CS 161 Section 6

CA: [name of CA]

Agenda

1. Dynamic Programming
2. Graphs
    a. Bellman-Ford
    b. Floyd-Warshall
3. Section Problems

Dynamic Programming

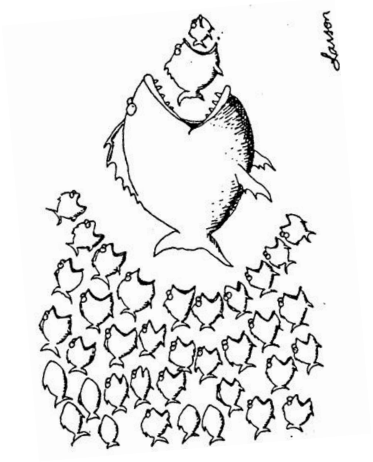
What is ***dynamic programming***?

- It is an algorithm design paradigm
  - like divide-and-conquer is an algorithm design paradigm.
- Usually it is for solving **optimization problems**
  - eg, ***shortest*** path, or ***longest*** common subsequence
  - (Fibonacci numbers aren't an optimization problem, but they are a good example…)

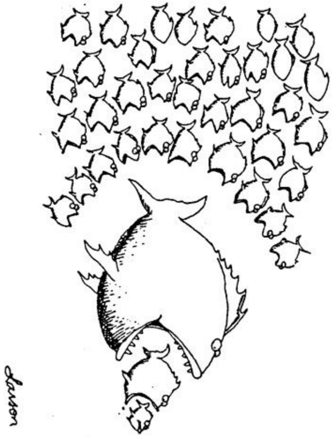
Bottom up approach

what we just saw.

- For Fibonacci:
- Solve the small problems first
  - fill in F[0],F[1]
- Then bigger problems
  - fill in F[2]
- …
- Then bigger problems
  - fill in F[n-1]
- Then finally solve the real problem.
  - fill in F[n]



Top down approach



- Think of it like a recursive algorithm.
- To solve the big problem:
  - Recurse to solve smaller problems
    - Those recurse to solve smaller problems
      - etc..

- The difference from divide and conquer:
  - **Memo-ization**
  - Keep track of what small problems you've already solved to prevent re-solving the same problem twice.

What have we learned?

- ***Dynamic programming:***
  - ○ Paradigm in algorithm design.
  - ○ Uses **optimal substructure**
  - ○ Uses **overlapping subproblems**
  - ○ Can be implemented **bottom-up** or **top-down.**
  - ○ It’s a fancy name for a pretty common-sense idea:

Don’t duplicate work if you don’t have to!

Longest Common Subsequence

- Subsequence:
  - BDFH is a **subsequence** of ABCDEFGH
- If X and Y are sequences, a **common subsequence** is a sequence which is a subsequence of both.
  - BDFH is a **common subsequence** of ABCDEFGH and of ABDFGHI
- A **longest common subsequence**…
  - …is a common subsequence that is longest.
  - The **longest common subsequence** of ABCDEFGH and ABDFGHI is ABDFGH.

Recipe for applying Dynamic Programming

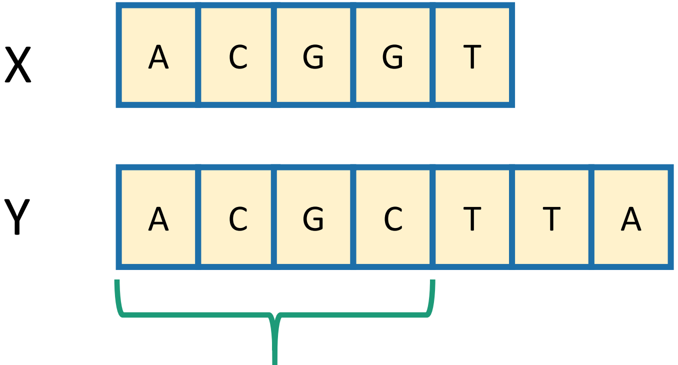
- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the length of the longest common subsequence.
- **Step 3:** Use dynamic programming to find the length of the longest common subsequence.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual LCS.

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.

Step 1: Optimal substructure

Prefixes:



X	A	C	G	G	T		
Y	A	C	G	C	T	T	A

Notation: denote this prefix **ACGC** by Y_4

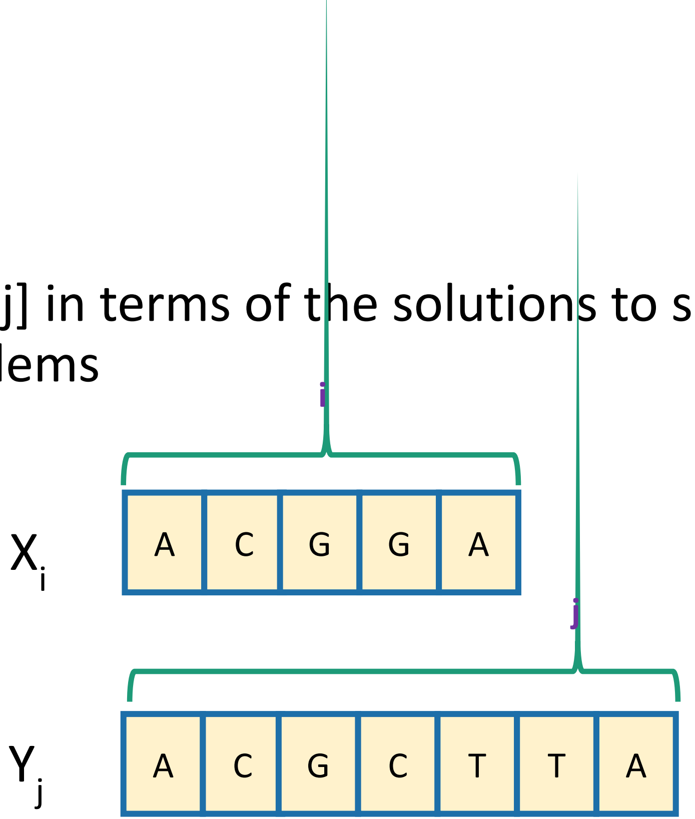
- Our sub-problems will be finding LCS's of prefixes to X and Y.
- Let C[i,j] = length_of_LCS(X_i, Y_j)

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the length of the longest common subsequence.

Goal

- Write C[i,j] in terms of the solutions to smaller sub-problems

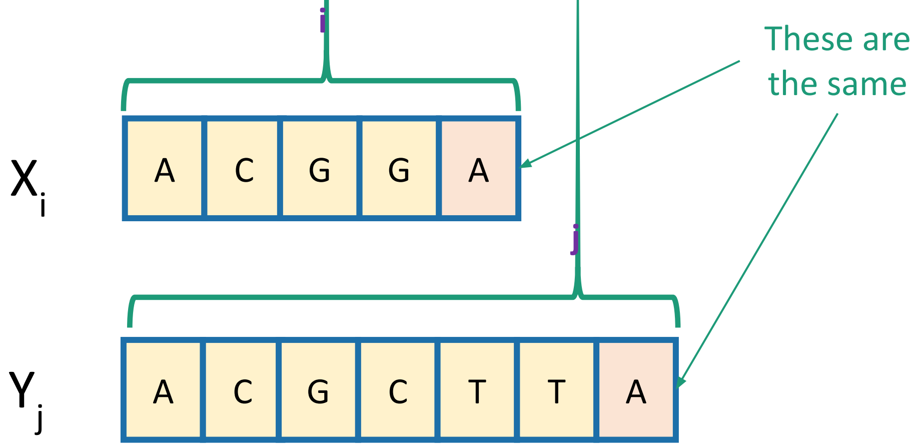


C[i,j] = length_of_LCS(X_i, Y_j)

Two cases

Case 1: X[i] = Y[j]

- Our sub-problems will be finding LCS's of prefixes to X and Y.
- Let C[i,j] = length_of_LCS(X_i, Y_j)



X_i A C G G A

Y_j A C G C T T A

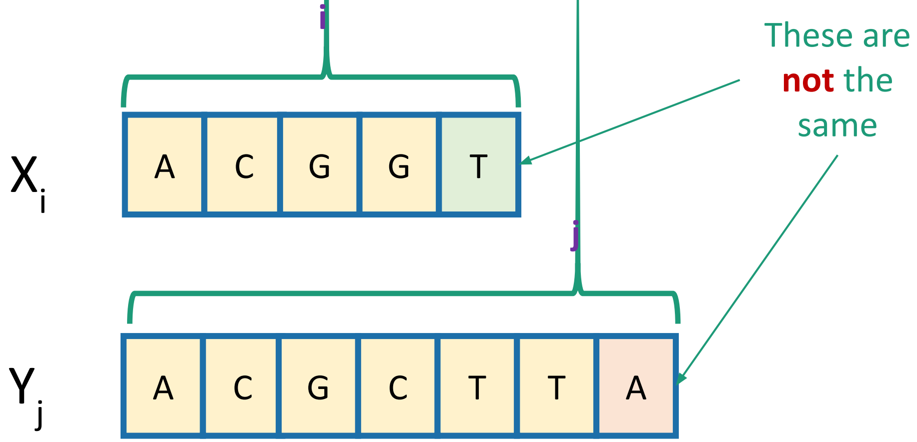
These are the same

- Then C[i,j] = 1 + C[i-1,j-1].
  - because LCS(X_i,Y_j) = LCS(X_{i-1},Y_{j-1}) followed by $\boxed{A}$

Two cases

Case 2: X[i] != Y[j]

- Our sub-problems will be finding LCS's of prefixes to X and Y.
- Let C[i,j] = length_of_LCS(X_i, Y_j)



These are **not** the same

- Then C[i,j] = max{ C[i-1,j], C[i,j-1] }.
  - either LCS(X_i,Y_j) = LCS(X_{i-1},Y_j) and $\boxed{T}$ is not involved,
  - or LCS(X_i,Y_j) = LCS(X_i,Y_{j-1}) and $\boxed{A}$ is not involved,
  - (maybe both are not involved, that's covered by the "or").

Recursive formulation

of the optimal solution

X_0							
Y_j	A	C	G	C	T	T	A

Case 0

- $$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1,j-1]+1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{\, C[i,j-1], C[i-1,j]\} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$$

Case 1

X_i	A	C	G	G	A		
Y_j	A	C	G	C	T	T	A

Case 2

X_i	A	C	G	G	T		
Y_j	A	C	G	C	T	T	A

Recipe for applying Dynamic Programming

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LCS DP OMG BBQ

- **LCS**(X, Y):
  - C[i,0] = C[0,j] = 0 for all i = 1,…,m, j=1,…n.
  - **For** i = 1,…,m and j = 1,…,n:
    - **If** X[i] = Y[j]:
      - C[i,j] = C[i-1,j-1] + 1
    - **Else:**
      - C[i,j] = max{ C[i,j-1], C[i-1,j] }

Running time: O(nm)

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{\, C[i,j-1], C[i-1,j]\} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$$

Example

X	A	C	G	G	A

Y	A	C	T	G

Y

X

		A	C	T	G
	0	0	0	0	0
A	0				
C	0				
G	0				
G	0				
A	0				

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i, j > 0 \\ \max\{\, C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i, j > 0 \end{cases}$$

Example

X	A	C	G	G	A

Y	A	C	T	G

Y

X		A	C	T	G
	0	0	0	0	0
A	0	1	1	1	1
C	0	1	2	2	2
G	0	1	2	2	3
G	0	1	2	2	3
A	0	1	2	2	3

So the LCS of X and Y has length 3.

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i, j > 0 \\ \max\{\, C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i, j > 0 \end{cases}$$

Recipe for applying Dynamic Programming

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Finding an LCS

- See lecture notes for pseudocode
- Takes time O(mn) to fill the table
- Takes time O(n + m) on top of that to recover the LCS
    - We walk up and left in an n-by-m array
    - We can only do that for n + m steps.
- Altogether, we can find LCS(X,Y) in time O(mn).

Time and Space complexity

- If we are only interested in the length of the LCS:
  - Since we go across the table one-row-at-a-time, we can only keep two rows if we want.
  - If we want to recover the LCS, we need to keep the whole table.

- Can we do better than $O(mn)$ time?
  - A bit better.
    - By a log factor or so.
  - But doing much better (e.g. $O(mn^{0.9})$) is an open problem!
    - If you can do it let me know ☺

What have we learned?

- We can find LCS(X,Y) in time O(nm)
  - if |Y|=n, |X|=m

- We went through the steps of coming up with a dynamic programming algorithm.
  - We kept a 2-dimensional table, breaking down the problem by decrementing the length of X and Y.

Graphs: Bellman-Ford and Floyd-Warshall

Shortest path DP by recipe

Substructure = path with small number of edges

- **Step 1:**

Optimal substructure: shortest path using $\leq$i edges

- **Step 2:**

Suppose we already know $d^i(s,u)$ for fixed s and all u

Recursive formulation: $d^{i+1}(s,v) = \min_u \{d^i(s,u)+w(u,v)\}$

last
vertex
before v

Optimal path
from s to u + from u to
v

- **Step 3+4:** Later…

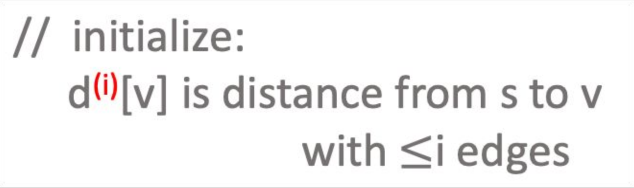
Step 3: write the algorithm

Bellman-Ford(G,s):

```
• $d^{(0)}[v] = \infty$ for all v in V
• $d^{(0)}[s] = 0$

• For i=0,…,n-2:
    • $d^{(i+1)}[v] = d^{(i)}[v]$ for all v in V      // baseline distance:
                                                      v doesn't need $(i+1)^{th}$ edge
    • For v in V:
        • For u in v.neighbors:
            • $d^{(i+1)}[v] \leftarrow \min(d^{(i+1)}[v], d^{(i)}[u] + edgeWeight(u,v))$
                                                      // found a better path through u

• Return $d^{(n-1)}$
```



// initialize:
$d^{(i)}[v]$ is distance from s to v
with $\leq$i edges

Bellman-Ford take-aways

n = # of vertices
m = # of edges

- Running time is O(mn)
  - For each of n rounds, update m edges.

```
For i=0,…,n-1:
    For u in V:
        For v in u.neighbors:
```

m = # of edges

$= \frac{1}{2} \sum_{v \in V} \text{degree}(v)$

- Works fine with negative edges.
- Does not work with negative cycles.
  - But it can detect negative cycles!

Note on implementation

- Don’t actually keep all n arrays around.
- Just keep two at a time: “last round” and “this round”

	Stanford	S.F.	Point Reyes	Yosemite	Yellowstone
~~$d^{(0)}$~~	~~0~~	~~∞~~	~~∞~~	~~∞~~	~~∞~~
~~$d^{(1)}$~~	~~0~~	~~**1**~~	~~∞~~	~~∞~~	~~**-3**~~
~~$d^{(2)}$~~	~~0~~	~~**-5**~~	~~**2**~~	~~**7**~~	~~**-3**~~
$d^{(3)}$	**-4**	-5	**-4**	**6**	**-3**
$d^{(4)}$	-4	-5	-4	6	**-7**

Only need these two in order to compute $d^{(4)}$

Floyd-Warshall Algorithm

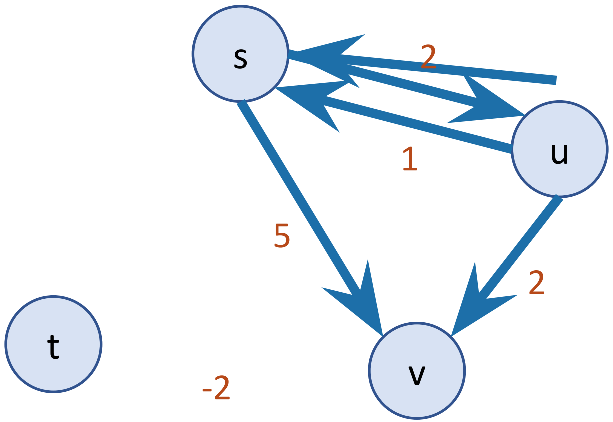
Another example of DP

- This is an algorithm for **All-Pairs Shortest Paths** (APSP)
  - That is, I want to know the shortest path from u to v for **ALL pairs** u,v of vertices in the graph.
  - Not just from a special single source s.

Destination

Source

	s	u	v	t
s	0	2	4	2
u	1	0	2	0
v			0	-2
t				0



Floyd-Warshall Algorithm

Another example of DP

- This is an algorithm for **All-Pairs Shortest Paths** (APSP)
  - That is, I want to know the shortest path from u to v for **ALL pairs** u,v of vertices in the graph.
  - Not just from a special single source s.
- Naïve solution:
  - For all s in G:
    - Run Bellman-Ford on G starting at s.
  - Time $O(n \cdot nm) = O(n^2m)$,
    - may be as bad as n^4 if $m=n^2$

Can we do better?

Floyd-Warshall algorithm

- Initialize n-by-n arrays $D^{(k)}$ for k = 0,…,n
  - $D^{(k)}[u,u] = 0$ for all u, for all k
  - $D^{(k)}[u,v] = \infty$ for all $u \neq v$, for all k
  - $D^{(0)}[u,v] = \text{weight}(u,v)$ for all (u,v) in E.
- **For** k = 1, …, n:
  - **For** pairs u,v in V^2:
    - $D^{(k)}[u,v] = \min\{ D^{(k-1)}[u,v], D^{(k-1)}[u,k] + D^{(k-1)}[k,v] \}$
- **Return** $D^{(n)}$

The base case checks out: the only path through zero other vertices are edges directly from u to v.

n = # of vertices
m = # of edges

We've basically just shown

- Theorem:

  If there are no negative cycles in a weighted directed graph G, then the Floyd-Warshall algorithm, running on G, returns a matrix $D^{(n)}$ so that:

  $$D^{(n)}[u,v] = \text{distance between u and v in G.}$$

- Running time: $O(n^3)$
  - Better than running Bellman-Ford n times!

Work out the details of a proof!

- Storage:
  - Need to store **two** n-by-n arrays, and the original graph.

As with Bellman-Ford, we don't really need to store all n of the $D^{(k)}$.

n = # of vertices
m = # of edges

Recap of today's lecture

- Shortest Path in weighted graph w/ dynamic programming

- **Bellman-Ford**: Single Source Shortest Path (SSSP)
  - **Optimal substructure**: shortest path with $\leq i$ edges
  - Run time: O(nm)

- **Floyd-Warshall**: All Pairs Shortest Path (APSP)
  - **Optimal substructure**: shortest path using vertices {1,…,k-1}
  - Run time O(n^3)

Thank you!