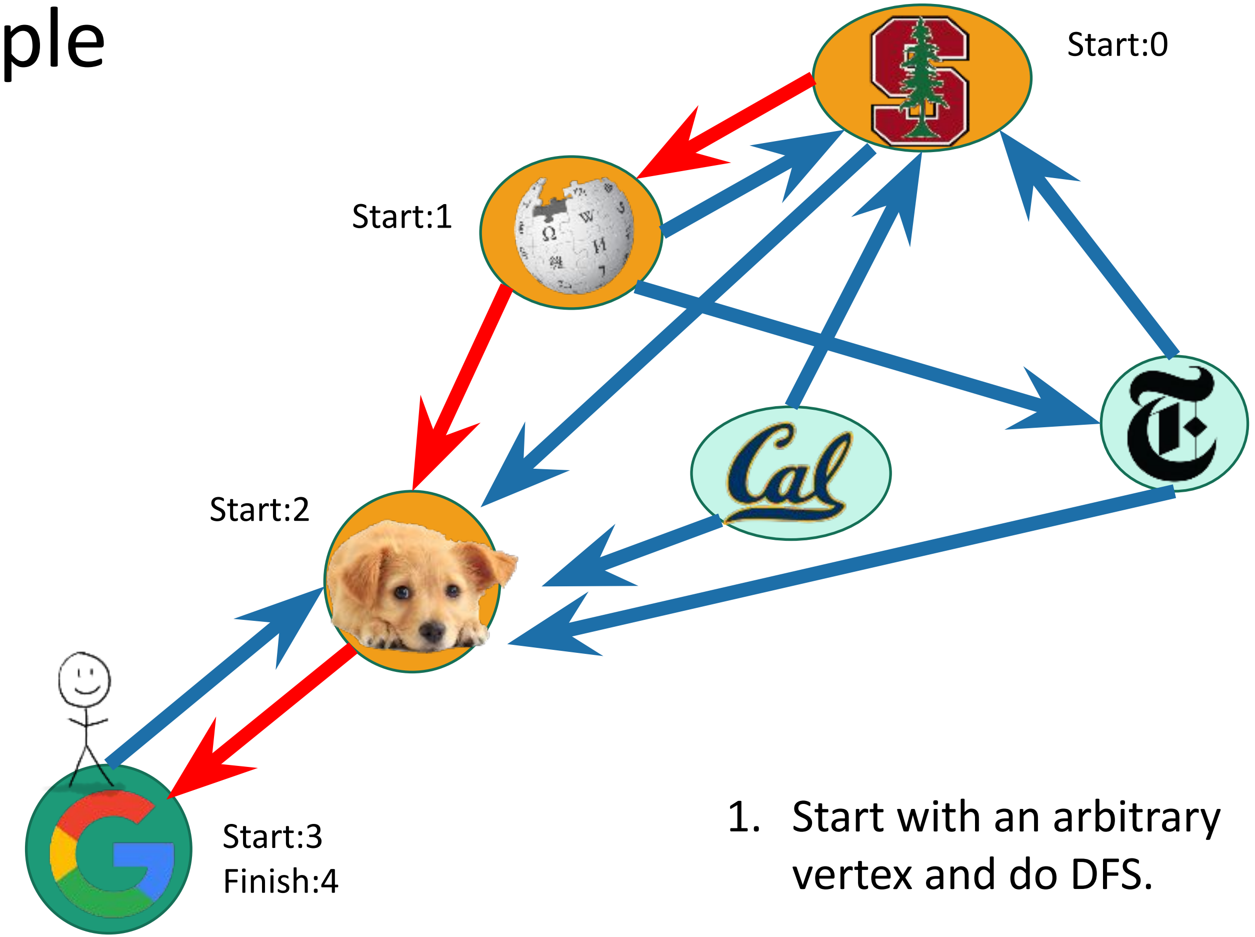

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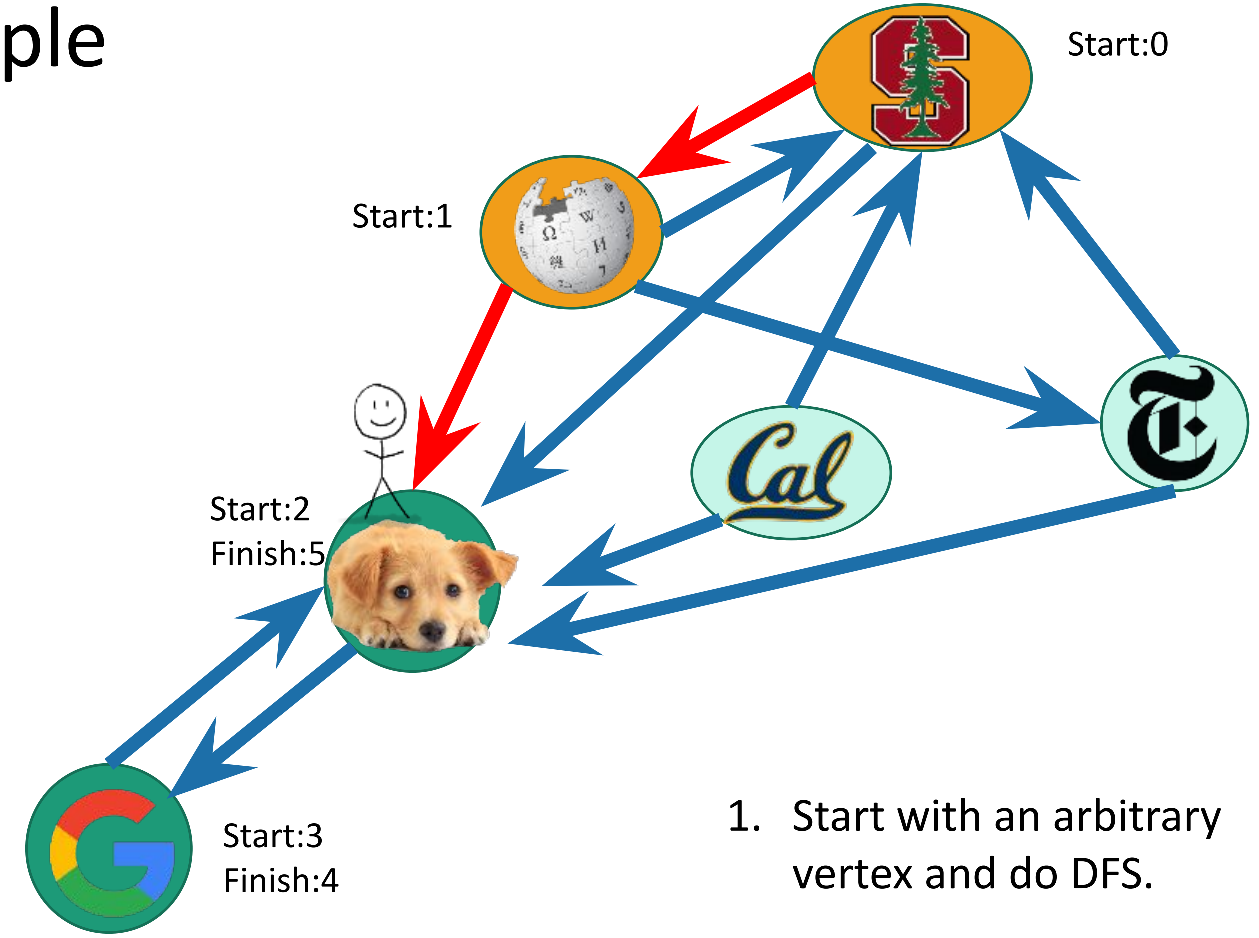

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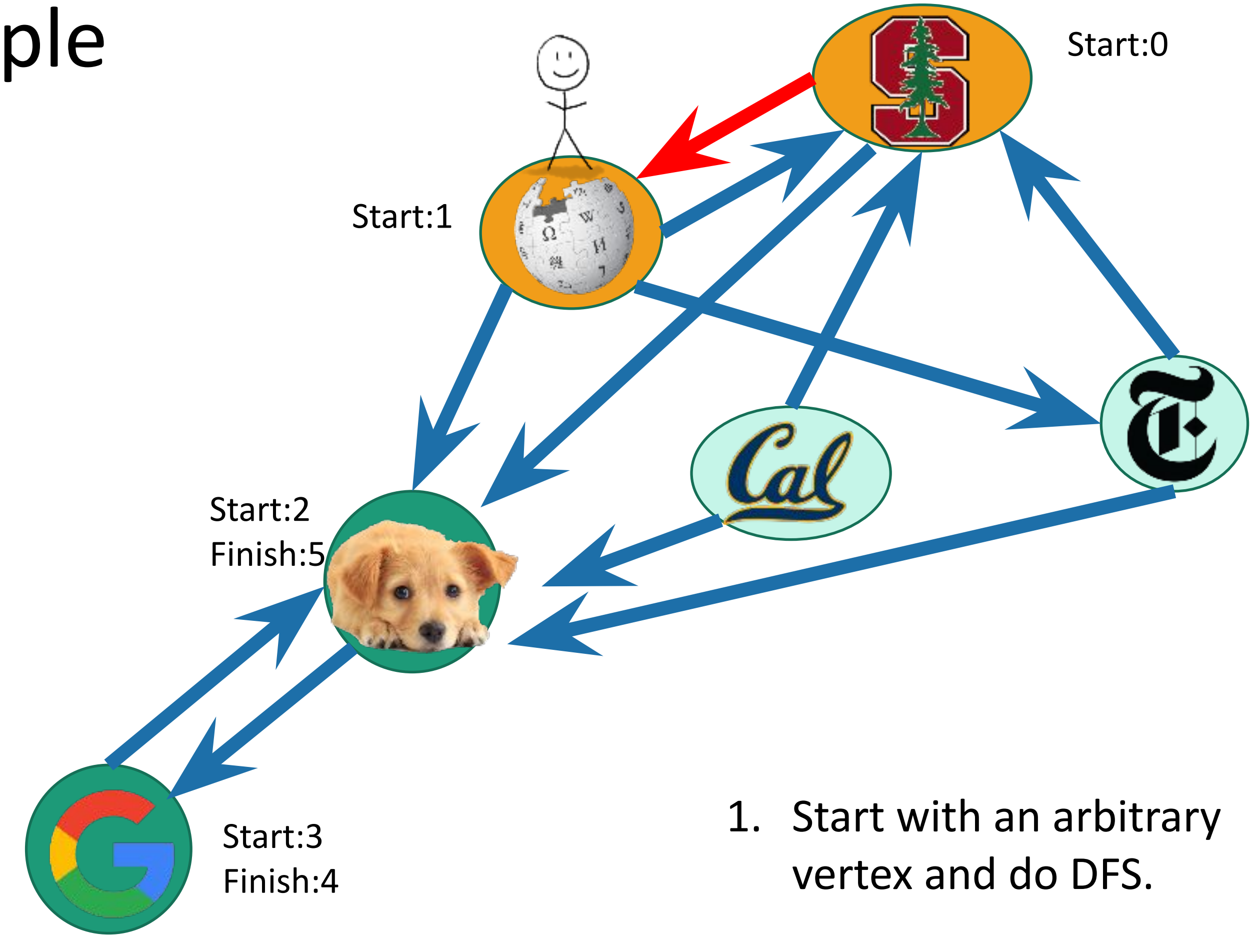

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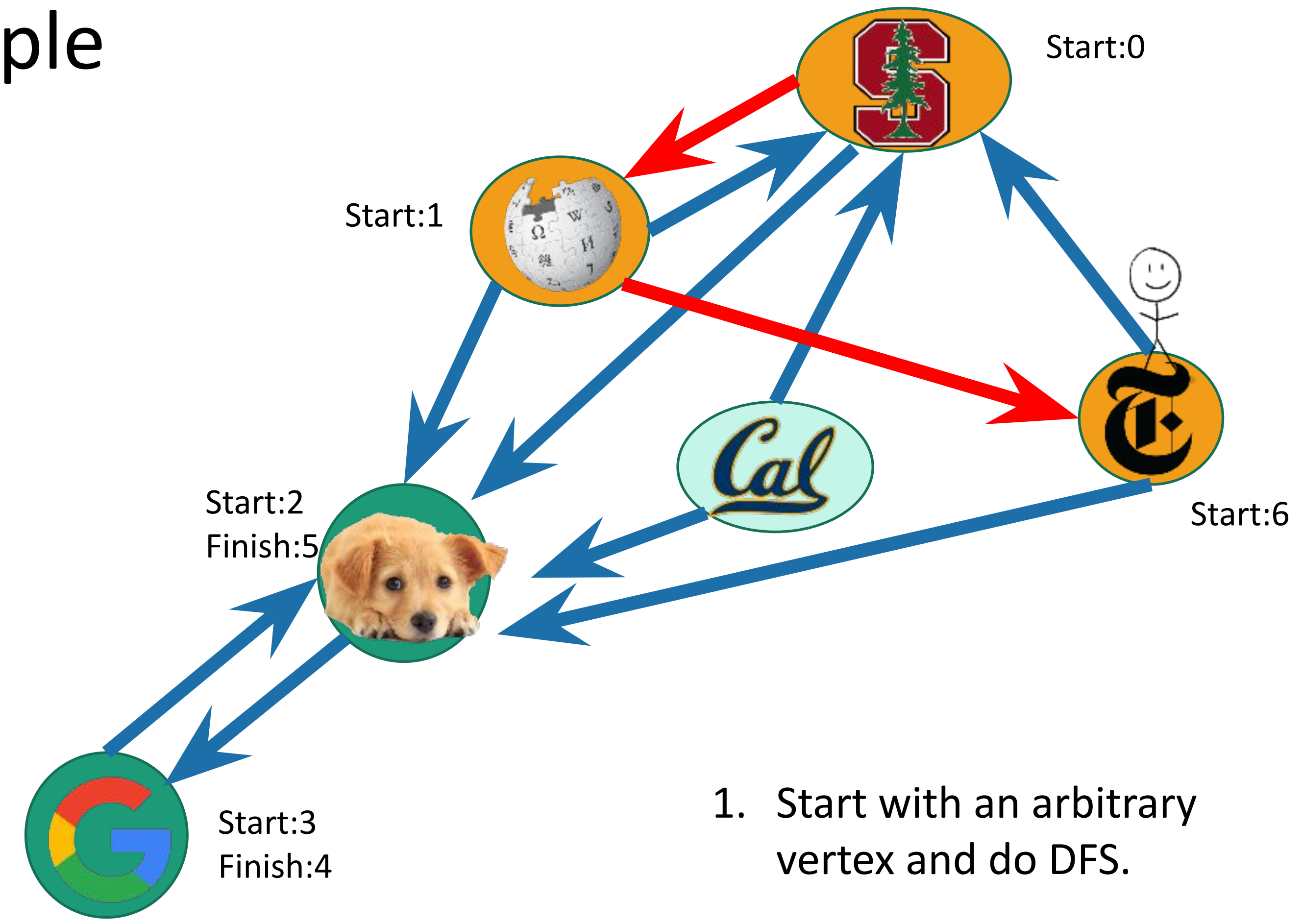

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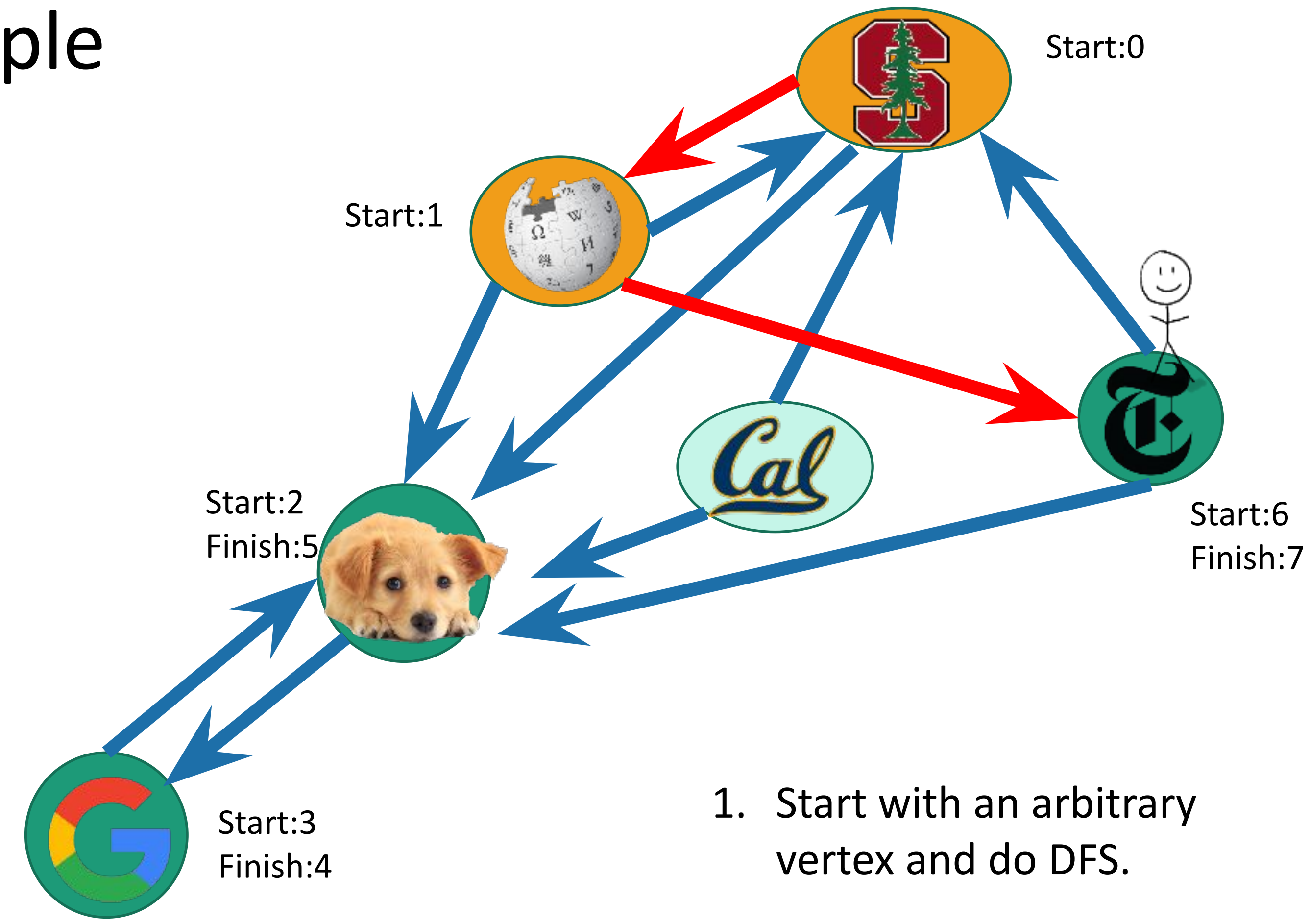

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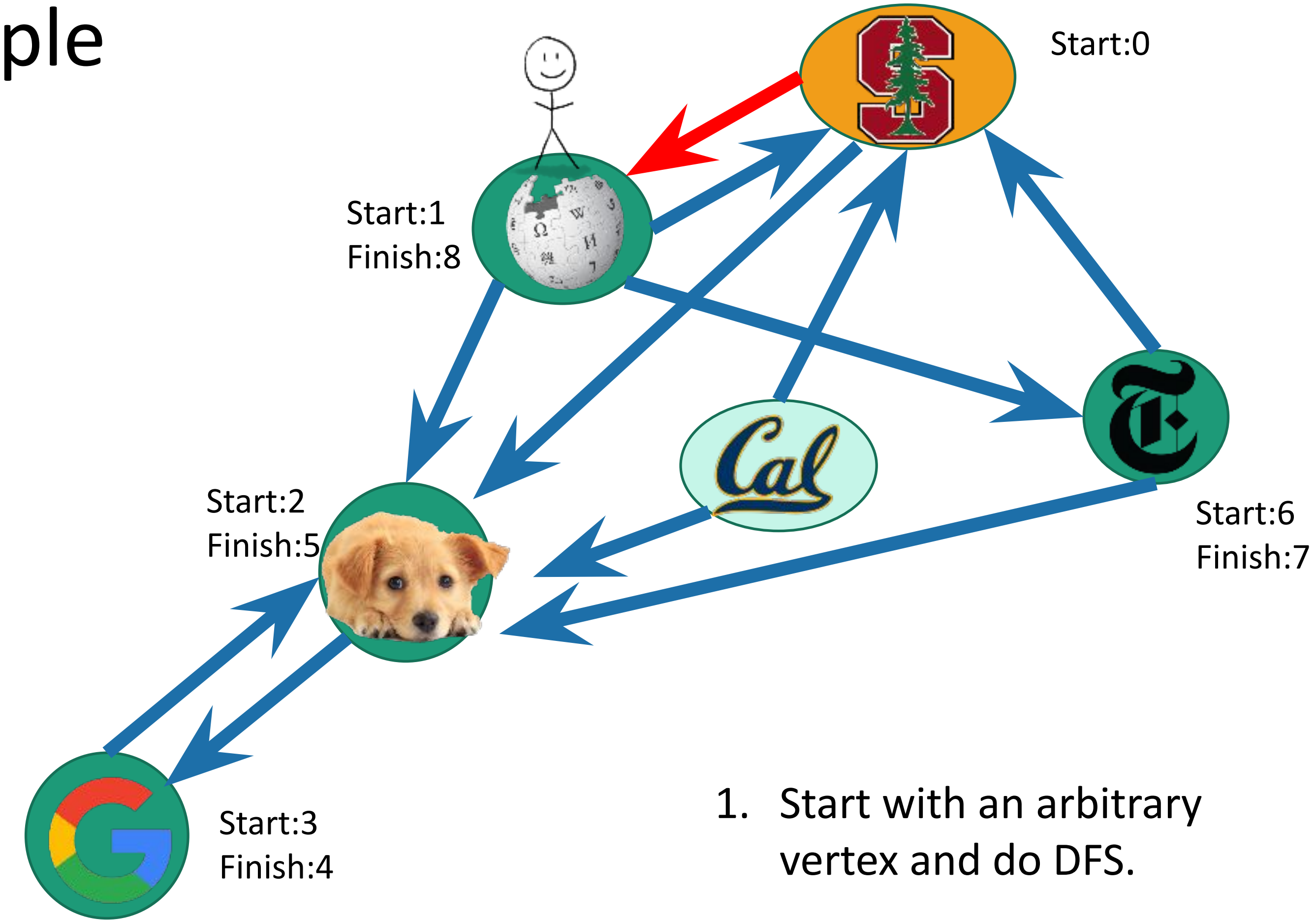

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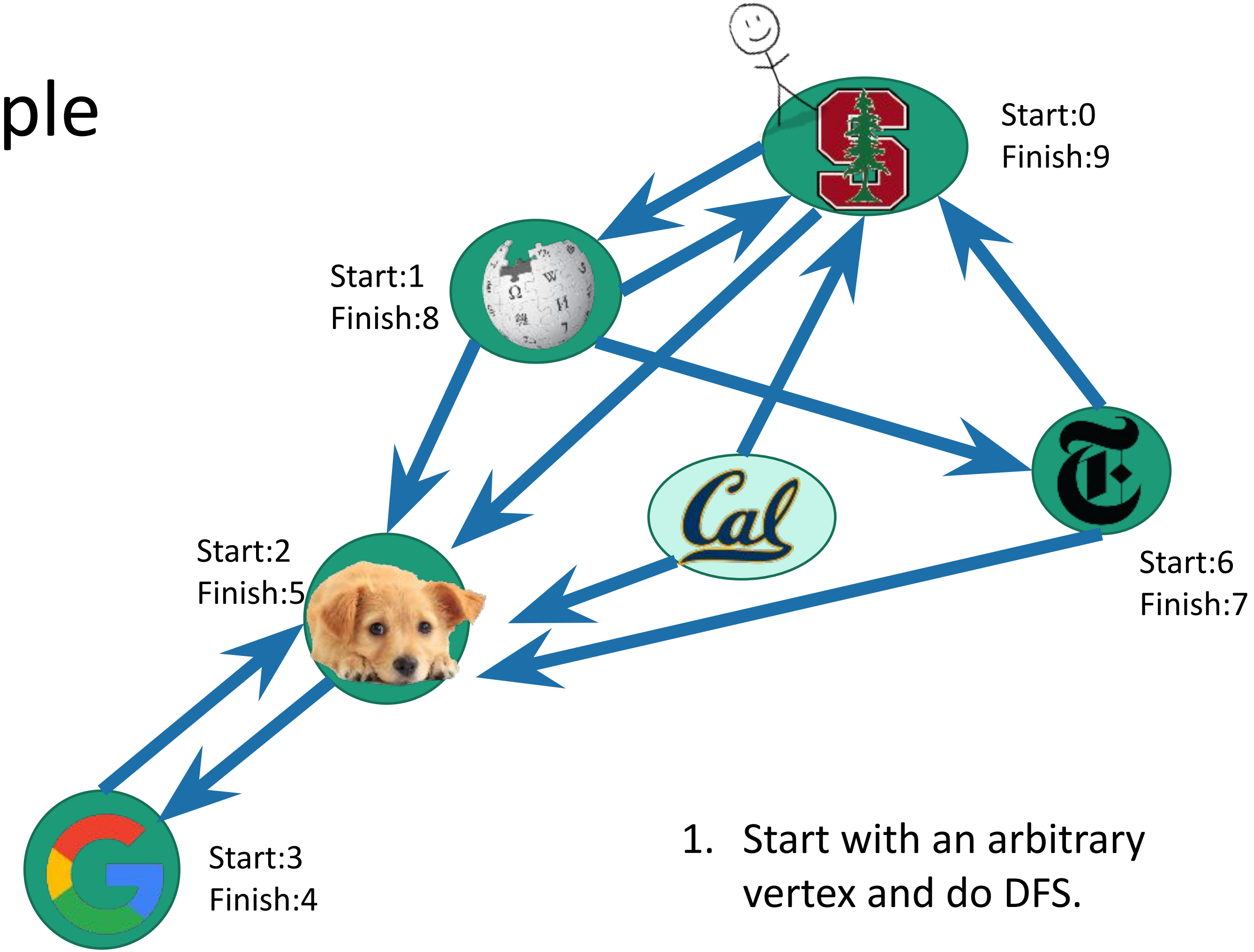

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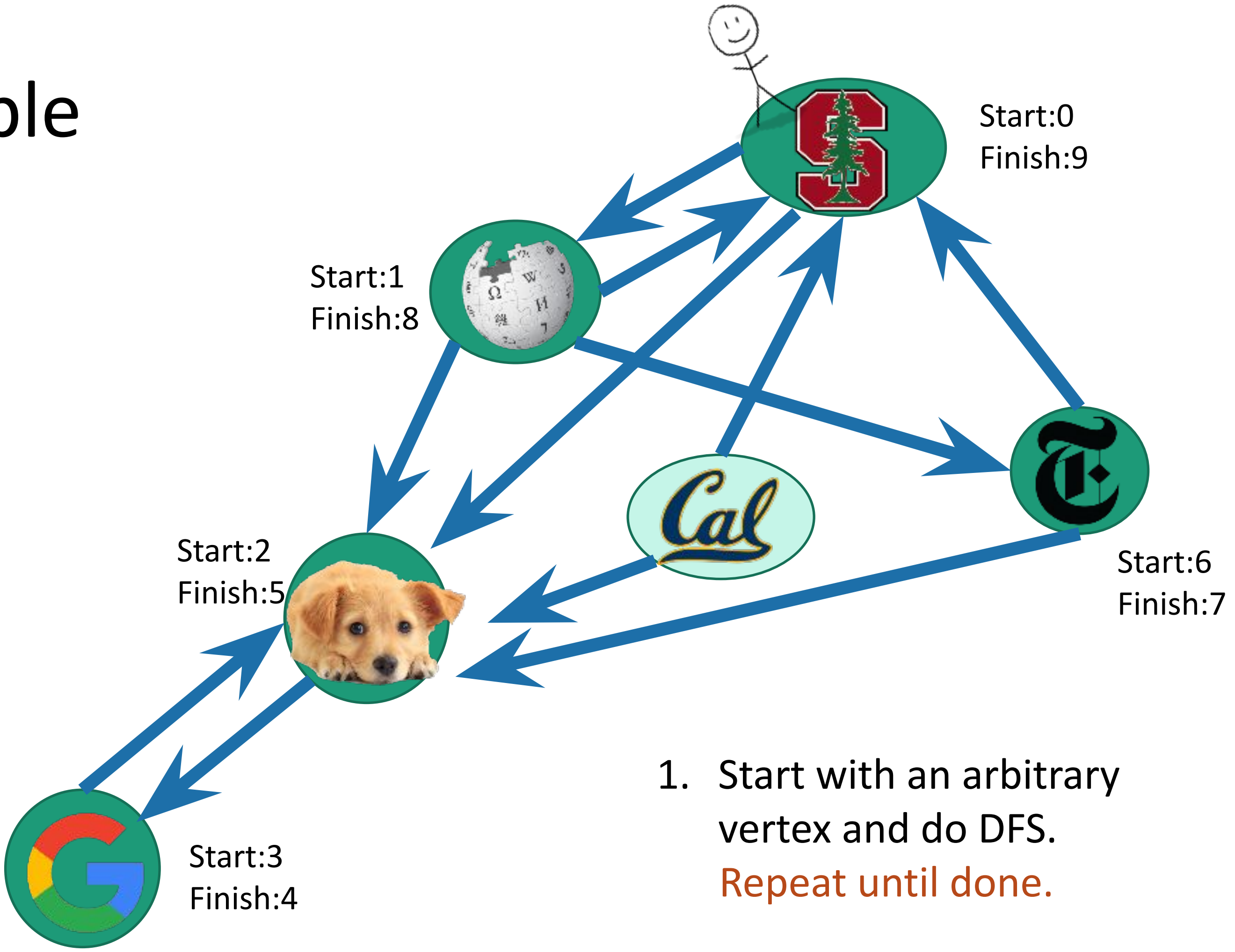
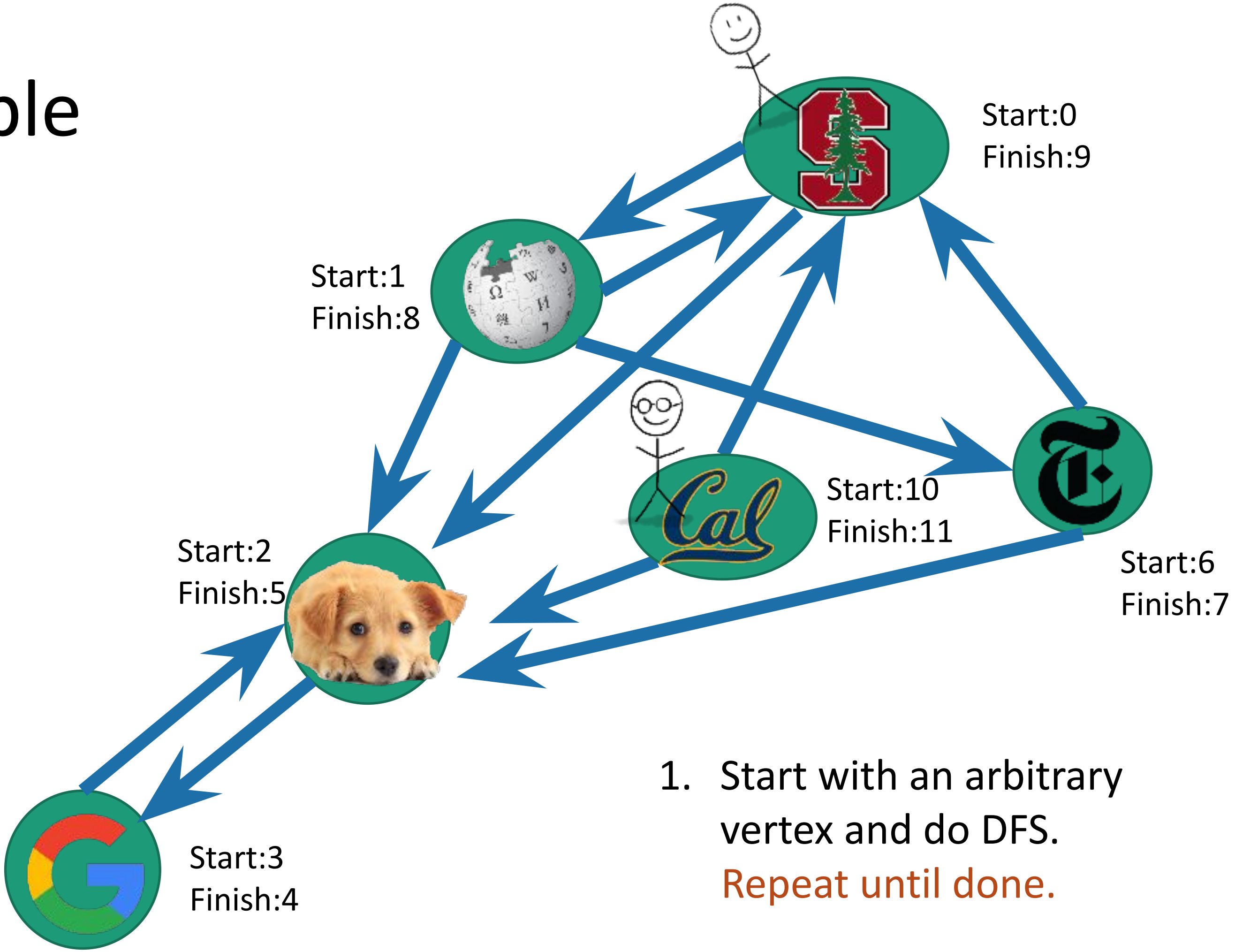


1. Start with an arbitrary vertex and do DFS. Repeat until done.

Example



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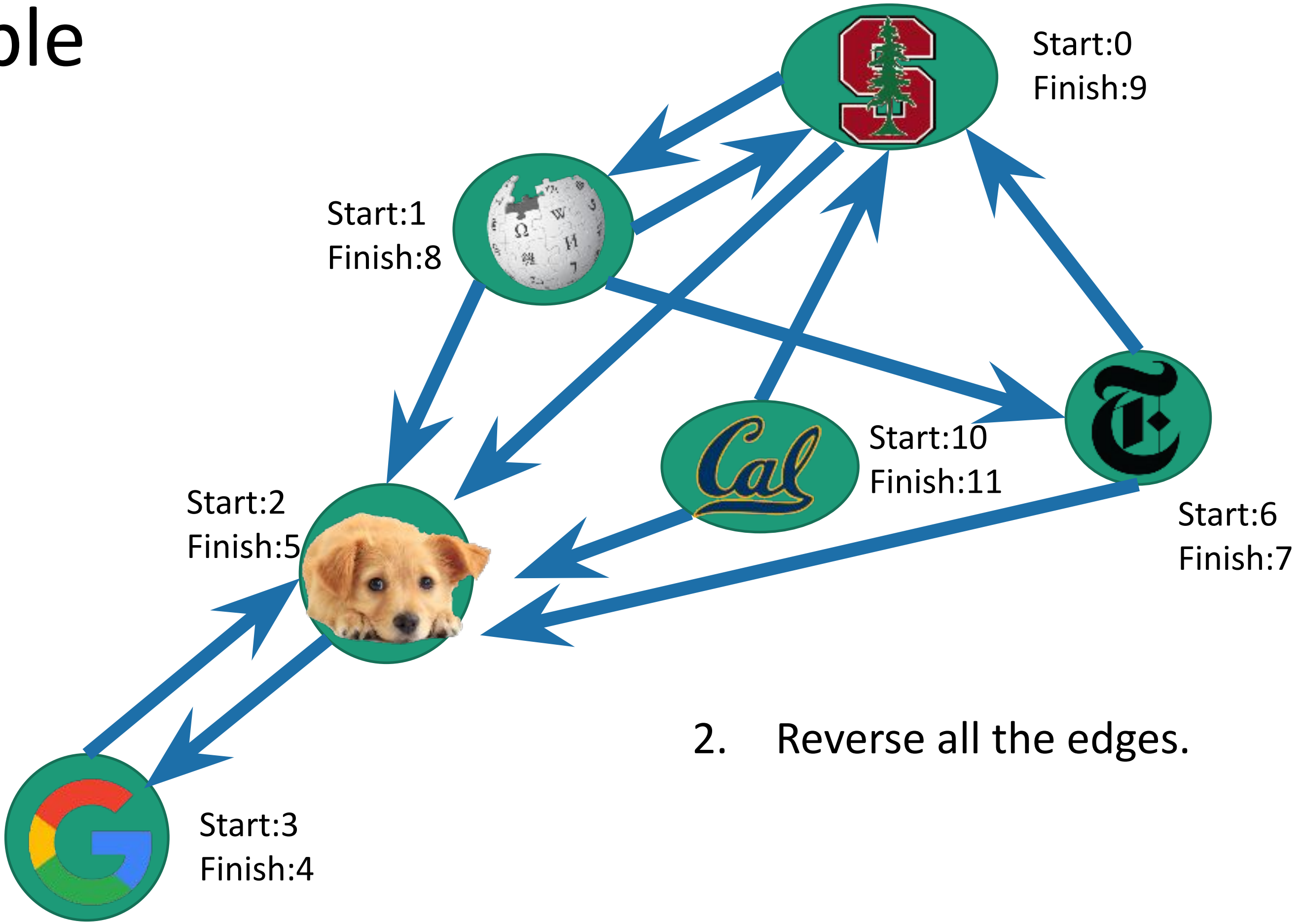

2. Reverse all the edges.

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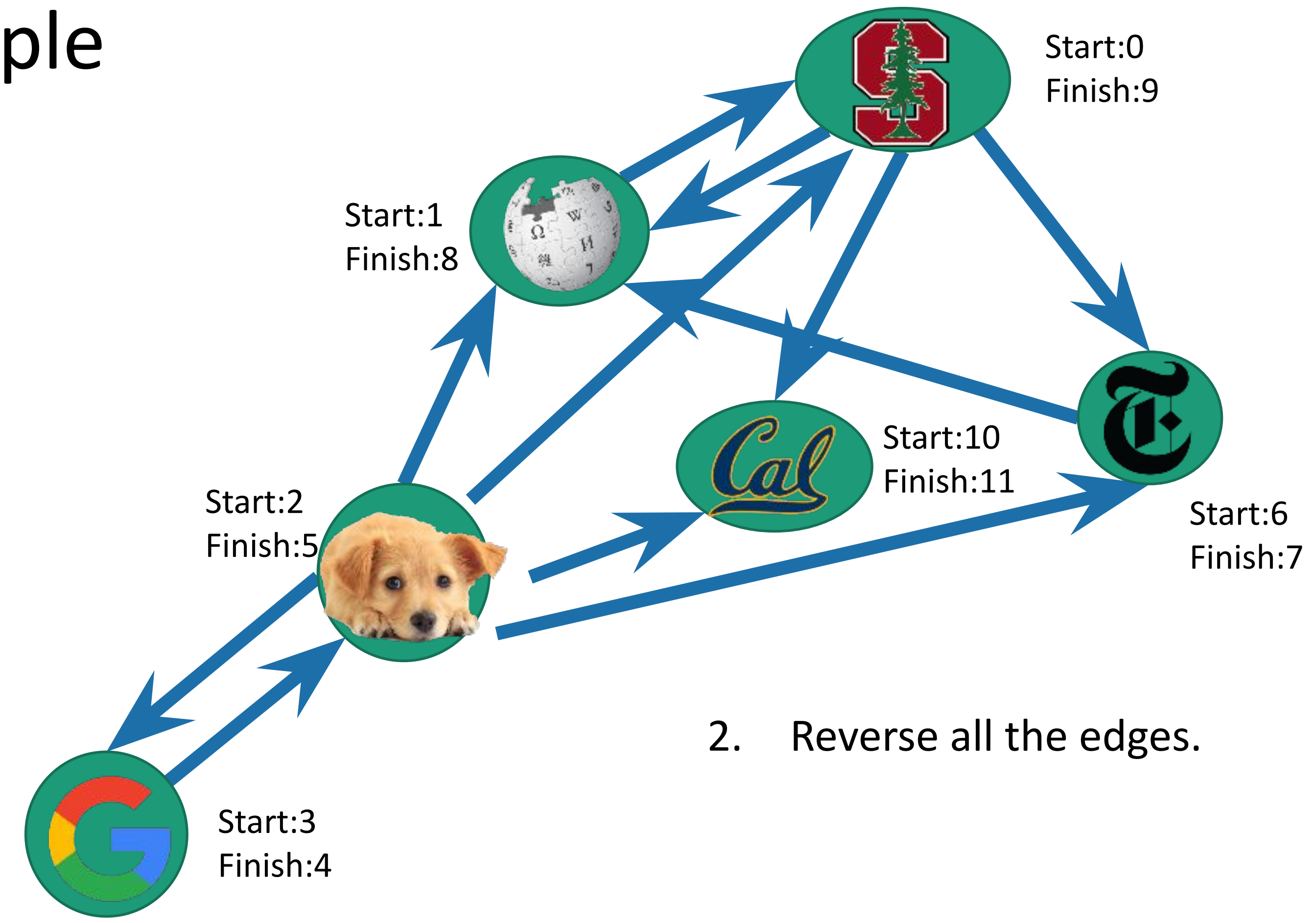

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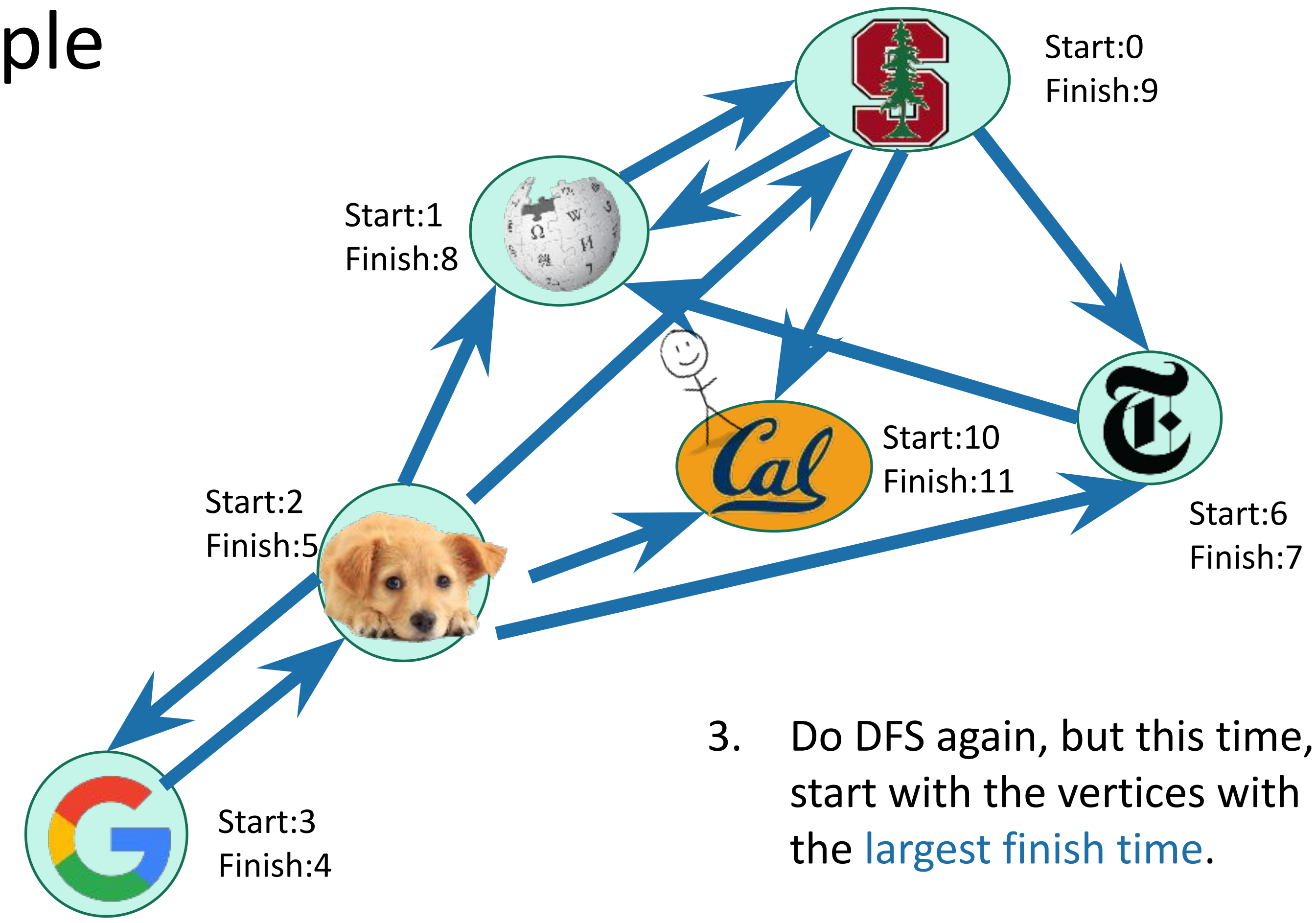
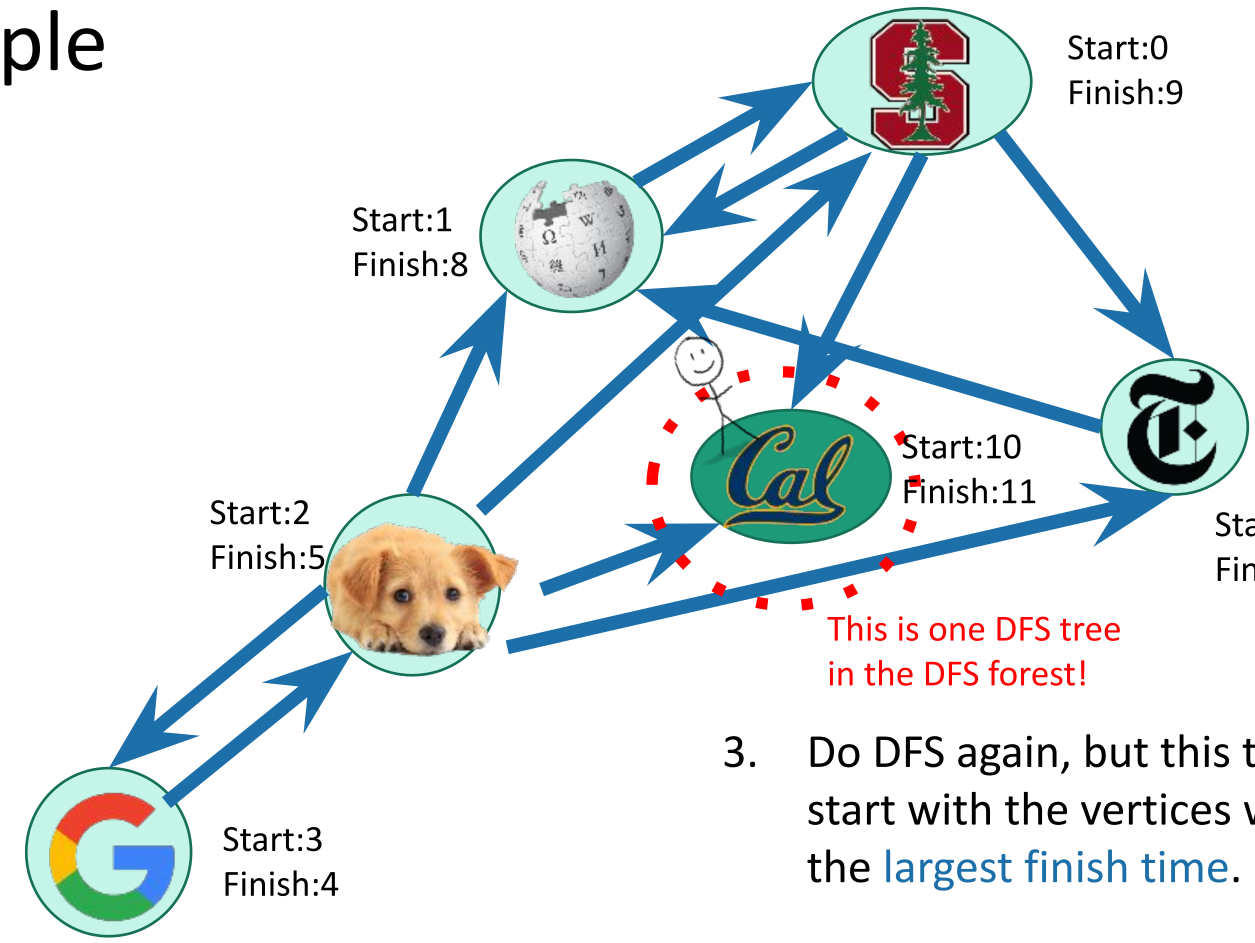


3. Do DFS again, but this time, start with the vertices with the largest finish time.

Example



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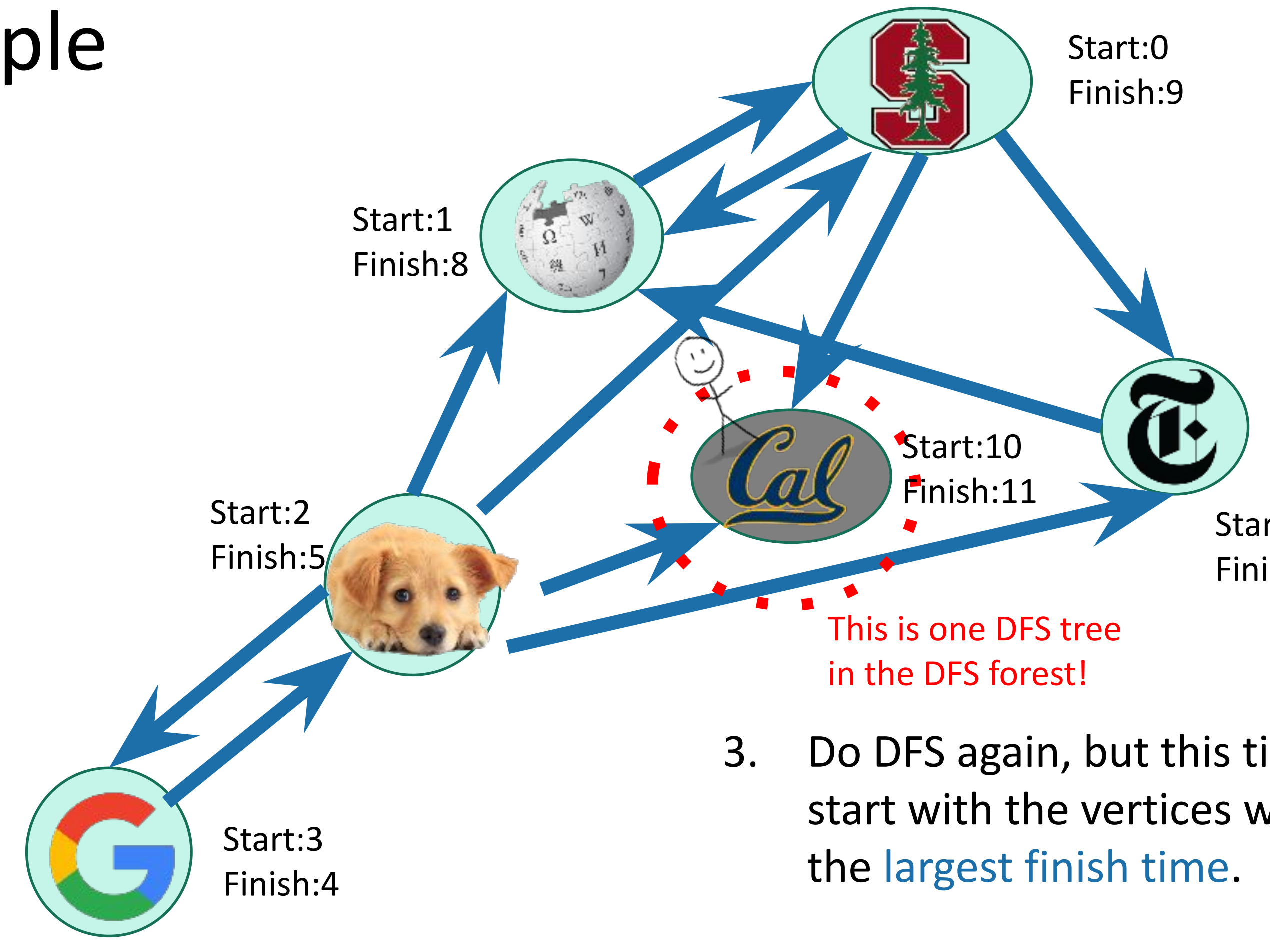

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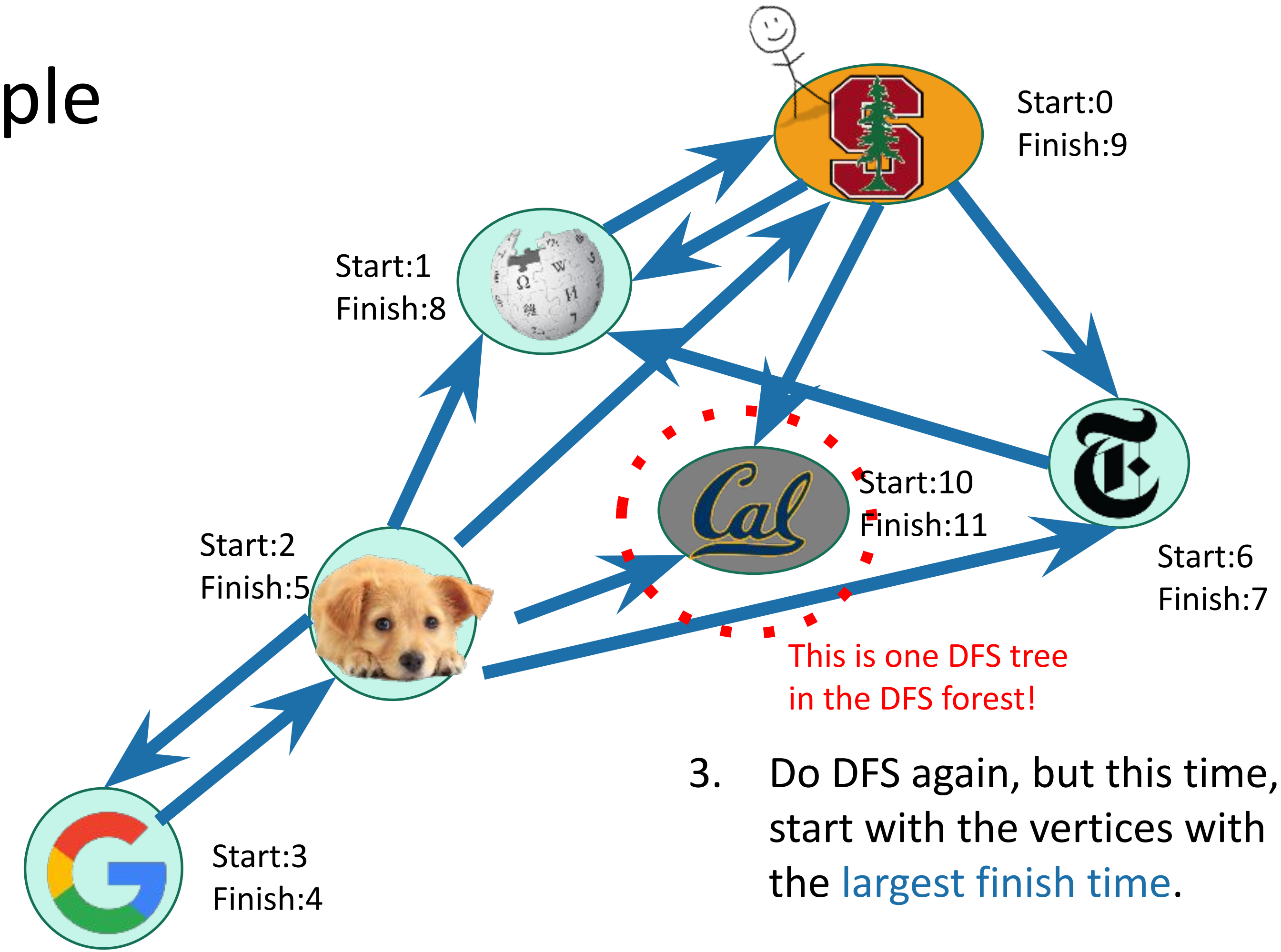

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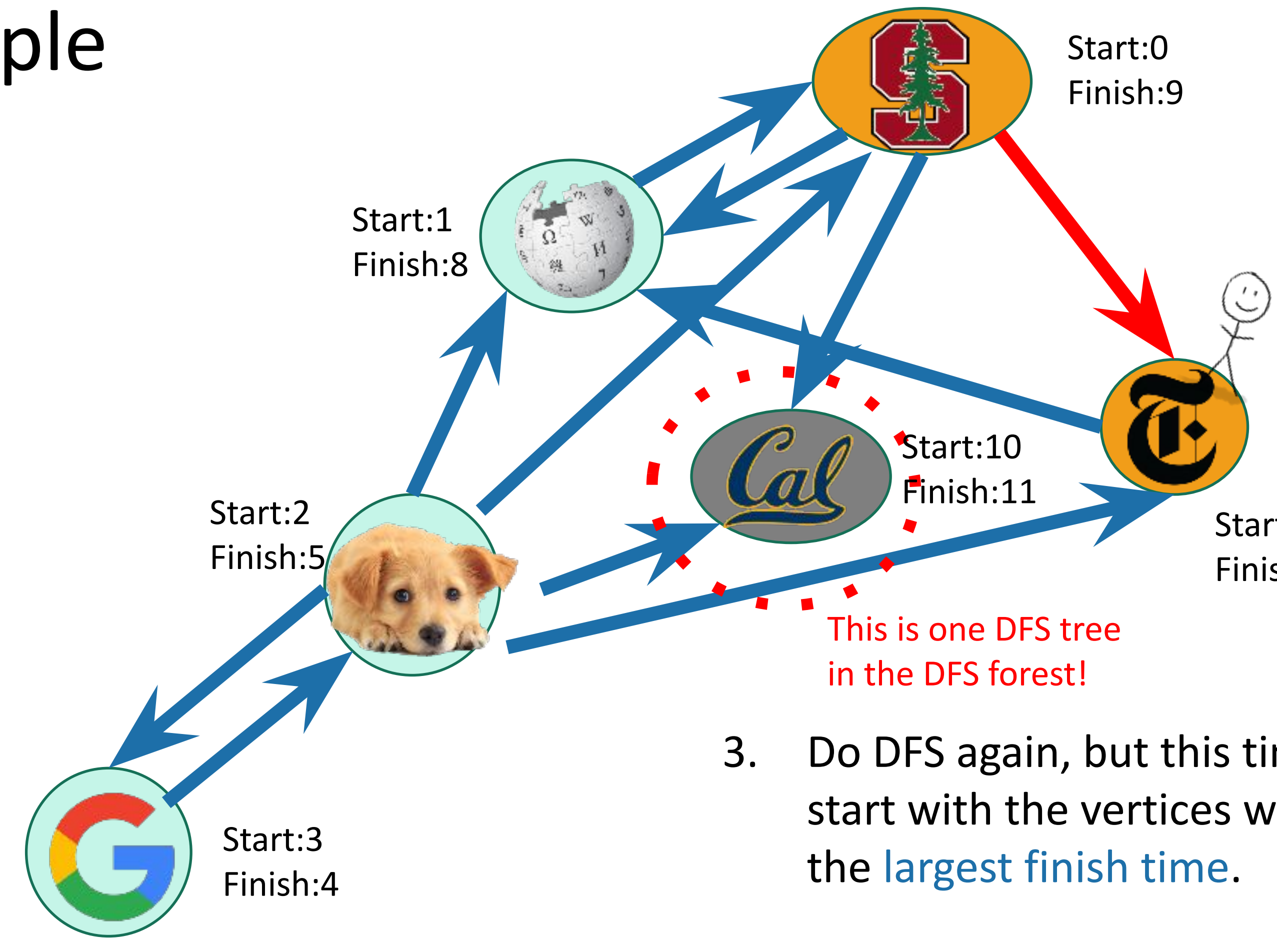

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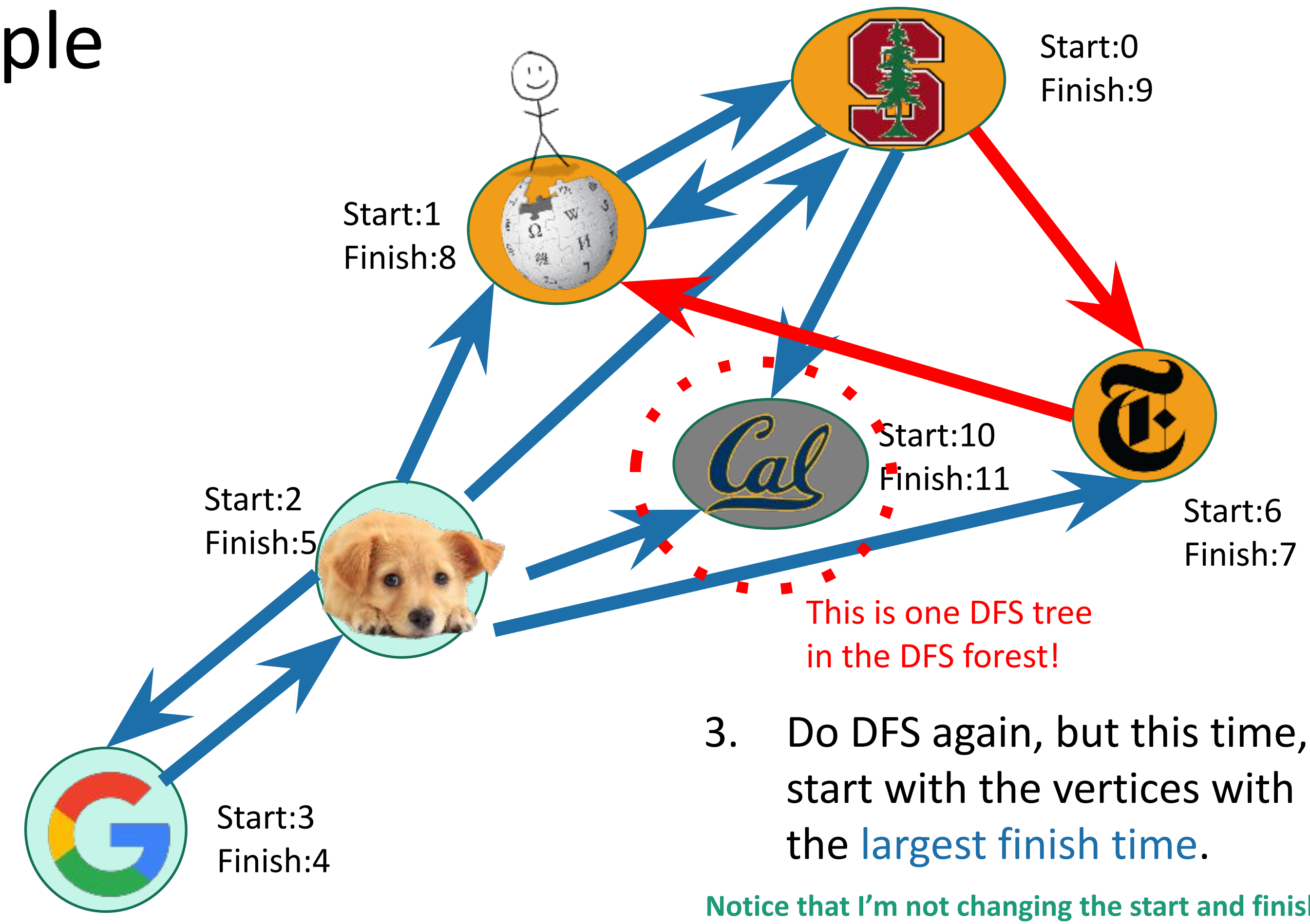

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Notice that I'm not changing the start and finish times – I'm keeping them from the first run.

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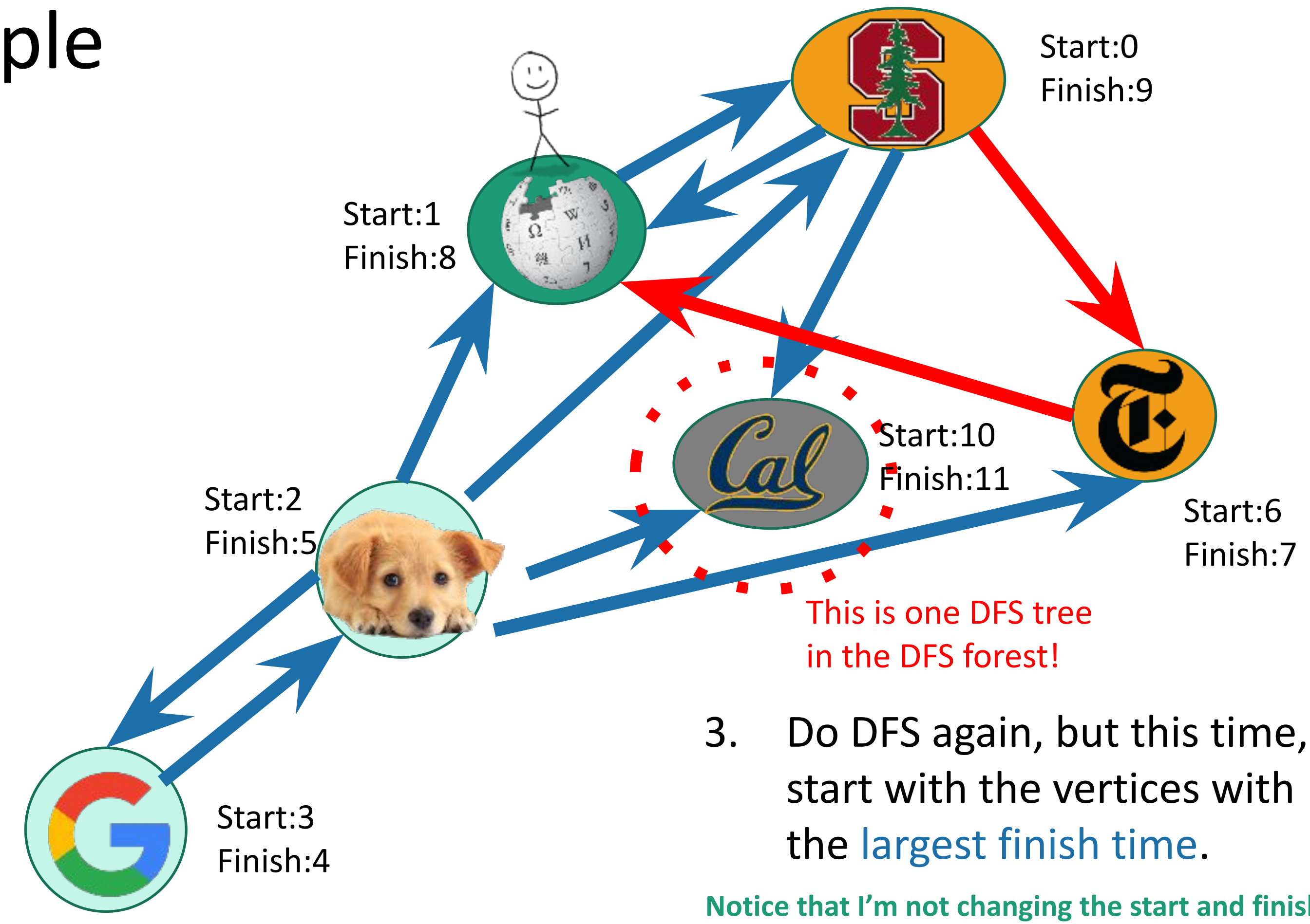

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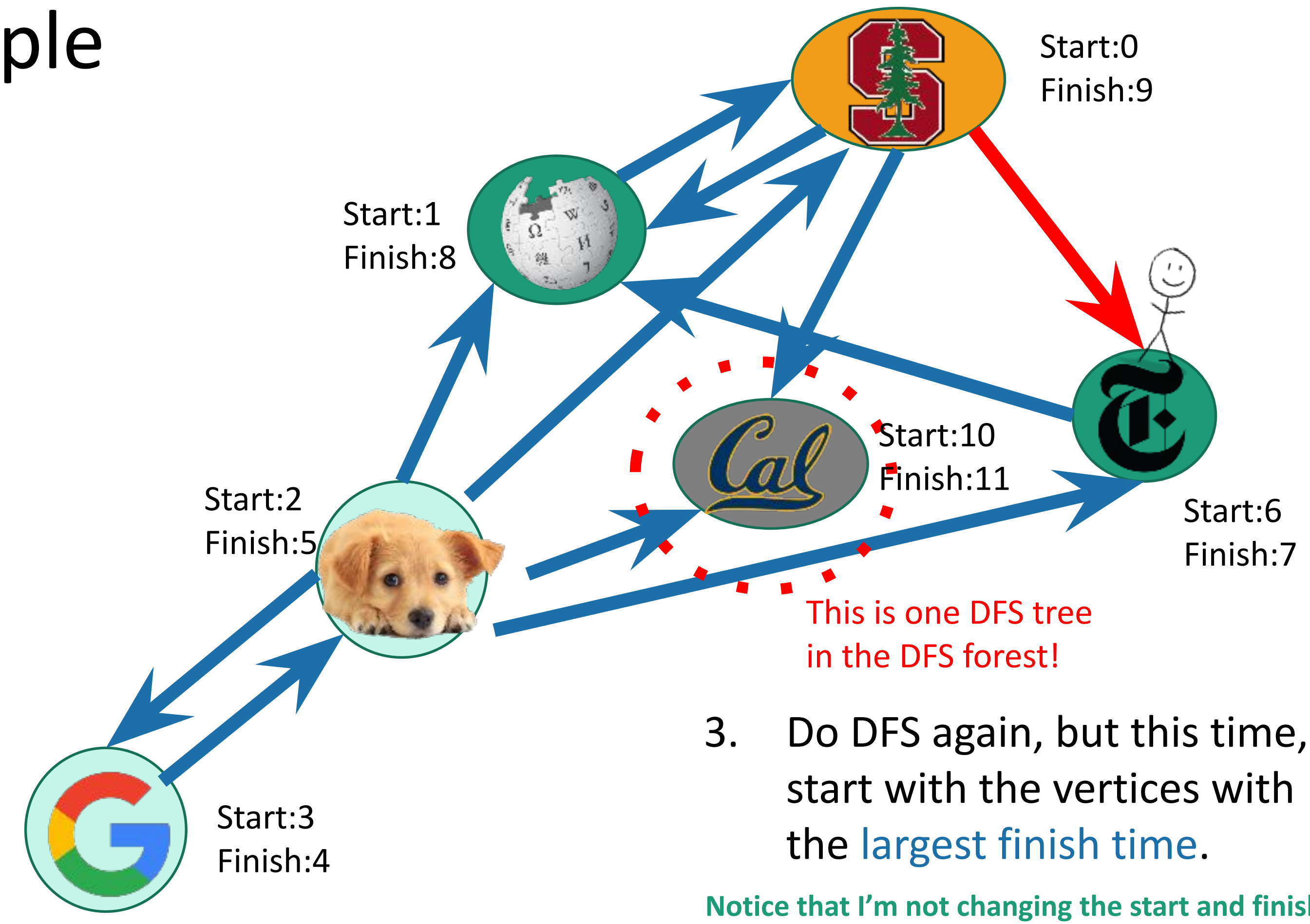

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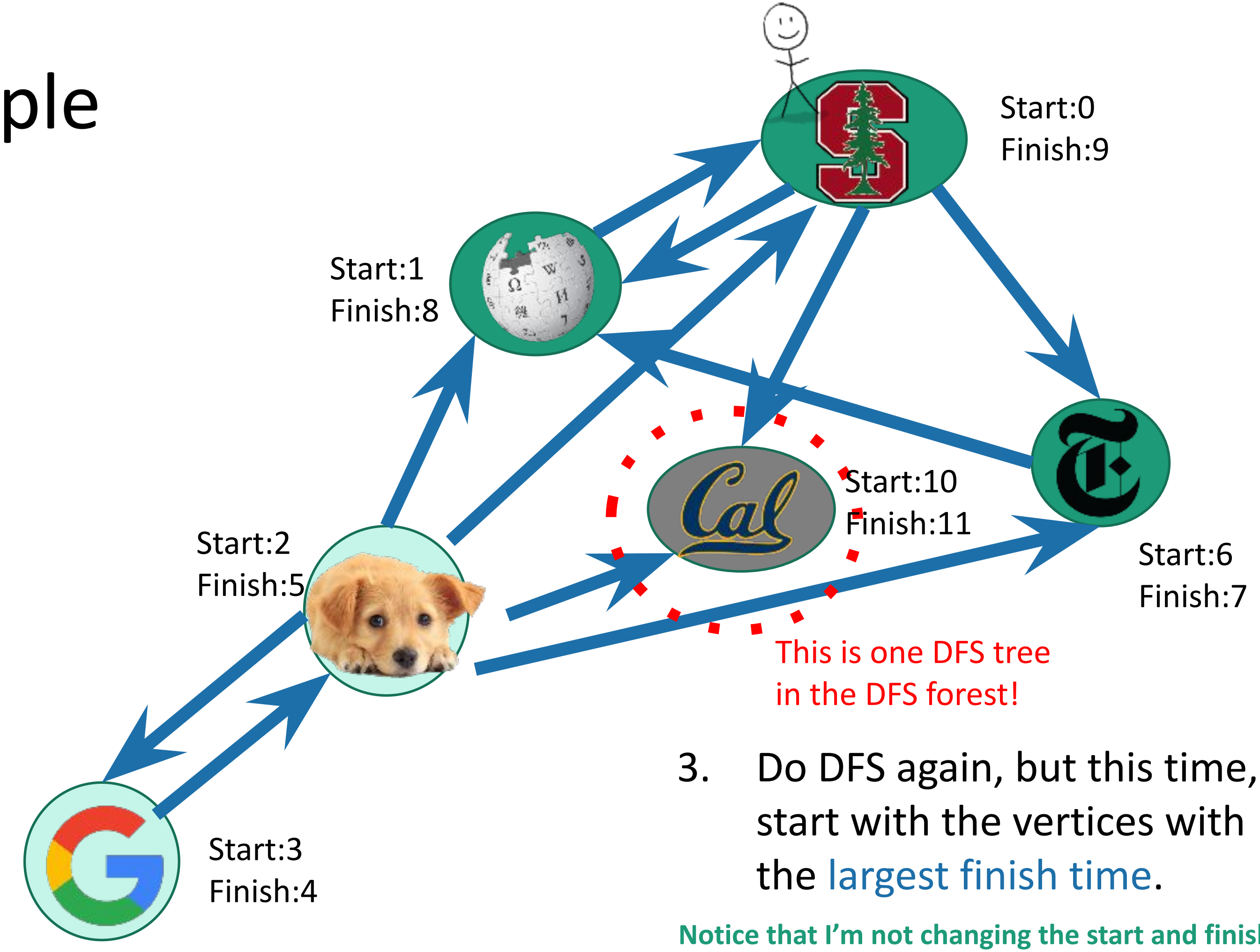

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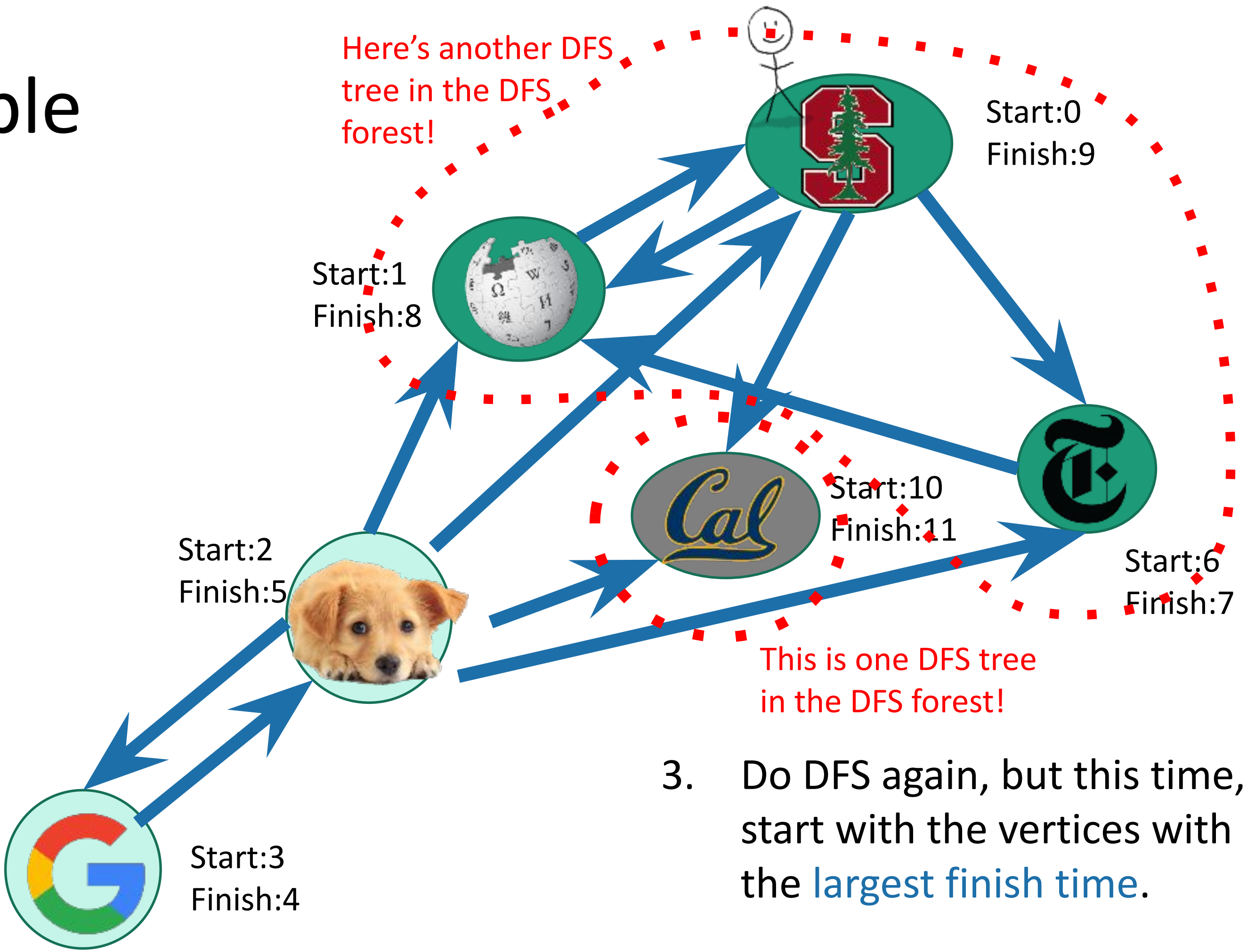

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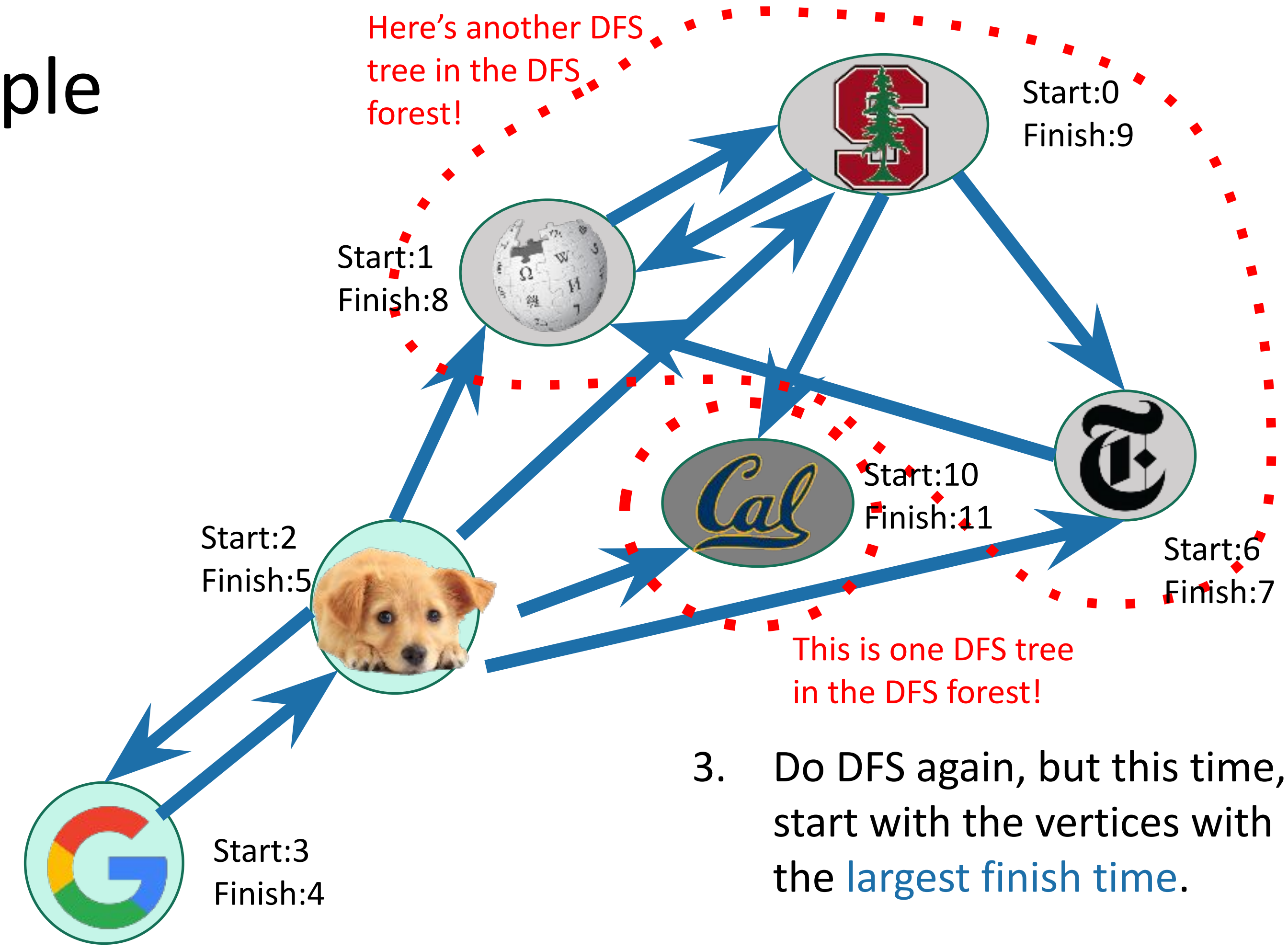

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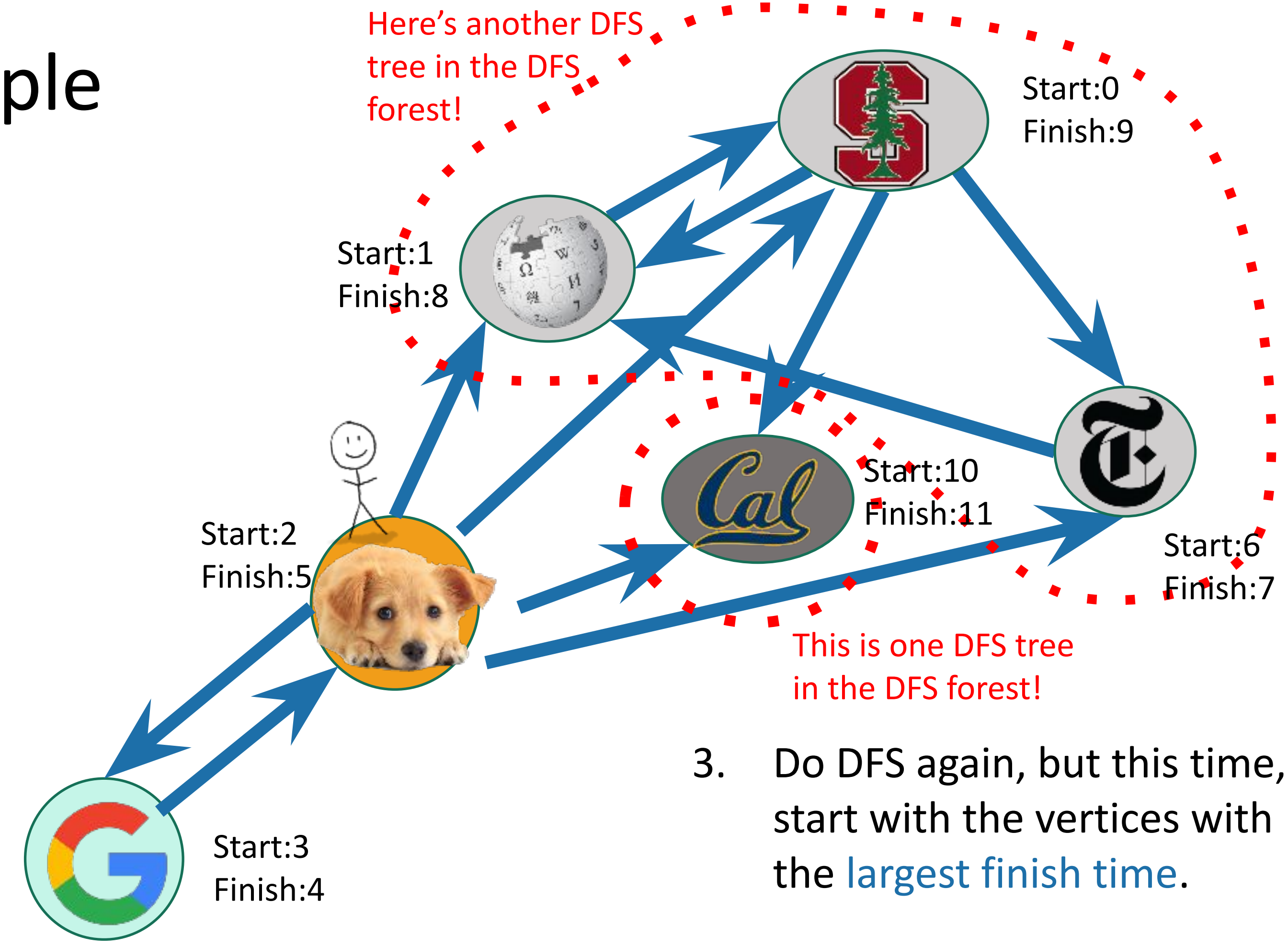

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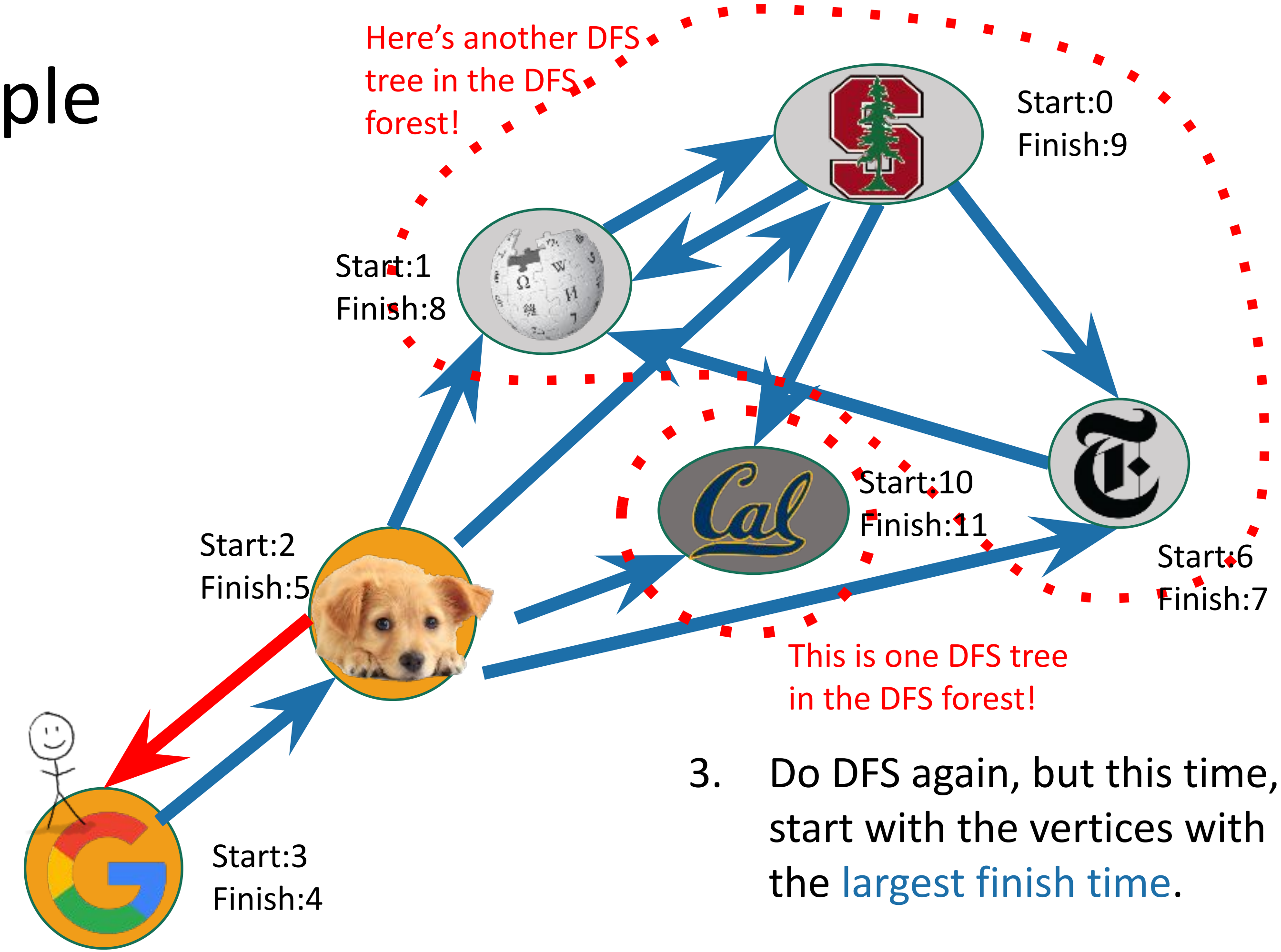

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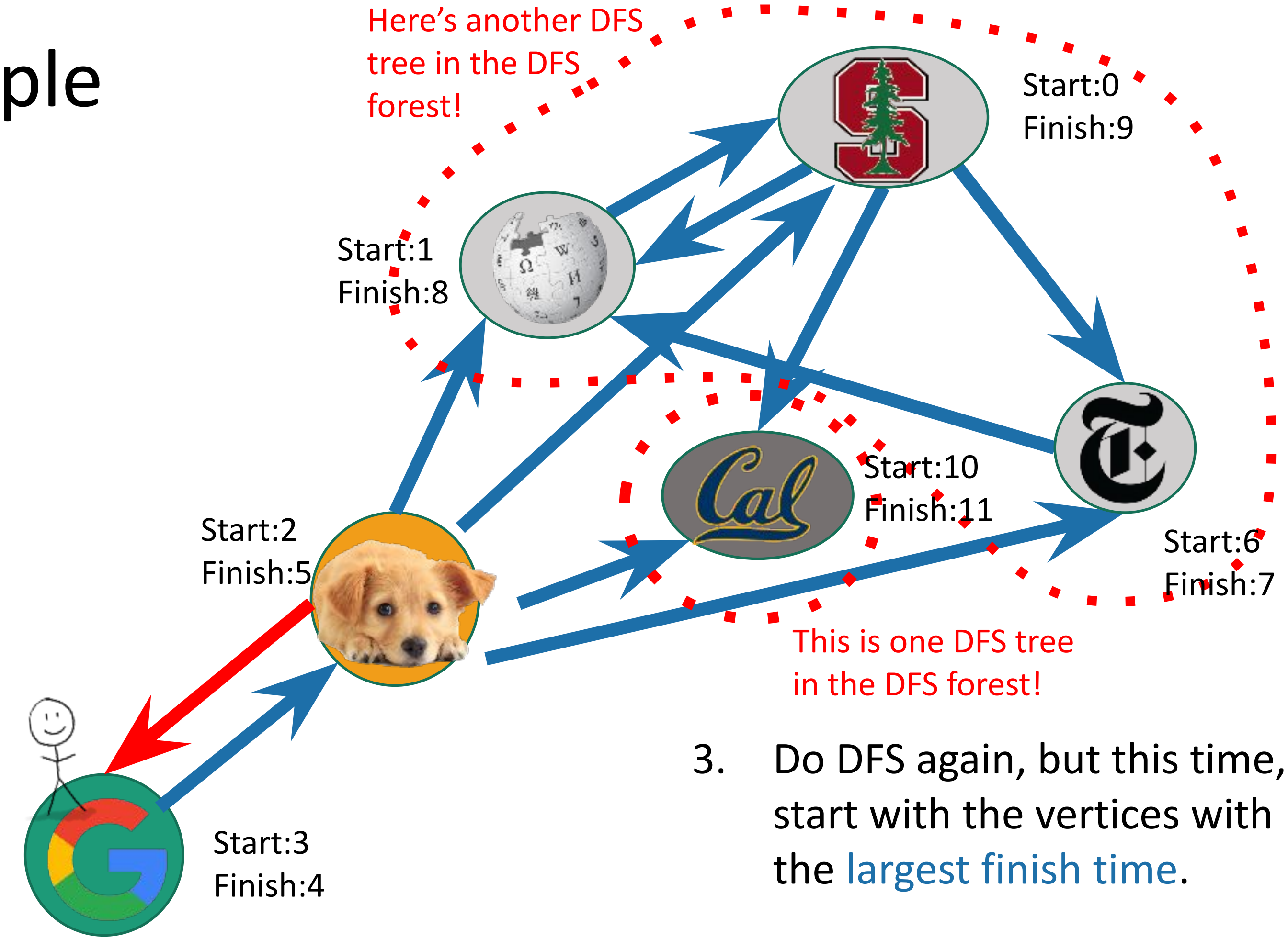

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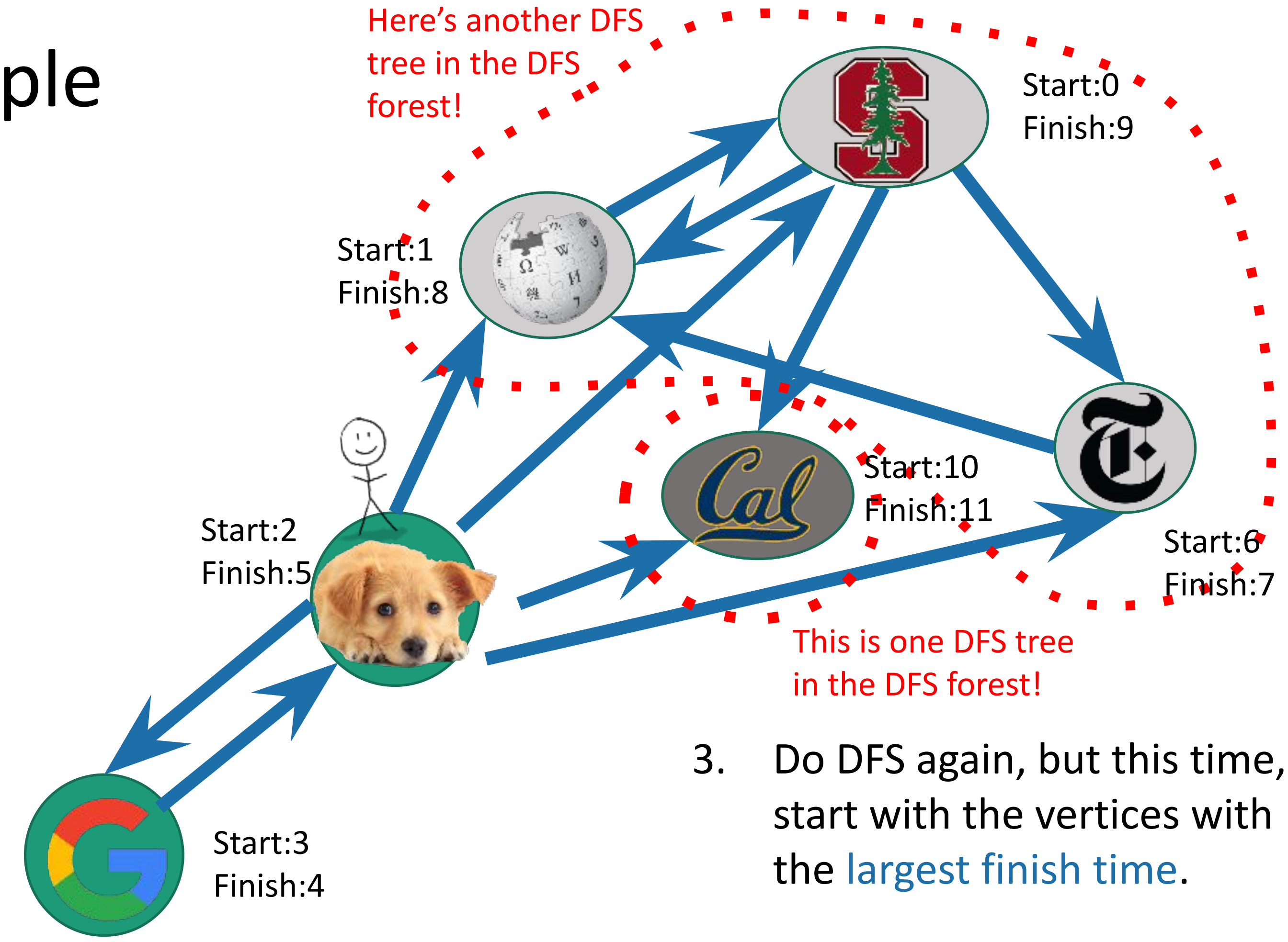
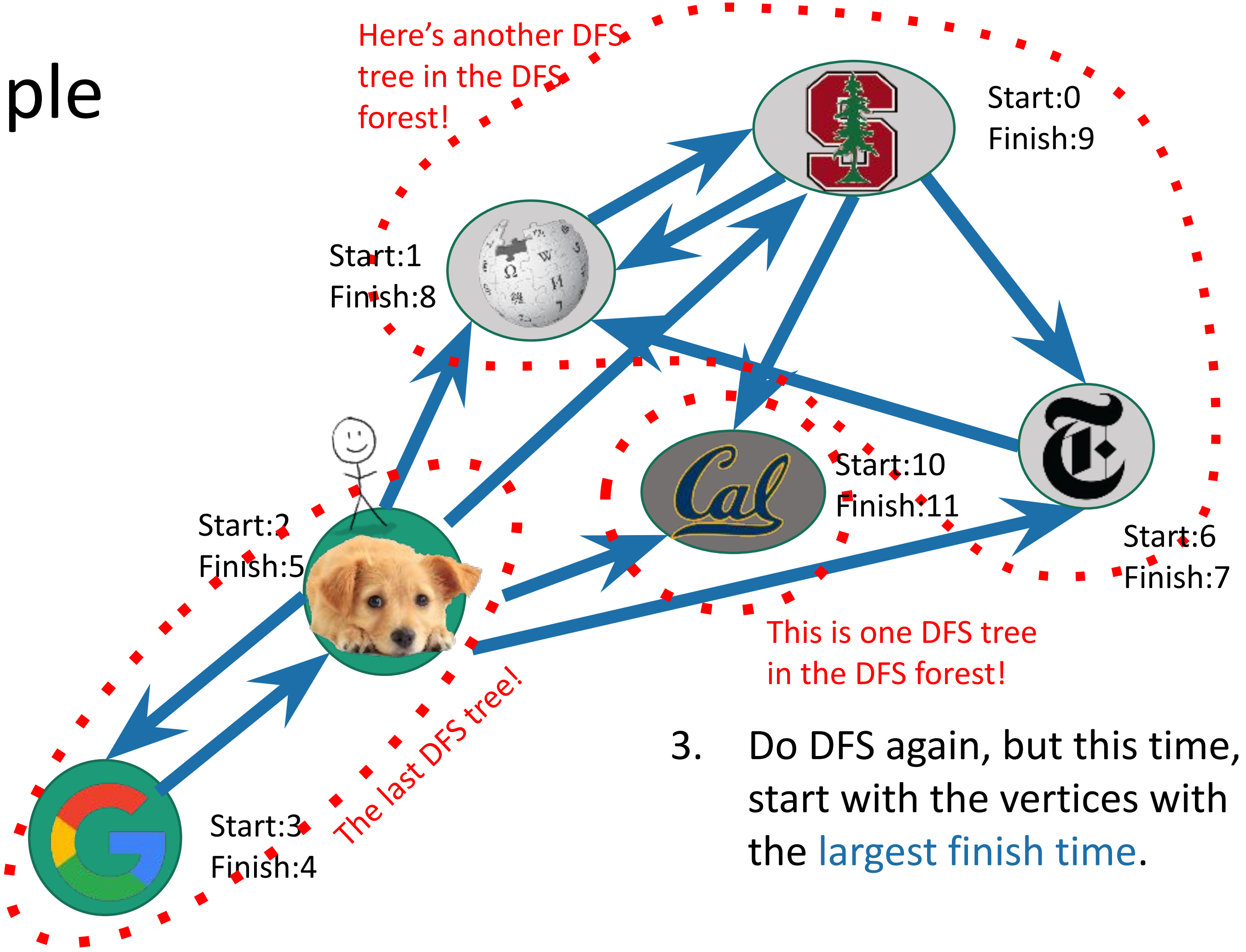
Example



3. Do DFS again, but this time, start with the vertices with the largest finish time.

Example



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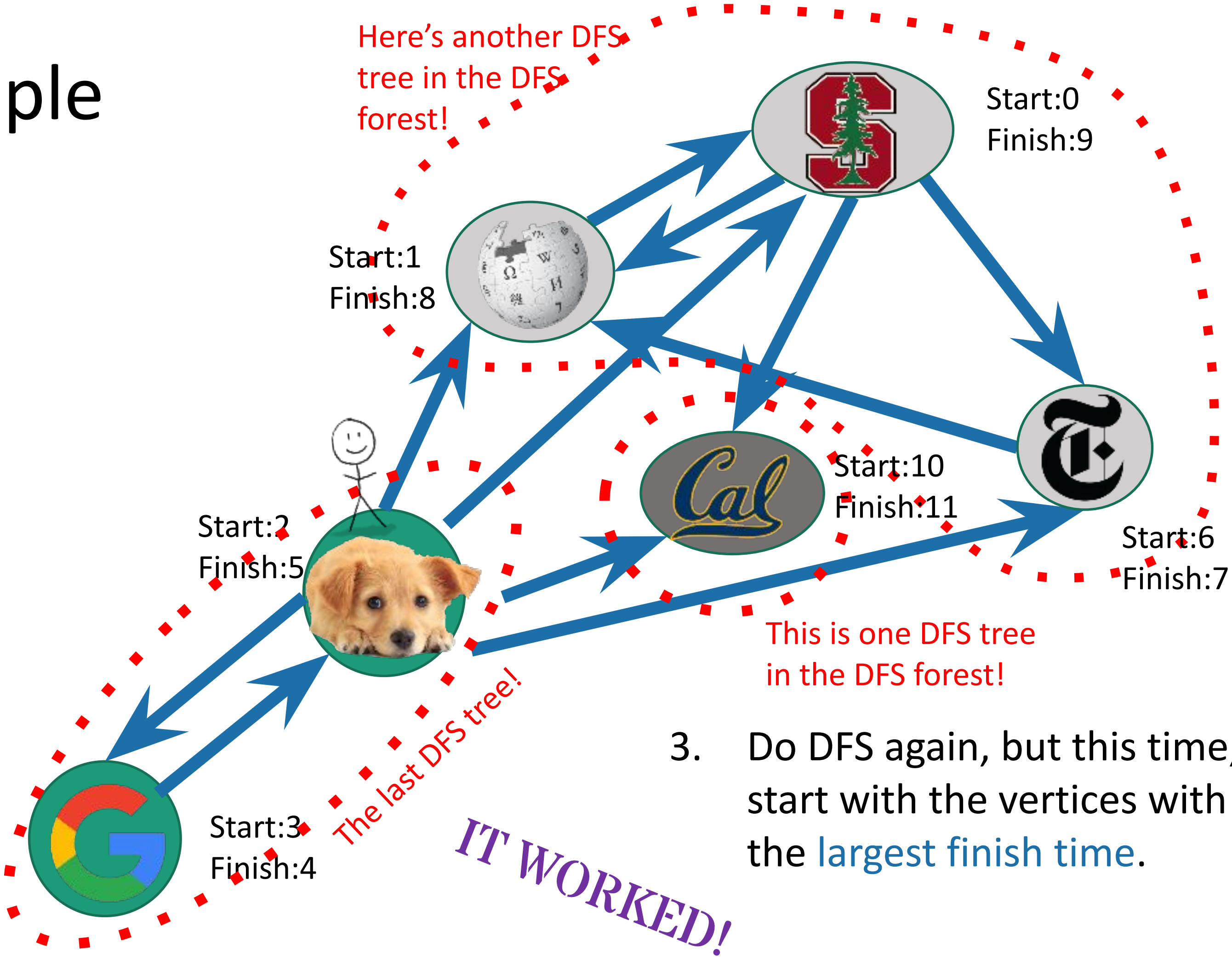
Example

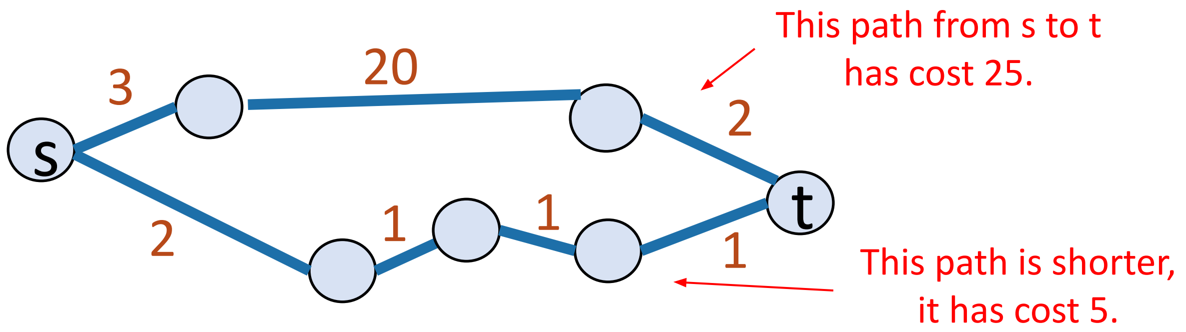


3. Do DFS again, but this time, start with the vertices with the largest finish time.

Shortest Path algorithms

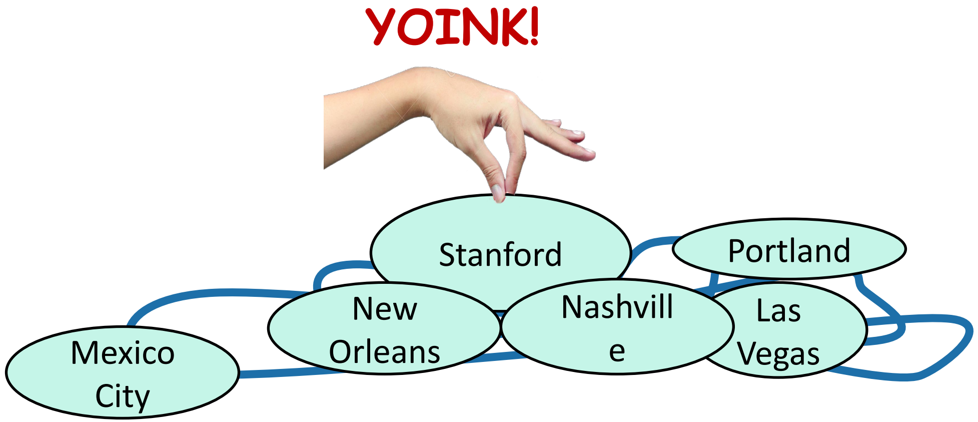
Shortest path problem

- What is the shortest path between u and v in a weighted graph?
  - the **cost** of a path is the sum of the weights along that path
  - The **shortest path** is the one with the minimum cost.

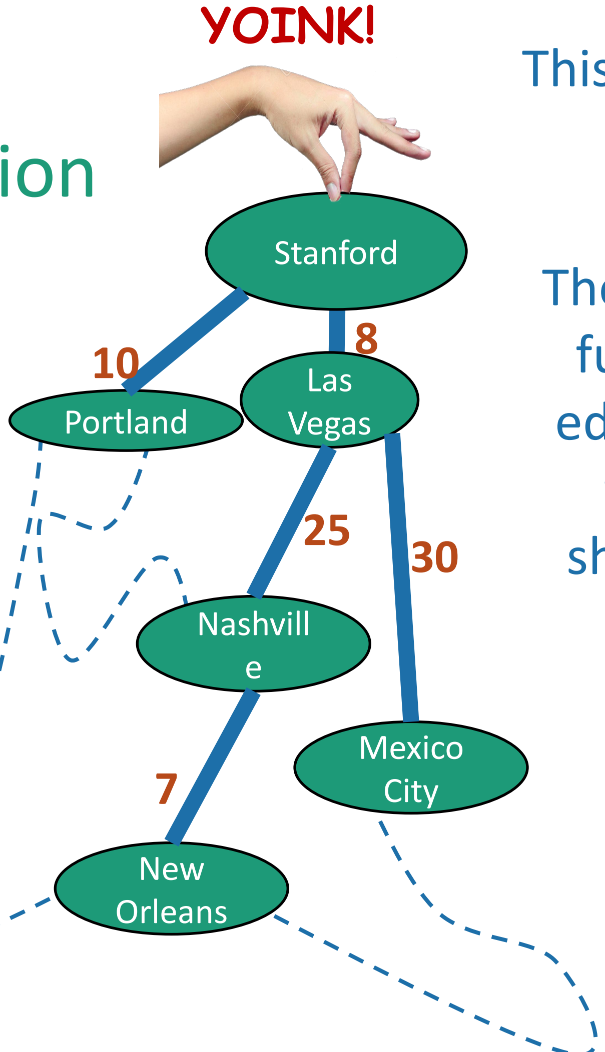

This path from s to t has cost 25.
3
20
2
s
t
2
1
1
1
This path is shorter, it has cost 5.

- The **distance** d(u,v) between two vertices u and v is the cost of the shortest path between u and v.

Dijkstra
visual intuition



Dijkstra
visual intuition



This creates a tree!

The highlighted + fully stretched edges along this tree are the shortest paths!

Dijkstra Pseudocode

Dijkstra(G,s):

```
• Set all vertices to not-sure
• d[v] = ∞ for all v in V (except s)
• d[s] = 0
• While there are not-sure nodes:
    • Pick the not-sure node u with the smallest estimate d[u].
    • Mark u as sure.
    • For v in u.neighbors:
        • d[v] ← min( d[v] , d[u] + edgeWeight(u,v))
• Now d(s, v) = d[v]
```

Shouldn't be used on graphs with negative edges!

Runtime? Will need to use Fibonacci heaps
to make operations highlighted operations run faster!

Need to use Fibonacci Heap to implement fastest Dijkstra

* - amortized time

	Sorted Arrays	Linked Lists	Binary Heap	Fibonacci Heap	Red-Black Trees
Search	O(log(n))	O(n)	O(n)	O(n)	O(log(n))
Delete	O(n)	Search +O(1)	Search +O(log(n))	Search +O(log(n))*	O(log(n))
Insert	O(n)	O(1)	O(log(n))	O(1)	O(log(n))
Extract-min	O(1)	O(n)	O(log(n))	O(log(n))*	O(log(n))
Decrease-key	O(n)	Search +O(1)	Search +O(log(n)	O(1)*	O(log(n)

Dijkstra with **Fibonacci Heap**

- T(extractMin) = O(log(n)) (amortized time*)
- T(decreaseKey) = O(1) (amortized time*)
- See CS166 for more! (or CLRS)

- Running time of Dijkstra

  = O(n(T(`extractMin`)) + m T(`decreaseKey`))

  = O(nlog(n) + m)

*This means that any sequence of d `extractMin` calls takes time at most O(dlog(n)). But a few of the d may take longer than O(log(n)) and some may take less time..

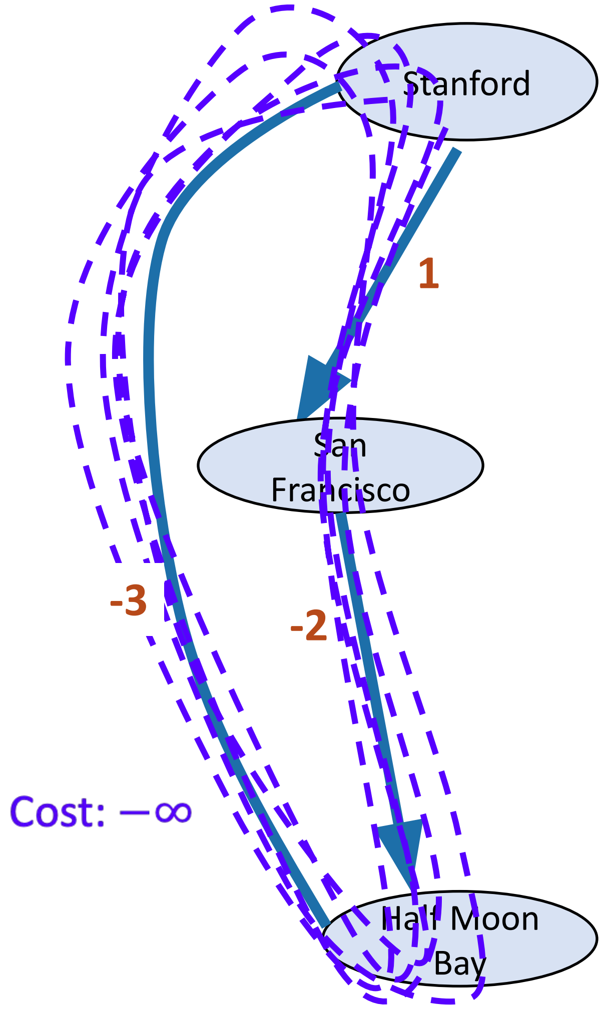
What about a Dynamic Programming approach to shortest path problems?

Assumption: no negative cycles

- Negative weights are possible (some algorithms won't work in that case!)
  - Example: negative costs because I pick up some passengers

- But if we have negative cycles…

- Shortest paths aren't defined if there are negative cycles!



Shortest path using DP

Substructure = path with small number of edges

- **Step 1:**

  Optimal substructure: shortest path using $\leq i$ edges

- **Step 2:**

Suppose we already know $d^i(s,u)$ for fixed s and all u

Recursive formulation: $d^{i+1}(s,v) = \min_u \{d^i(s,u) + w(u,v)\}$

last vertex before v

Optimal path from s to u + from u to v

Step 3: write the algorithm

The DP algorithm is exactly Bellman-Ford!

Bellman-Ford(G,s):

```
• $d^{(0)}[v] = \infty$ for all v in V
• $d^{(0)}[s] = 0$                         // initialize:
                                             $d^{(i)}[v]$ is distance from s to v
                                                    with $\leq$i edges

• For i=0,…,n-2:
    • $d^{(i+1)}[v] = d^{(i)}[v]$ for all v in V    // baseline distance:
                                                      v doesn't need (i+1)th edge
    • For v in V:
        • For u in v.neighbors:
            • $d^{(i+1)}[v] \leftarrow \min(d^{(i+1)}[v], d^{(i)}[u] + edgeWeight(u,v))$
                                                    // found a better path through u

• Return $d^{(n-1)}$
```

n = # of vertices
m = # of edges

Bellman-Ford take-aways

- Running time is O(mn)
  - For each of n rounds, update m edges.

```
• For i=0,…,n-1:
    • For u in V:
        • For v in u.neighbors:
```

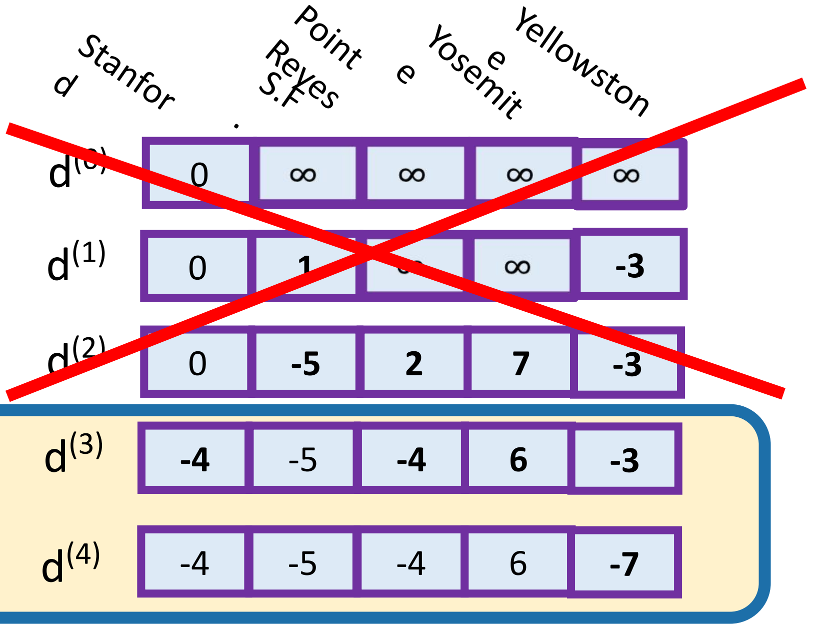
m = # of edges

$$= \frac{1}{2} \sum_{v \in V} \text{degree}(v)$$

- Works fine with negative edges.
- Does not work with negative cycles.
  - But it can detect negative cycles!

Note on implementation

- Don't actually keep all n arrays around.
- Just keep two at a time: "last round" and "this round"



	Stanford	S.F.	Point Reyes	Yosemite	Yellowstone
~~$d^{(0)}$~~	~~0~~	~~∞~~	~~∞~~	~~∞~~	~~∞~~
~~$d^{(1)}$~~	~~0~~	~~**1**~~	~~∞~~	~~∞~~	~~**-3**~~
~~$d^{(2)}$~~	~~0~~	~~**-5**~~	~~**2**~~	~~**7**~~	~~**-3**~~
$d^{(3)}$	**-4**	-5	**-4**	**6**	**-3**
$d^{(4)}$	-4	-5	-4	6	**-7**

Only need these two in order to compute $d^{(4)}$

Thank you!