Hashing and Sketching

Part One

Randomized Data Structures

- Randomization is a powerful tool for improving efficiency and solving problems under seemingly impossible constraints.
- Over the next three lectures, we'll explore a sampler of data structures that give a feel for the breadth of what's out there.
- You can easily spend an entire academic career just exploring this space; take CS265 for more on randomized algorithms!

Where We're Going

- ***Hashing and Sketching (This Week)***
  - Using hash functions to count without counting.
- ***Cuckoo Hashing (Next Week)***
  - Hashing with worst-case O(1) lookups, along with a splash of random hypergraph theory.

Outline for Today

- ***Hash Functions***
  - Understanding our basic building blocks.
- ***Frequency Estimation***
  - Estimating how many times we've seen something.
- ***Concentration Inequalities***
  - "Correct on expectation" versus "correct with high probability."
- ***Probability Amplification***
  - Increasing our confidence in our answers.

Preliminaries: ***Hash Functions***

Hashing in Practice

- Hash functions are used extensively in programming and software engineering:
  - They make hash tables possible: think C++ `std::hash`, Python's `__hash__`, or Java's `Object.hashCode()`.
  - They're used in cryptography: SHA-256, HMAC, etc.
- ***Question:*** When we're in Theoryland, what do we mean when we say "hash function?"

Hashing in Theoryland

- In Theoryland, a hash function is a function from some domain called the ***universe*** (typically denoted $\mathscr{U}$) to some codomain.
- The codomain is usually a set of the form

$$\boldsymbol{[m]} = \{0, 1, 2, 3, \ldots, m - 1\}$$

$$h : \mathscr{U} \to [m]$$

Hashing in Theoryland

- ***Intuition:*** No matter how clever you are with designing a specific hash function, that hash function isn't random, and so there will be pathological inputs.
  - You can formalize this with the pigeonhole principle.
- ***Idea:*** Rather than finding the One True Hash Function, we'll assume we have a collection of hash functions to pick from, and we'll choose which one to use randomly.

Families of Hash Functions

- A ***family*** of hash functions is a set $\mathscr{H}$ of hash functions with the same domain and codomain.
- We can then introduce randomness into our data structures by sampling a random hash function from $\mathscr{H}$.
- ***Key Point:*** The randomness in our data structures almost always derives from the random choice of hash functions, not from the data.

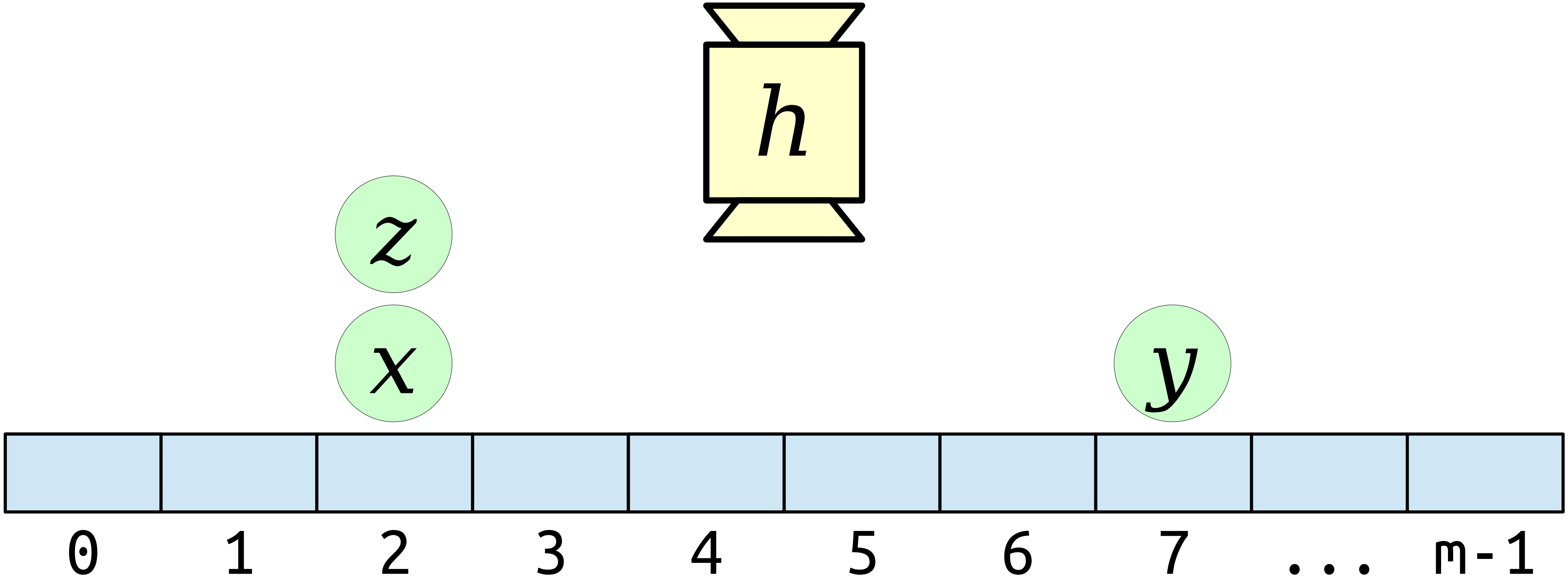
Data is adversarial.
Hash function selection is random.

- ***Question:*** What makes a family of hash functions $\mathscr{H}$ a "good" family of hash functions?

Goal: If we pick $h \in \mathscr{H}$ uniformly at random, then h should distribute elements uniformly randomly.

Problem: A hash function that distributes n elements uniformly at random over $[m]$ requires $\Omega(n \log m)$ space in the worst case.

Question: Do we actually need true randomness? Or can we get away with something weaker?

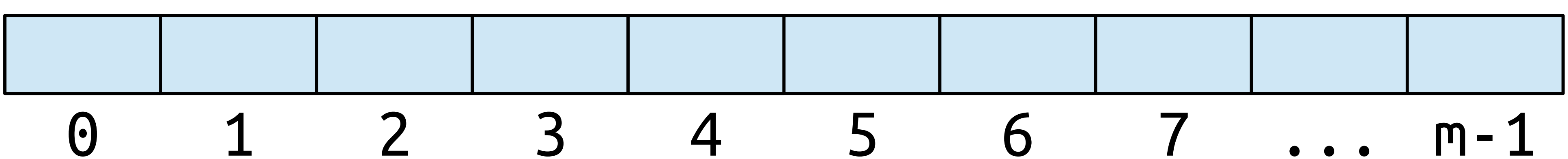


Distribution Property:
Each element should have an equal probability of being placed in each slot.

For any $x \in \mathscr{U}$ and random $h \in \mathscr{H}$, the value of $h(x)$ is uniform over its codomain.

Find an "obviously bad" family of hash functions that satisfies the distribution property.

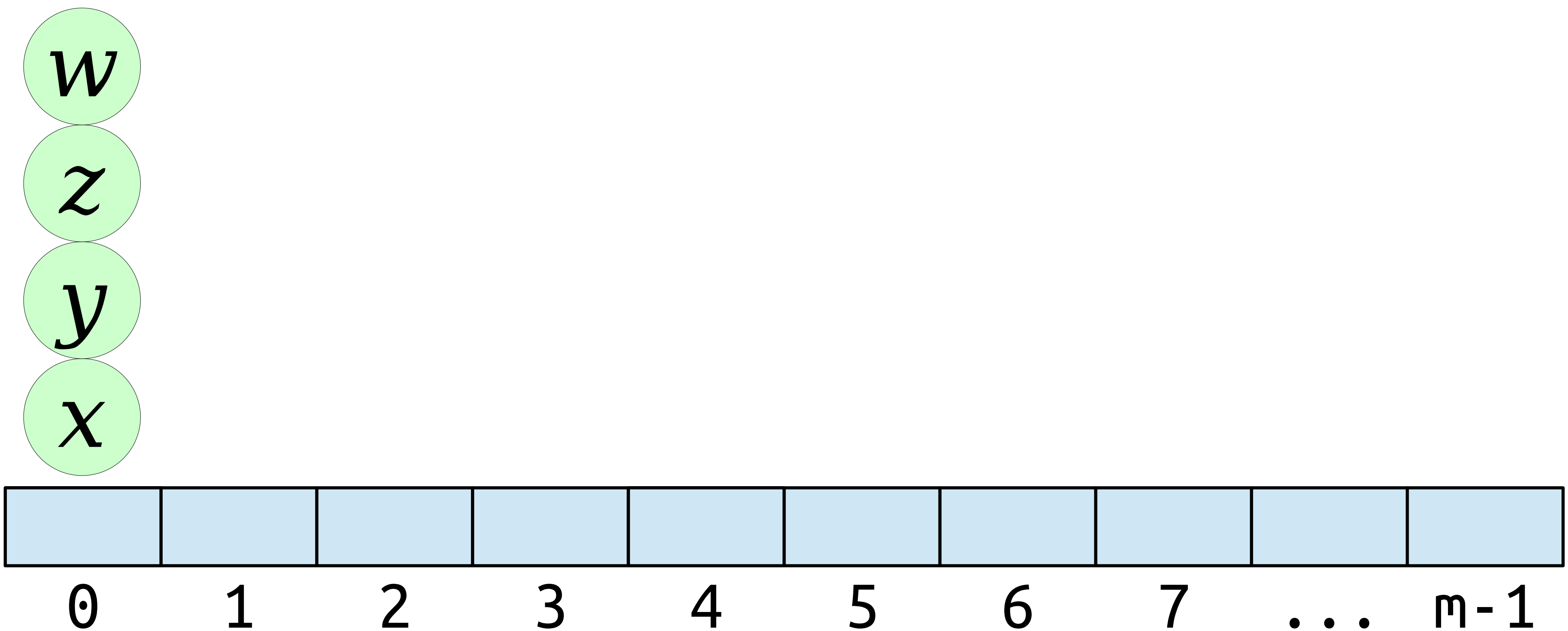
Formulate a hypothesis!



Distribution Property: Each element should have an equal probability of being placed in each slot.

For any $x \in \mathscr{U}$ and random $h \in \mathcal{H}$, the value of $h(x)$ is uniform over its codomain.

Problem: This rule doesn't guarantee that elements are spread out.



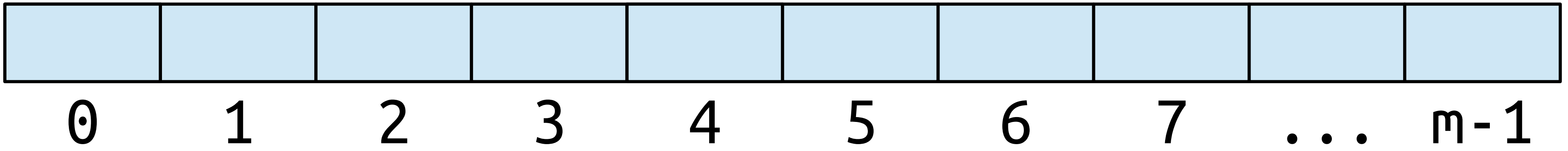
Distribution Property:
Each element should have an equal probability of being placed in each slot.

For any $x \in \mathscr{U}$ and random $h \in \mathcal{H}$, the value of $h(x)$ is uniform over its codomain.

Independence Property:
Where one element is placed shouldn't impact where a second goes.

For any distinct $x, y \in \mathscr{U}$ and random $h \in \mathcal{H}$, $h(x)$ and $h(y)$ are independent random variables.

A family of hash functions $\mathscr{H}$ is called ***2-independent*** (or ***pairwise independent***) if it satisfies the distribution and independence properties.



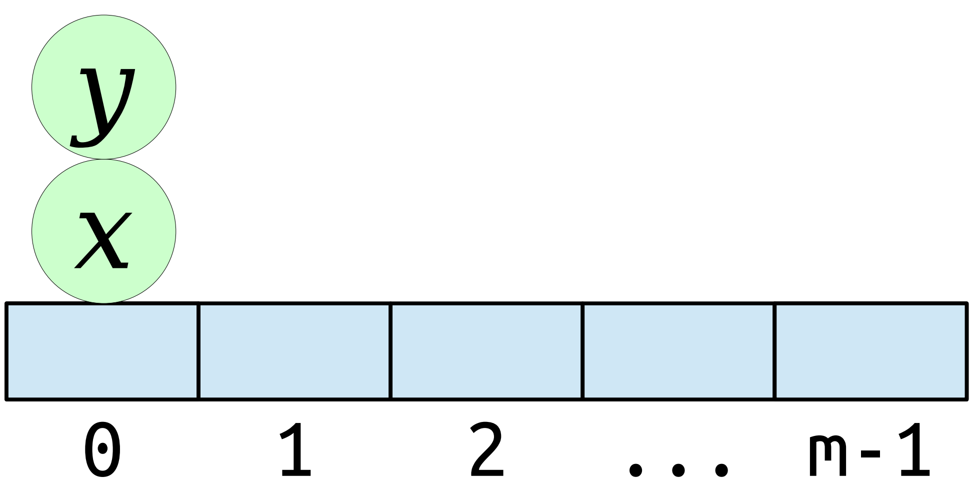
For any $x \in \mathscr{U}$ and random $h \in \mathscr{H}$, the value of $h(x)$ is uniform over its codomain.

For any distinct $x, y \in \mathscr{U}$ and random $h \in \mathscr{H}$, $h(x)$ and $h(y)$ are independent random variables.

Intuition: 2-independence means any pair of elements is unlikely to collide.

$$\Pr[h(x) = h(y)]$$

$$= \sum_{i=0}^{m-1} \Pr[h(x) = i \wedge h(y) = i]$$



Question: Where did these elements collide with one another?

For any $x \in \mathscr{U}$ and random $h \in \mathcal{H}$, the value of $h(x)$ is uniform over its codomain.

For any distinct $x, y \in \mathscr{U}$ and random $h \in \mathcal{H}$, $h(x)$ and $h(y)$ are independent random variables.

Intuition:
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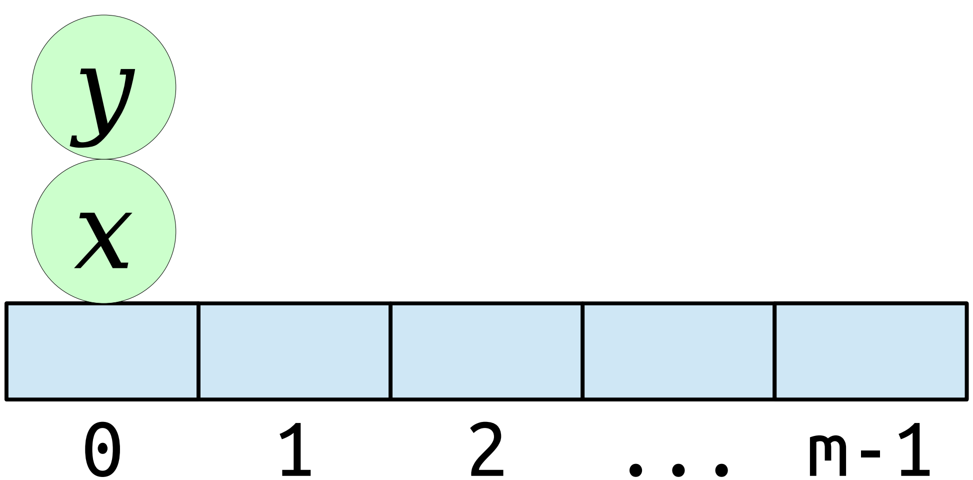
\end{aligned}
$$



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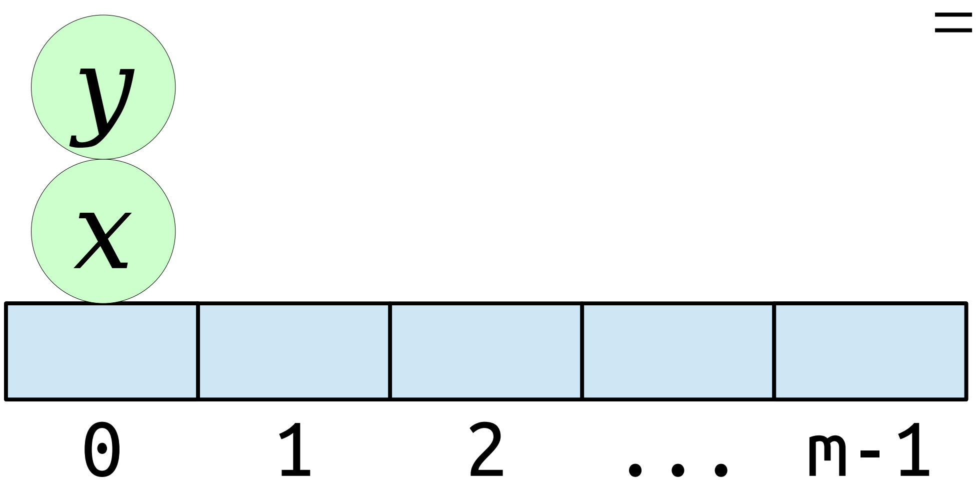
= &\sum_{i=0}^{m-1} \frac{1}{m^2}
\end{aligned}
$$



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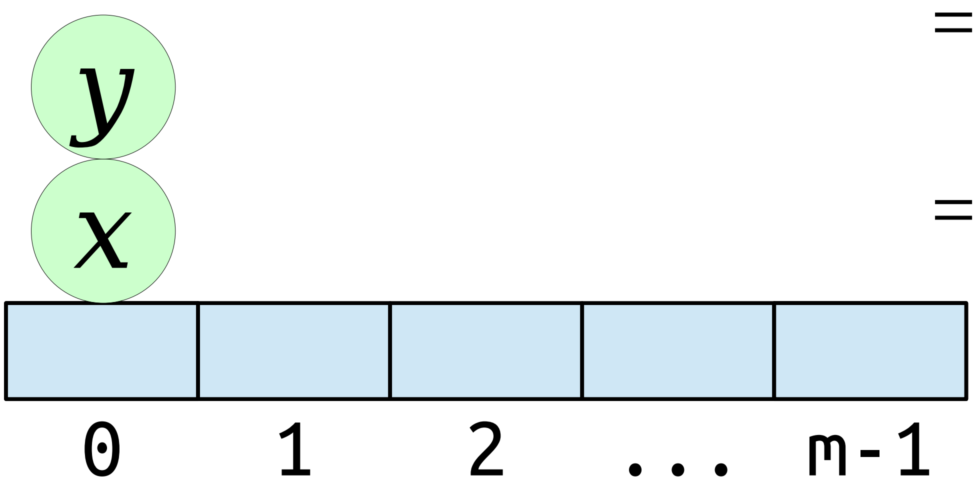


This is the same as if h were a truly random function.

For more on hashing outside of Theoryland, check out ***[this Stack Exchange post]***.

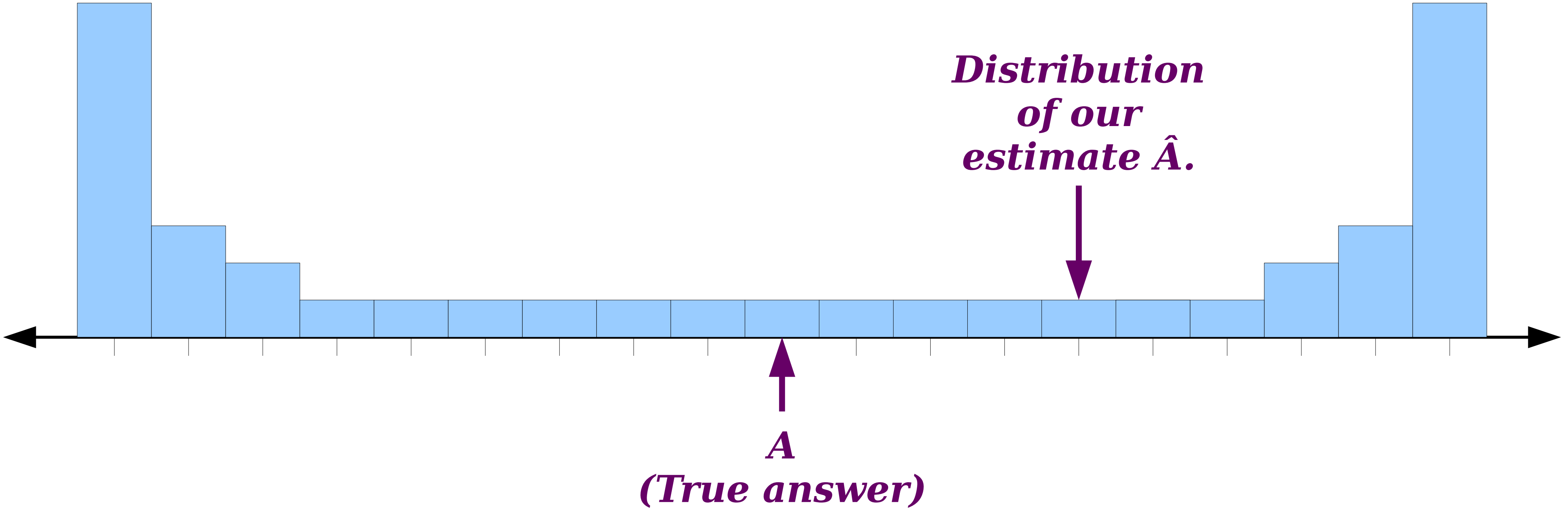
Approximating Quantities

What makes for a good "approximate" solution?

Let ***A*** be the true answer. Let $\boldsymbol{\hat{A}}$ be a random variable denoting our estimate.

This would not make for a good estimate. However, we have $\mathrm{E}[\hat{A}] = A$.

Observation 1: Being correct in expectation isn't sufficient.

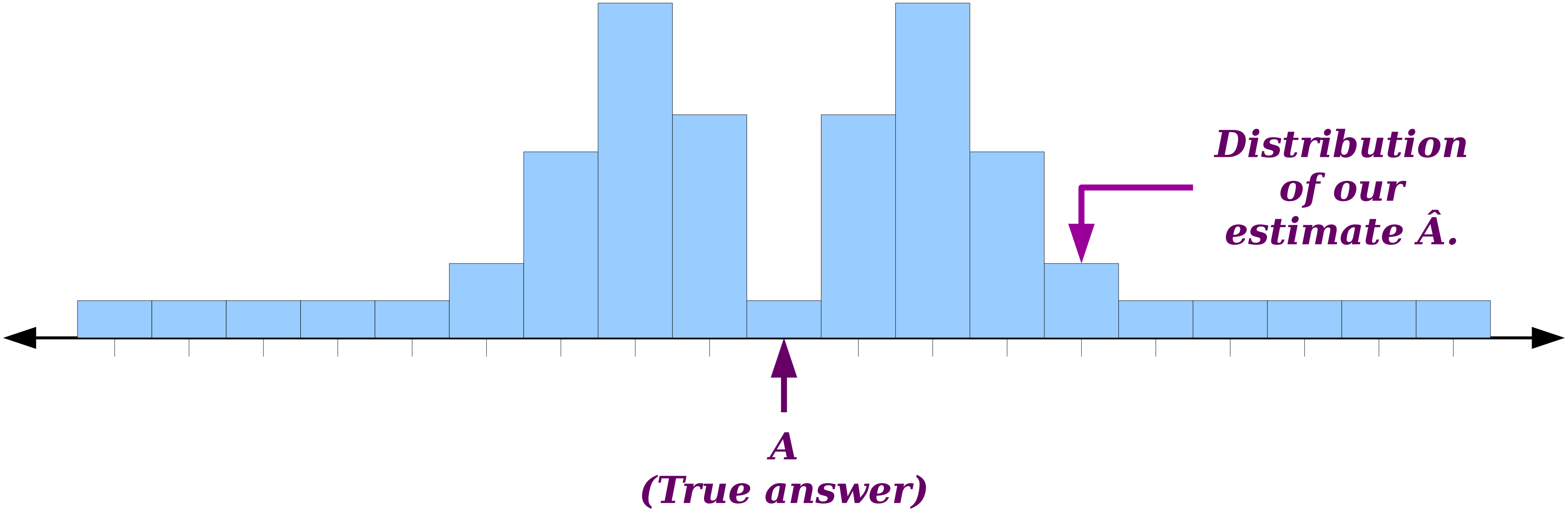


What does it mean for an approximation to be "good"?

Let $\boldsymbol{A}$ be the true answer. Let $\boldsymbol{\hat{A}}$ be a random variable denoting our estimate.

It's unlikely that we'll get the right answer, but we're probably going to be close.

Observation 2: The difference $|\hat{A} - A|$ between our estimate and the truth should ideally be small.

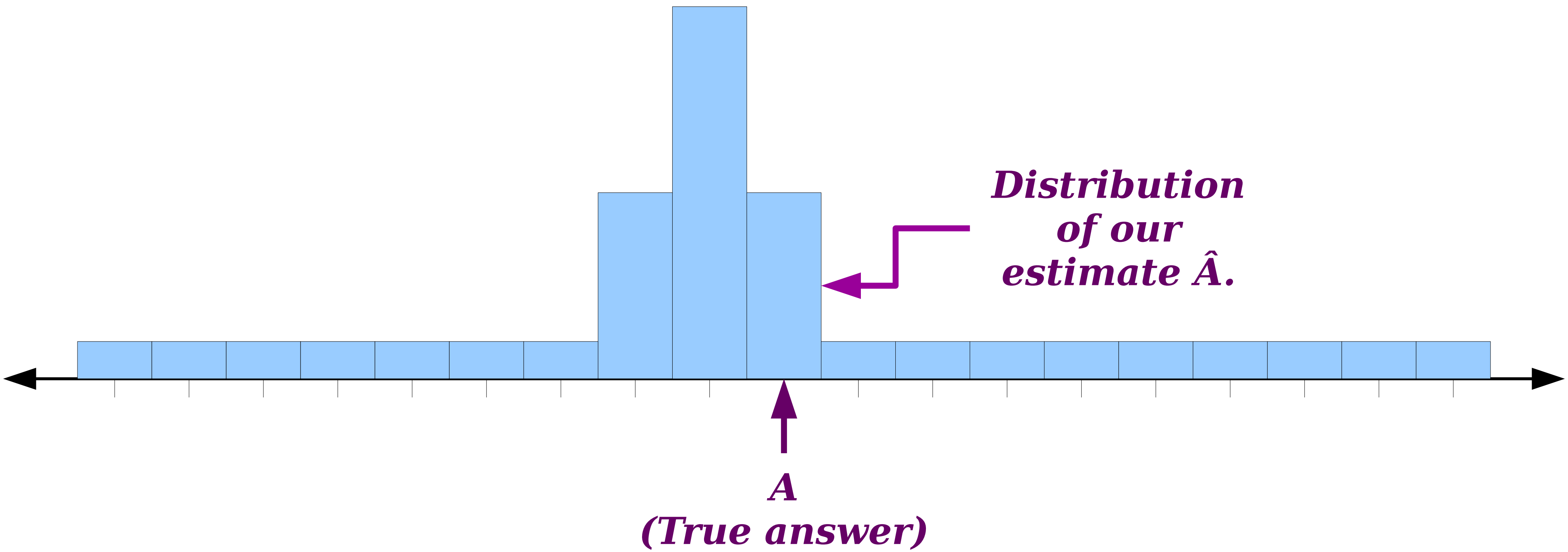


What does it mean for an approximation to be "good"?

Let A be the true answer. Let $\hat{A}$ be a random variable denoting our estimate.

This estimate skews low, but it's very close to the true value.

Observation 3: An estimate doesn't have to be unbiased to be useful.

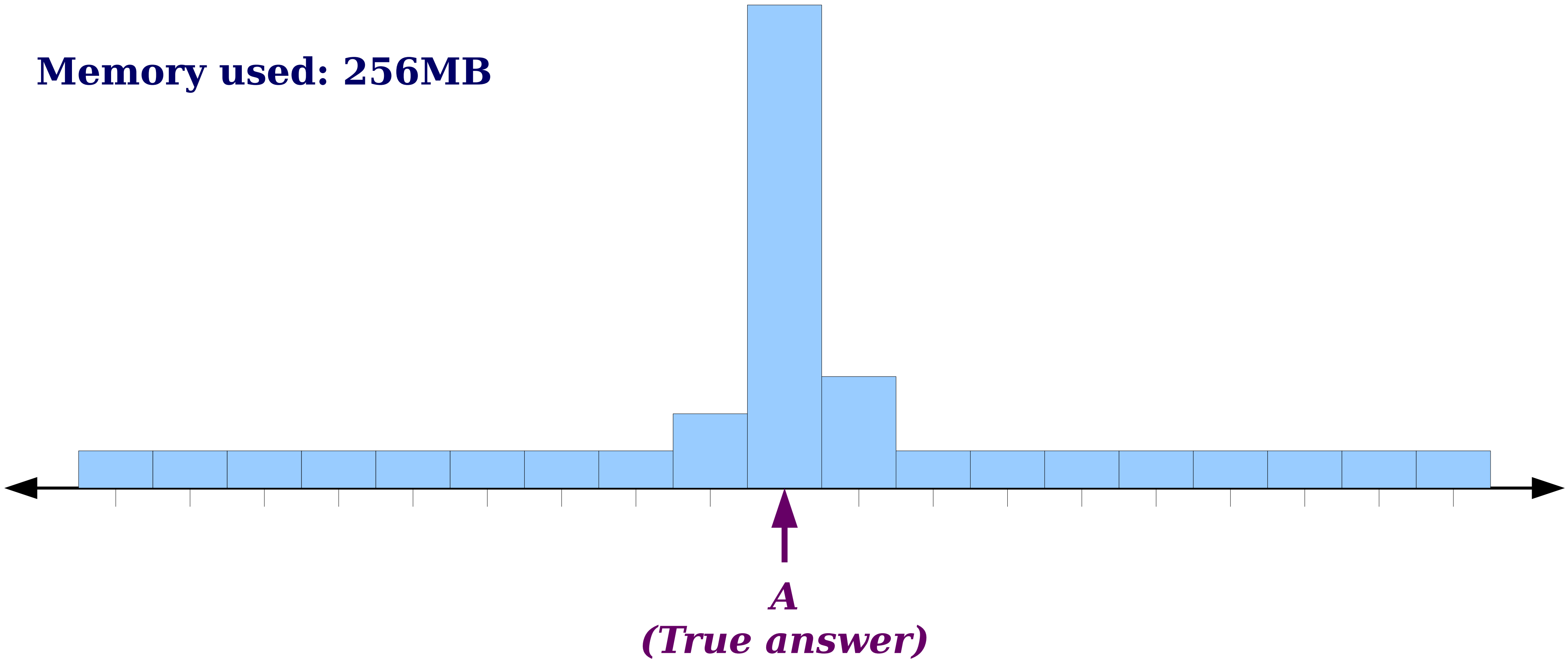


What does it mean for an approximation to be "good"?

Let $\boldsymbol{A}$ be the true answer. Let $\hat{\boldsymbol{A}}$ be a random variable denoting our estimate.

The more resources we allocate, the better our estimate should be.

Observation 4: A good approximation should be tunable.



What does it mean for an approximation to be "good"?

We have two user-provided values

$$\varepsilon \in (0, 1]$$
$$\delta \in (0, 1]$$

where ε represents ***accuracy*** and δ represents ***confidence***.

Goal: Make an estimator $\hat{A}$ for some quantity A where

With probability at least $1 - \delta$, — ***Probably***

$|\hat{A} - A| \leq \varepsilon \cdot size(input)$ — ***Approximately Correct***

for some measure of the size of the input.

What does it mean for an approximation to be "good"?

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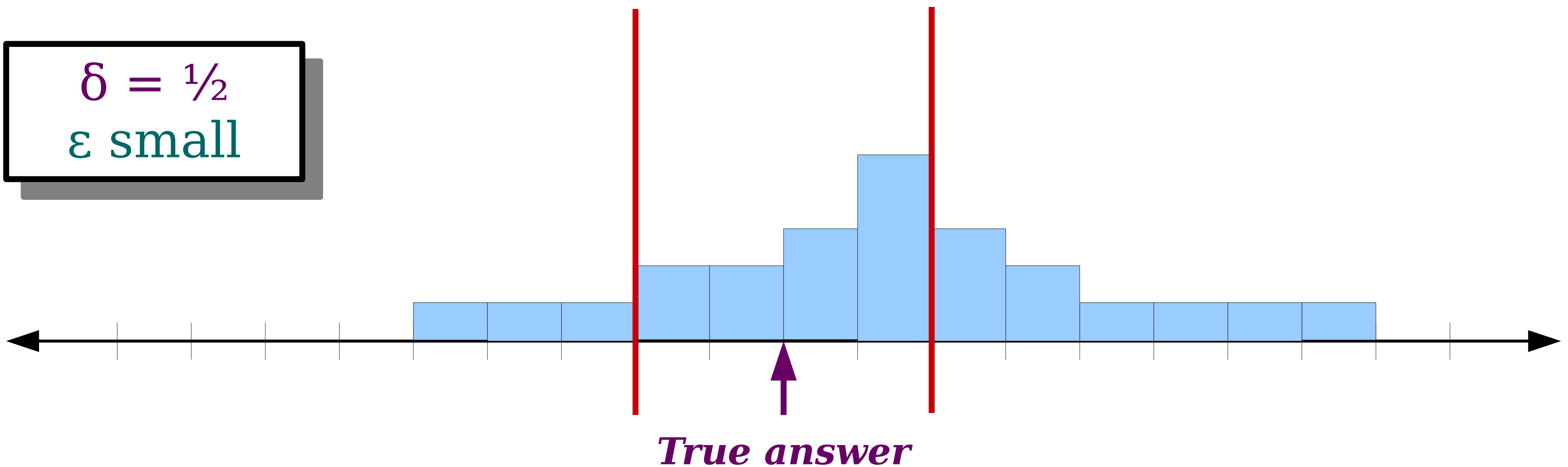
for some measure of the size of the input.

$\delta = ½$

ε small



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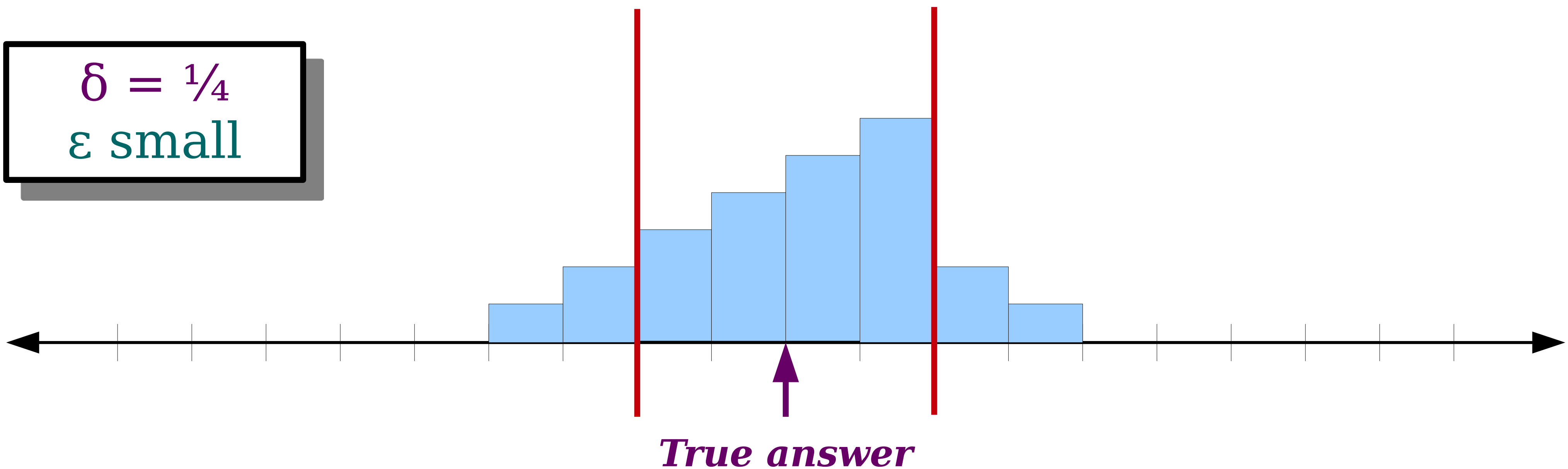
for some measure of the size of the input.

$\delta = \frac{1}{4}$

ε small



What does it mean for an approximation to be "good"?

Goal: Make an estimator $\hat{A}$ for some quantity A where

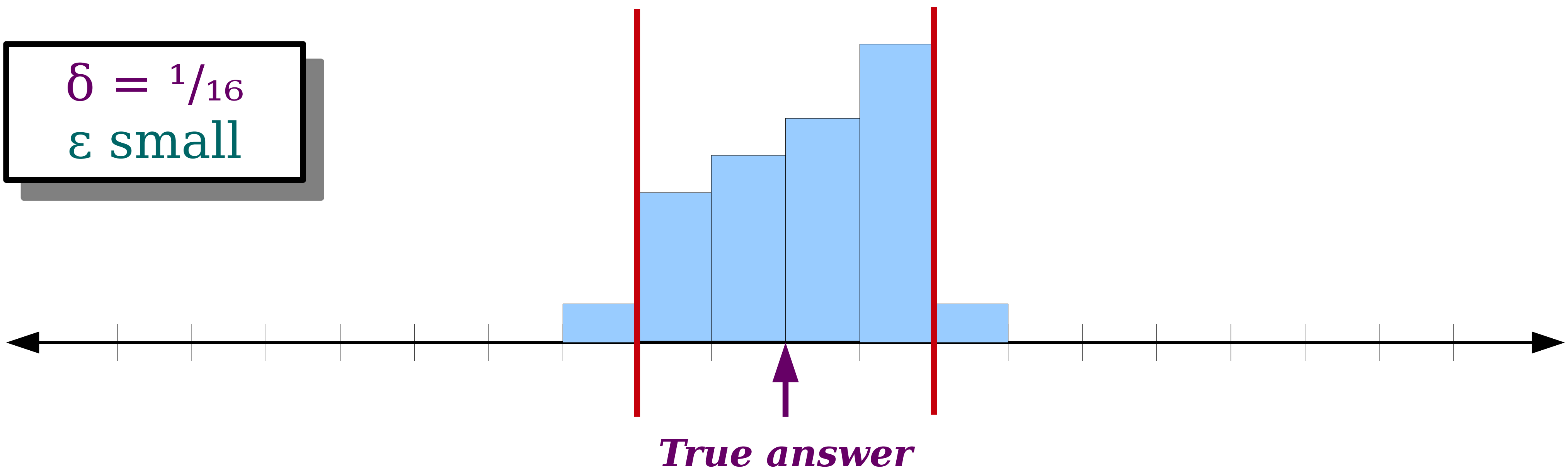
With probability at least $1 - \delta$, **Probably**

$|A - \hat{A}| \leq \varepsilon \cdot size(input)$ **Approximately Correct**

for some measure of the size of the input.

$\delta = {}^{1}/_{16}$

ε small



What does it mean for an approximation to be "good"?

Time-Out for Announcements!

PS2 / IA2

- PS1 and IA1 were due today at 3:15PM.
  - Need more time? You can submit during the grace period, which ends tomorrow at 3:15PM.
- PS2 and IA2 go out today. They’re due next Thursday at the start of class.
  - Explore balanced trees, data structure isometries, and the Method of Four Russians!

Final Project Logistics

- We’ve posted information about the CS166 final project to the course website.
- The brief summary:
  - You’ll work in teams of three or four.
  - You’ll pick a data structure and become an expert on it.
  - You’ll put together an explanatory article that guides readers on a magical journey to understanding.
  - You’ll do something “interesting” with the topic, broadly construed.
  - You’ll meet with the course staff for a Q&A session to discuss your writeup, “interesting” component, and the topic at large.
- We hope you have fun with this one – you’ll learn a ton in the process of working through this!

Final Project Logistics

- Your first deliverable is a project proposal, which is due next Thursday at the start of class.
  - Because we need to do topic matchmaking, there is no grace period for the project proposal.
- What you need to do:
  - Select a team of 3 – 4 people.
  - Give us an ordered list of your top four project topics, along with two sources for each topic. (One source per topic must be a research paper.)
- We’ve compiled an extensive list of recommended project topics. It’s available up on the course website.

Back to CS166!

Frequency Estimation

Frequency Estimators

- A ***frequency estimator*** is a data structure supporting the following operations:
  - ***increment***(x), which increments the number of times that x has been seen, and
  - ***estimate***(x), which returns an estimate of the frequency of x.
- Using BSTs, we can solve this in space $\Theta(n)$ with worst-case $O(\log n)$ costs on the operations.
- Using hash tables, we can solve this in space $\Theta(n)$ with expected $O(1)$ costs on the operations.

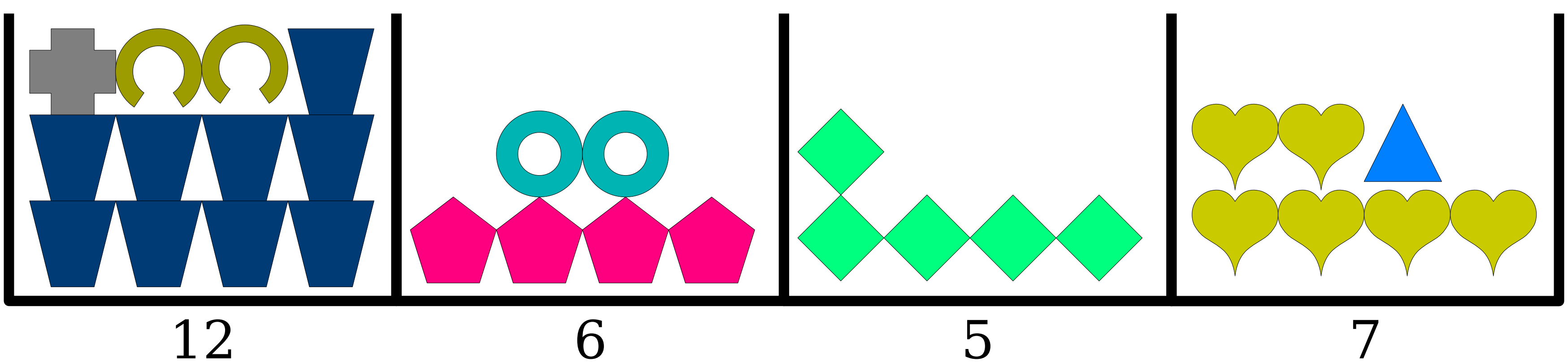
Frequency Estimators

- Frequency estimation has many applications:
  - Search engines: Finding frequent search queries.
  - Network routing: Finding common source and destination addresses.
- In these applications, $\Theta(n)$ memory can be impractical.
- ***Goal:*** Get *approximate* answers to these queries in sublinear space.

The Count-Min Sketch

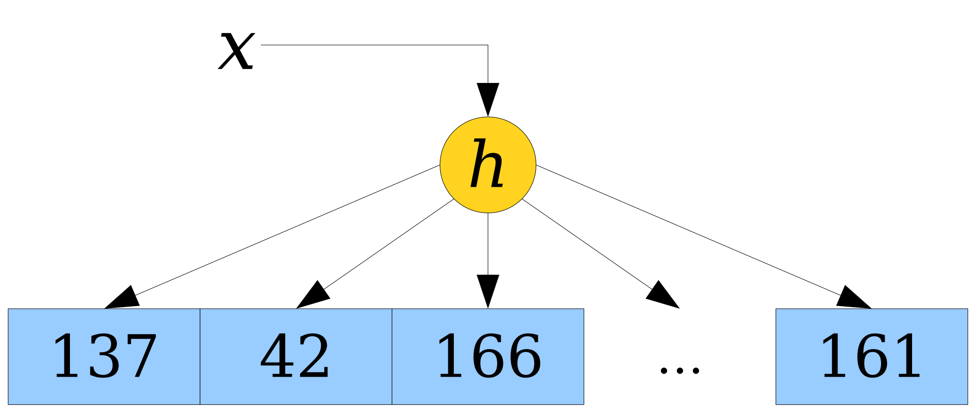
Revisiting the Exact Solution

- In the exact solution to the frequency estimation problem, we maintained a single counter for each distinct element. This is too space-inefficient.
- ***Idea:*** Store a fixed number of counters and assign a counter to each $x_i \in \mathscr{U}$. Multiple x_i's might be assigned to the same counter.
- To ***increment***(x), increment the counter for x.
- To ***estimate***(x), read the value of the counter for x.



Our Initial Structure

- We can model "assigning each x_i to a counter" by using hash functions.
- Pick a number of counters w (for "width;" more on that later). We'll choose the exact value of w later.
- Choose, from a family of 2-independent hash functions $\mathscr{H}$, a uniformly-random hash function $h : \mathscr{U} \to [w]$.
- Create an array **count** of w counters, each initially zero.
- To ***increment***(x), increment **count**$[h(x)]$.
- To ***estimate***(x), return **count**$[h(x)]$.



Analyzing our Structure

Some Notation

- Let x_1, x_2, x_3, … denote the list of distinct items whose frequencies are being stored.
- Let $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, … denote the frequencies of those items.
  - e.g. $\boldsymbol{a}_i$ is the true number of times x_i is seen.
- Let $\hat{\boldsymbol{a}}_1$, $\hat{\boldsymbol{a}}_2$, $\hat{\boldsymbol{a}}_3$, … denote the estimate our data structure gives for the frequency of each item.
  - e.g. $\hat{\boldsymbol{a}}_i$ is our estimate for how many times x_{i} has been seen.
- ***Important detail:*** the $\boldsymbol{a}_i$ values are not random variables (data are chosen adversarially), while the $\hat{\boldsymbol{a}}_i$ values are random variables (they depend on a randomly-sampled hash function).

Our Goal

- We want to show that, with high probability, our estimate isn't too far from the correct value.
- Mathematically, we want to look at the expression $\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i$ and show that there is a "high probability" that this is "small enough."
- We need to pin down what "high probability" and "small enough" mean. To do that, let's first work out, mathematically, what $\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i$ is.

Idea: Think of our element frequencies $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, … as a vector

$$\boldsymbol{a} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \ldots].$$

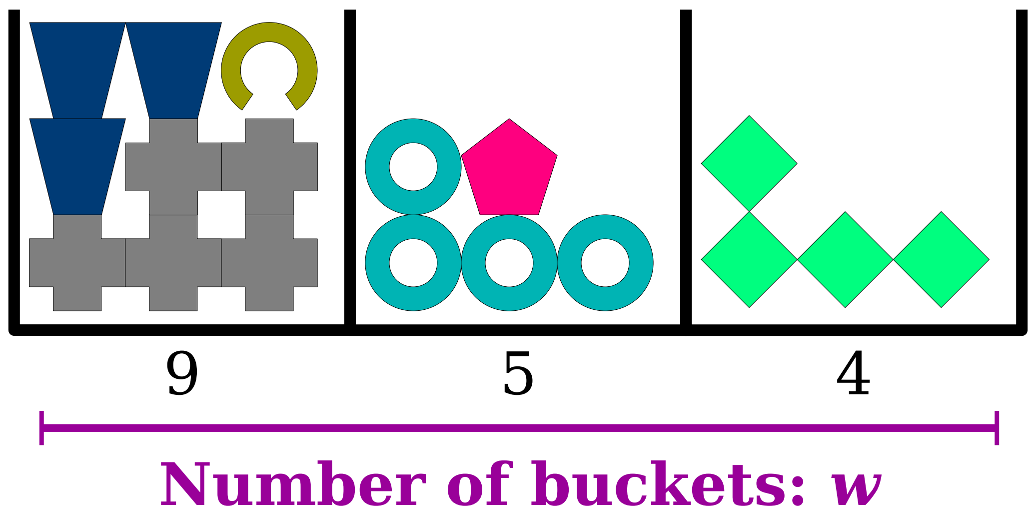
The total number of objects is the sum of the vector entries.

This is called the ***L_1 norm*** of $\boldsymbol{a}$, and is denoted $\|\boldsymbol{a}\|_1$:

$$\|\boldsymbol{a}\|_1 = \sum_i |\boldsymbol{a}_i|$$

There are $\|\boldsymbol{a}\|_1$ total elements distributed across w buckets. We're using a 2-independent hash family.

Reasonable guess: each bin has $\|\boldsymbol{a}\|_1 / w$ elements in it, so $E[\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i] \leq \|\boldsymbol{a}\|_1 / w$



Question: Intuitively, what should we expect our approximation error to be?

Analyzing this Structure

- Let's look at $\hat{\boldsymbol{a}}_i$ = **count**[$h(x_i)$] for some choice of x_i.
- For each element x_j:
  - If $h(x_i) = h(x_j)$, then x_j contributes $\boldsymbol{a}_j$ to **count**[$h(x_i)$].
  - If $h(x_i) \neq h(x_j)$, then x_j contributes 0 to **count**[$h(x_i)$].
- To pin this down precisely, let's define a set of random variables X_1, X_2, …, as follows:

$$X_j = \begin{cases} 1 & \text{if } h(x_i) = h(x_j) \\ 0 & \text{otherwise} \end{cases}$$

Each of these variables is called an ***indicator random variable***, since it "indicates" whether some event occurs.

Analyzing this Structure

- Let's look at $\hat{\boldsymbol{a}}_i$ = `count`[$h(x_i)$] for some choice of x_i.
- For each element x_j:
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$$X_j = \begin{cases} 1 & \text{if } h(x_i) = h(x_j) \\ 0 & \text{otherwise} \end{cases}$$

- The value of $\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i$ is then given by

$$\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i = \sum_{j \neq i} \boldsymbol{a}_j X_j$$

$$
\begin{aligned}
\mathrm{E}[\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i] &= \mathrm{E}\Big[\sum_{j \neq i} \boldsymbol{a}_j X_j\Big] \\
&= \sum_{j \neq i} \mathrm{E}[\boldsymbol{a}_j X_j]
\end{aligned}
$$

This follows from ***linearity of expectation***. We'll use this property extensively over the next few days.

$$
\begin{aligned}
\mathrm{E}[\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i] &= \mathrm{E}[\sum_{j \neq i} \boldsymbol{a}_j X_j] \\
&= \sum_{j \neq i} \mathrm{E}[\boldsymbol{a}_j X_j] \\
&= \sum_{j \neq i} \boldsymbol{a}_j \mathrm{E}[X_j]
\end{aligned}
$$

The values of $\boldsymbol{a}_j$ are not random. ***The randomness comes from our choice of hash function.***

$$
\begin{aligned}
\mathrm{E}[\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i] &= \mathrm{E}\Big[\sum_{j \neq i} \boldsymbol{a}_j X_j\Big] \\
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&= \sum_{j \neq i} \boldsymbol{a}_j \mathrm{E}[X_j]
\end{aligned}
$$

$$
\mathrm{E}[X_j] = 1 \cdot \Pr[h(x_i) = h(x_j)] + 0 \cdot \Pr[h(x_i) \neq h(x_j)]
$$

$$
\boxed{X_j = \begin{cases} 1 & \text{if } h(x_i) = h(x_j) \\ 0 & \text{otherwise} \end{cases}}
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&= \Pr[h(x_i) = h(x_j)]
\end{aligned}
$$

If X is an indicator variable for some event $\mathcal{E}$, then **E[X] = Pr[$\mathcal{E}$]**. This is really useful when using linearity of expectation!

$$
\begin{aligned}
\mathrm{E}[\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i] &= \mathrm{E}[\sum_{j \neq i} \boldsymbol{a}_j X_j] \\
&= \sum_{j \neq i} \mathrm{E}[\boldsymbol{a}_j X_j] \\
&= \sum_{j \neq i} \boldsymbol{a}_j \mathrm{E}[X_j]
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}[X_j] &= 1 \cdot \Pr[h(x_i) = h(x_j)] + 0 \cdot \Pr[h(x_i) \neq h(x_j)] \\
&= \Pr[h(x_i) = h(x_j)] \\
&= \frac{1}{w}
\end{aligned}
$$

Hey, we saw this earlier!

$$
\begin{aligned}
E[\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i] &= E\Big[\sum_{j \neq i} \boldsymbol{a}_j X_j\Big] \\
&= \sum_{j \neq i} E[\boldsymbol{a}_j X_j] \\
&= \sum_{j \neq i} \boldsymbol{a}_j E[X_j] \\
&= \sum_{j \neq i} \frac{\boldsymbol{a}_j}{w}
\end{aligned}
$$

$$
\begin{aligned}
E[X_j] &= 1 \cdot \Pr[h(x_i) = h(x_j)] + 0 \cdot \Pr[h(x_i) \neq h(x_j)] \\
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&= \sum_{j \neq i} \mathrm{E}[\boldsymbol{a}_j X_j] \\
&= \sum_{j \neq i} \boldsymbol{a}_j \mathrm{E}[X_j] \\
&= \sum_{j \neq i} \frac{\boldsymbol{a}_j}{w} \\
&\leq \frac{\|\boldsymbol{a}\|_1}{w}
\end{aligned}
$$

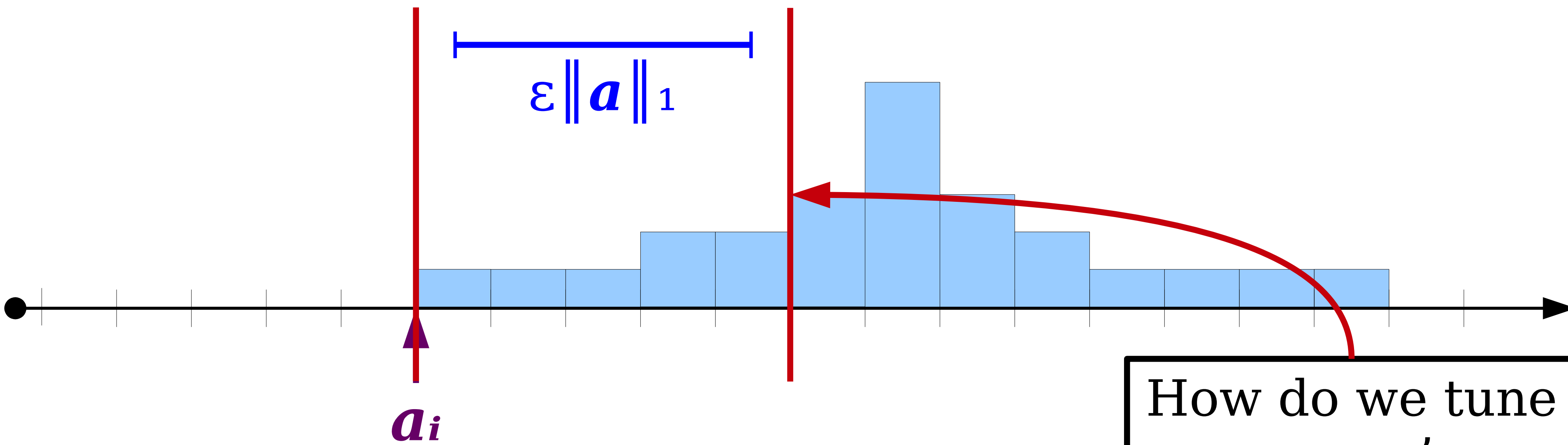
$$
\begin{aligned}
\mathrm{E}[X_j] &= 1 \cdot \Pr[h(x_i) = h(x_j)] + 0 \cdot \Pr[h(x_i) \neq h(x_j)] \\
&= \Pr[h(x_i) = h(x_j)] \\
&= \frac{1}{w}
\end{aligned}
$$

Goal: Make an estimator $\hat{\boldsymbol{a}}$ for some quantity $\boldsymbol{a}$ where

With probability at least 1 – δ, — ***Probably***

$|\hat{\boldsymbol{a}} - \boldsymbol{a}| \leq \varepsilon \cdot \mathit{size}(\mathit{input})$ — ***Approximately Correct***

for some measure of the size of the input.



How do we tune w so we're likely to fall in this range?

$$\mathrm{E}[\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i] \leq \frac{\|\boldsymbol{a}\|_1}{w}$$

$$\Pr\left[\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i \;>\; \varepsilon \|\boldsymbol{a}\|_1\right]$$

$$\leq \frac{\mathrm{E}\left[\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i\right]}{\varepsilon \|\boldsymbol{a}\|_1}$$

We don't know the exact distribution of this random variable.

However, we have a ***one-sided error***: our estimate can never be lower than the true value. This means that $\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i \geq 0$.

Markov's inequality says that if X is a nonnegative random variable, then

$$\Pr[X \geq c] \leq \frac{\mathrm{E}[X]}{c}.$$

$$\Pr\left[\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i \;>\; \varepsilon \|\boldsymbol{a}\|_1\right]$$

$$\leq \frac{\mathrm{E}\left[\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i\right]}{\varepsilon \|\boldsymbol{a}\|_1}$$

$$\leq \frac{\|\boldsymbol{a}\|_1}{w} \cdot \frac{1}{\varepsilon \|\boldsymbol{a}\|_1}$$

$$\boxed{\mathrm{E}\left[\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i\right] \;\leq\; \frac{\|\boldsymbol{a}\|_1}{w}}$$

$$
\begin{aligned}
& \Pr\left[\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i > \varepsilon \left\|\boldsymbol{a}\right\|_1\right] \\
\leq \quad & \frac{\mathrm{E}\left[\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i\right]}{\varepsilon \left\|\boldsymbol{a}\right\|_1} \\
\leq \quad & \frac{\left\|\boldsymbol{a}\right\|_1}{w} \cdot \frac{1}{\varepsilon \left\|\boldsymbol{a}\right\|_1} \\
= \quad & \frac{1}{\varepsilon\, w}
\end{aligned}
$$

Goal: Make an estimator $\hat{\boldsymbol{a}}$ for some quantity $\boldsymbol{a}$ where

With probability at least 1 – δ, ← ***Probably***

$|\hat{\boldsymbol{a}} - \boldsymbol{a}| \leq \varepsilon \cdot size(input)$ ← ***Approximately Correct***

for some measure of input size.

$$\Pr[\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i > \varepsilon \|\boldsymbol{a}\|_1] \leq \frac{1}{\varepsilon w}$$

Initial Idea:

Pick $w = \varepsilon^{-1} \cdot \delta^{-1}$. Then

$$\Pr[\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i > \varepsilon \|\boldsymbol{a}\|_1] \leq \delta$$

Suppose we're counting 1,000 distinct items.

If we want our estimate to be within $\varepsilon\|\boldsymbol{a}\|_1$ of the true value with 99.9% probability, how much memory do we need?

Answer: **1,000 · $\boldsymbol{\varepsilon^{-1}}$**.

Can we do better?

Goal: Make an estimator $\hat{\boldsymbol{a}}$ for some quantity $\boldsymbol{a}$ where

With probability at least 1 – δ,

$|\hat{\boldsymbol{a}} - \boldsymbol{a}| \leq \varepsilon \cdot \mathit{size}(\mathit{input})$

for some measure of input size.

Probably

Approximately Correct

$$\Pr[\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i > \varepsilon \|\boldsymbol{a}\|_1] \leq \frac{1}{\varepsilon w}$$

We could choose $w = k \cdot \varepsilon^{-1}$ for any constant k to get a failure probability of at most k^{-1}. The choice of e is (mostly) arbitrary.

Revised Idea: Pick $w = e \cdot \varepsilon^{-1}$. Then

$$\Pr[\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i > \varepsilon \|\boldsymbol{a}\|_1] < e^{-1}$$

Goal: Make an estimator $\hat{\boldsymbol{a}}$ for some quantity $\boldsymbol{a}$ where

With probability at least $1 - \delta$, — ***Probably***

$|\hat{\boldsymbol{a}} - \boldsymbol{a}| \le \varepsilon \cdot size(input)$ — ***Approximately Correct***

for some measure of input size.

$$\Pr[\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i > \varepsilon \|\boldsymbol{a}\|_1] \le \frac{1}{\varepsilon w}$$

Revised Idea: Pick $w = e \cdot \varepsilon^{-1}$. Then

$$\Pr[\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i > \varepsilon \|\boldsymbol{a}\|_1] < e^{-1}$$

This simple data structure, by itself, is likely to be wrong.

What happens if we run a bunch of copies of this approach in parallel?

Running in Parallel

- Let's run ***d*** copies of our data structure in parallel with one another.
- Each row has its hash function sampled uniformly at random from our hash family.
- Each time we ***increment*** an item, we perform the corresponding ***increment*** operation on each row.

$w = \lceil e \cdot \varepsilon^{-1} \rceil$

h_1	31	41	59	26	53	…	58
h_2	27	18	28	18	28	…	45
h_3	16	18	3	39	88	…	75
…	…						
h_d	69	31	47	18	5	…	59

$d = ??$

Running in Parallel

- Imagine we call ***estimate***(x) on each of our estimators and get back these estimates.
- We need to give back a single number.
- ***Question:*** How should we aggregate these numbers into a single estimate?

Formulate a hypothesis!

Estimator 1:	*Estimator 2:*	*Estimator 3:*	*Estimator 4:*	*Estimator 5:*
137	271	166	103	261

Running in Parallel

- Imagine we call ***estimate***(x) on each of our estimators and get back these estimates.
- We need to give back a single number.
- ***Question:*** How should we aggregate these numbers into a single estimate?

Discuss with your neighbors!

Estimator 1: 137	*Estimator 2:* 271	*Estimator 3:* 166	*Estimator 4:* 103	*Estimator 5:* 261

Running in Parallel

- Imagine we call ***estimate***(x) on each of our estimators and get back these estimates.
- We need to give back a single number.
- ***Question:*** How should we aggr into a single estimate?

Intuition: The smallest estimate returned has the least "noise," and that's the best guess for the frequency.

Estimator 1: 137	*Estimator 2:* 271	*Estimator 3:* 166	*Estimator 4:* 103	*Estimator 5:* 261

$$\Pr\left[\min\left\{\hat{\boldsymbol{a}}_{ij}\right\} - \boldsymbol{a}_i > \varepsilon\|\boldsymbol{a}\|_1\right]$$

$$= \Pr\left[\bigwedge_{j=1}^{d}\left(\hat{\boldsymbol{a}}_{ij} - \boldsymbol{a}_i > \varepsilon\|\boldsymbol{a}\|_1\right)\right]$$

The only way the minimum estimate is inaccurate is if *every* estimate is inaccurate.

Let $\hat{\boldsymbol{a}}_{ij}$ be the estimate from the jth copy of the data structure.

Our final estimate is $\min\{\hat{\boldsymbol{a}}_{ij}\}$

$$\Pr\left[\min\{\hat{\boldsymbol{a}}_{ij}\} - \boldsymbol{a}_i > \varepsilon\|\boldsymbol{a}\|_1\right]$$

$$= \Pr\left[\bigwedge_{j=1}^{d}\left(\hat{\boldsymbol{a}}_{ij} - \boldsymbol{a}_i > \varepsilon\|\boldsymbol{a}\|_1\right)\right]$$

$$= \prod_{j=1}^{d}\Pr\left[\hat{\boldsymbol{a}}_{ij} - \boldsymbol{a}_i > \varepsilon\|\boldsymbol{a}\|_1\right]$$

Each copy of the data structure is independent of the others.

Let $\hat{\boldsymbol{a}}_{ij}$ be the estimate from the jth copy of the data structure.

Our final estimate is $\min\{\hat{\boldsymbol{a}}_{ij}\}$

$$\Pr\left[\min\{\hat{\boldsymbol{a}}_{ij}\} - \boldsymbol{a}_i > \varepsilon\|\boldsymbol{a}\|_1\right]$$

$$= \Pr\left[\bigwedge_{j=1}^{d}\left(\hat{\boldsymbol{a}}_{ij} - \boldsymbol{a}_i > \varepsilon\|\boldsymbol{a}\|_1\right)\right]$$

$$= \prod_{j=1}^{d} \Pr\left[\hat{\boldsymbol{a}}_{ij} - \boldsymbol{a}_i > \varepsilon\|\boldsymbol{a}\|_1\right]$$

$$\leq \prod_{j=1}^{d} e^{-1}$$

$$\Pr[\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i \geq \varepsilon\|\boldsymbol{a}\|_1] \leq e^{-1}$$

Let $\hat{\boldsymbol{a}}_{ij}$ be the estimate from the jth copy of the data structure.

Our final estimate is min $\{\hat{\boldsymbol{a}}_{ij}\}$

$$
\begin{aligned}
& \Pr\left[\min\left\{\,\hat{\boldsymbol{a}}_{ij}\,\right\} - \boldsymbol{a}_i \;>\; \varepsilon\left\|\boldsymbol{a}\right\|_1\right] \\
= \;& \Pr\left[\bigwedge_{j=1}^{d}\left(\hat{\boldsymbol{a}}_{ij} - \boldsymbol{a}_i \;>\; \varepsilon\left\|\boldsymbol{a}\right\|_1\right)\right] \\
= \;& \prod_{j=1}^{d}\Pr\left[\hat{\boldsymbol{a}}_{ij} - \boldsymbol{a}_i \;>\; \varepsilon\left\|\boldsymbol{a}\right\|_1\right] \\
\leq \;& \prod_{j=1}^{d} e^{-1} \\
= \;& e^{-d}
\end{aligned}
$$

Let $\hat{\boldsymbol{a}}_{ij}$ be the estimate from the jth copy of the data structure.

Our final estimate is $\min\left\{\hat{\boldsymbol{a}}_{ij}\right\}$

Goal: Make an estimator $\hat{\boldsymbol{a}}$ for some quantity $\boldsymbol{a}$ where

With probability at least 1 – δ,

$|\hat{\boldsymbol{a}} - \boldsymbol{a}| \leq \varepsilon \cdot size(input)$

for some measure of input size.

Probably

Approximately Correct

$$\Pr\left[\min\{\hat{\boldsymbol{a}}_{ij}\} - \boldsymbol{a}_i > \varepsilon \|\boldsymbol{a}\|_1\right] \leq e^{-d}$$

Idea: Choose d = -ln δ.
(Equivalently: d = ln δ^{-1}.) Then

$$\Pr\left[\min\{\hat{\boldsymbol{a}}_{ij}\} - \boldsymbol{a}_i > \varepsilon \|\boldsymbol{a}\|_1\right] \leq \delta$$

The Count-Min Sketch

$w = \lceil e \cdot \varepsilon^{-1} \rceil$

$d = \lceil \ln \delta^{-1} \rceil$

h_1	31	41	59	26	53	...	58
h_2	27	18	28	18	28	...	45
h_3	16	18	3	39	88	...	75
...				...			
h_d	69	31	47	18	5	...	59

Sampled uniformly and independently from a 2-independent family of hash functions

The Count-Min Sketch

h_1	32	41	59	26	53	…	58
h_2	27	18	28	19	28	…	45
h_3	16	19	3	39	88	…	75
…	…						
h_d	69	31	47	18	5	…	60

```
increment(x):
   for i = 1 … d:
      count[i][hᵢ(x)]++
```

```
estimate(x):
   result = ∞
   for i = 1 … d:
      result = min(result, count[i][hᵢ(x)])
   return result
```

The Count-Min Sketch

- Update and query times are $\Theta(d)$, which is $\Theta(\log \delta^{-1})$.
- Space usage: $\Theta(\varepsilon^{-1} \cdot \log \delta^{-1})$ counters.
  - Each individual estimator has $\Theta(\varepsilon^{-1})$ counters, and we run $\Theta(\log \delta^{-1})$ copies in parallel.
- This is a *major* improvement over our earlier approach that used $\Theta(\varepsilon^{-1} \cdot \delta^{-1})$ counters.
- This can be *significantly* better than just storing a raw frequency count!
- Provides an estimate to within $\varepsilon \| \boldsymbol{a} \|_1$ with probability at least $1 - \delta$.

Major Ideas From Today

- ***2-independent hash families*** are useful when we want to keep collisions low.
- A “good” approximation of some quantity should have tunable ***confidence*** and ***accuracy*** parameters.
- ***Sums of indicator variables*** are useful for deriving expected values of estimators.
- ***Concentration inequalities*** like ***Markov’s inequality*** are useful for showing estimators don’t stay too much from their expected values.
- Good estimators can be built from multiple parallel copies of weaker estimators.

Next Time

- ***Count Sketches***
  - An alternative frequency estimator with different time/space bounds.
- ***Cardinality Estimation***
  - Estimating how many different items you’ve seen in a data stream.