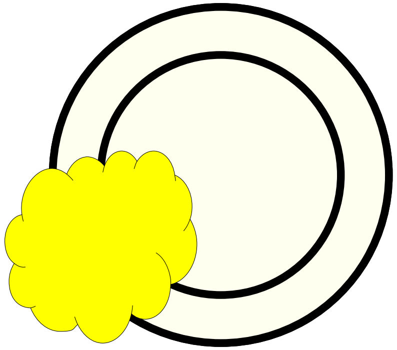
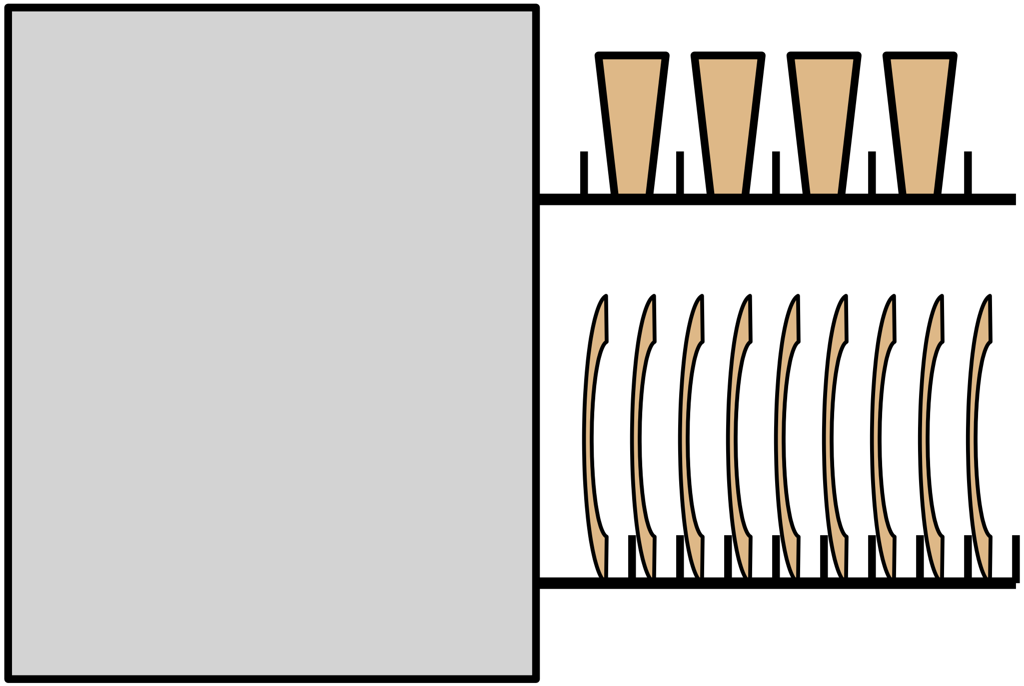
# Amortized Analysis

# A Motivating Analogy

# Doing the Dishes

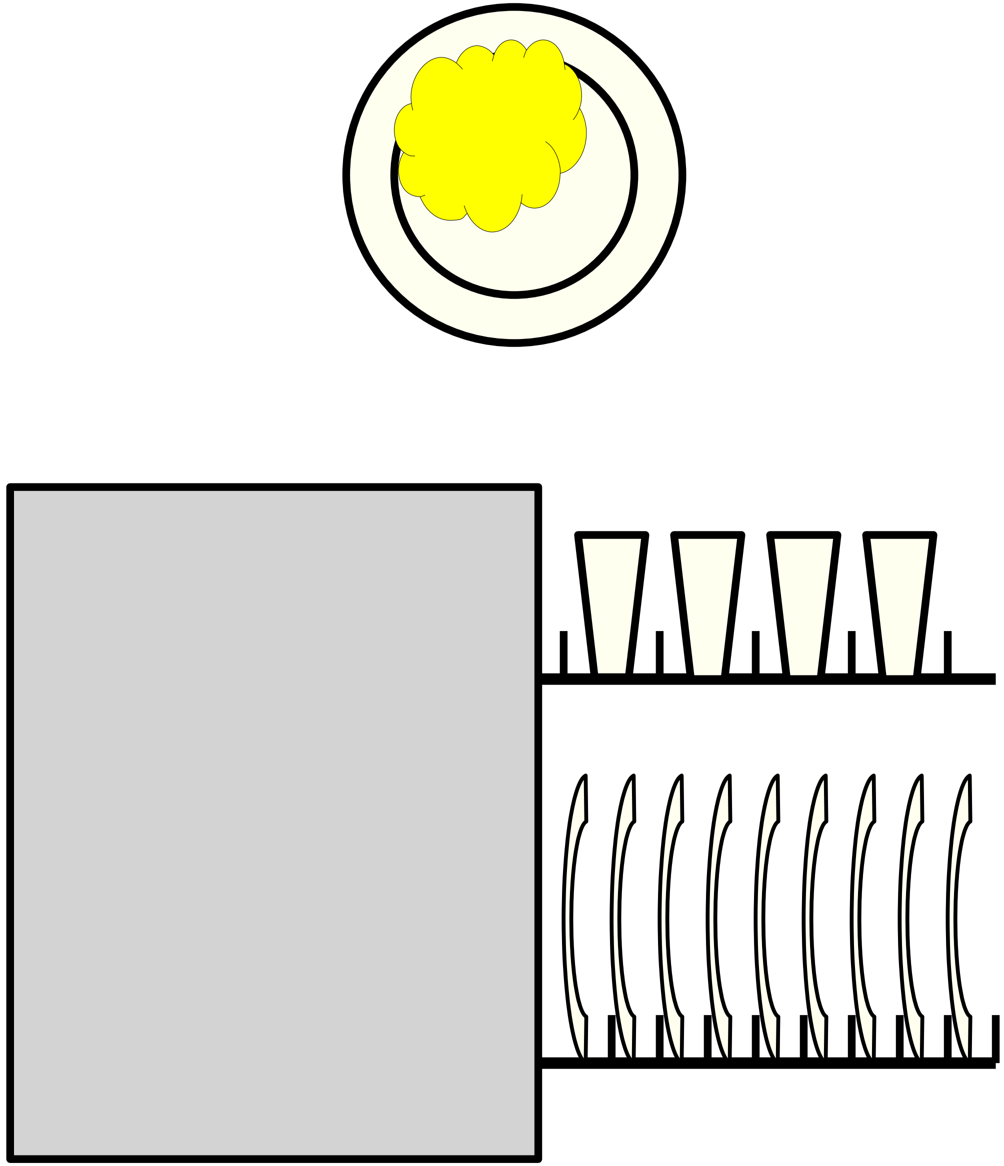
- What do I do with a dirty dish or kitchen utensil?
- ***Option 1:*** Wash it by hand.
- ***Option 2:*** Put it in the dishwasher rack, then run the dishwasher if it's full.





# Doing the Dishes

- Washing every individual dish and utensil by hand is *way* slower than using the dishwasher, but I always have access to my plates and kitchen utensils.
- Running the dishwasher is faster in aggregate, but means I may have to wait a bit for dishes to be ready.



***Key Idea:*** Design data structures that trade *per-operation efficiency* for *overall efficiency*.

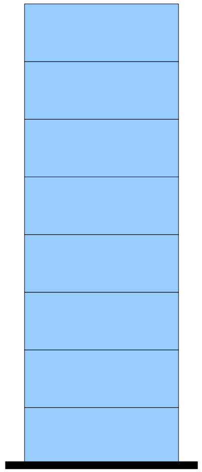
# Where We're Going

- ***Amortized Analysis (Today)***
  - A little accounting trickery never hurt anyone, right?
- ***Scapegoat Trees (Tuesday)***
  - Building a balanced BST, lazily.
- ***Tournament Heaps (Next Thursday)***
  - A fast, flexible priority queue that's a great building block for more complicated structures.
- ***Abdication Heaps (Next Tuesday)***
  - A priority queue optimized for graph algorithms that, at least in theory, leads to optimal implementations.

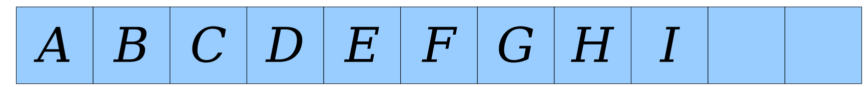
# Outline for Today

- ***Amortized Analysis***
  - Trading worst-case efficiency for aggregate efficiency.
- ***Examples of Amortization***
  - Three motivating data structures and algorithms.
- ***Potential Functions***
  - Quantifying messiness and formalizing costs.
- ***Performing Amortized Analyses***
  - How to show our examples are indeed fast.

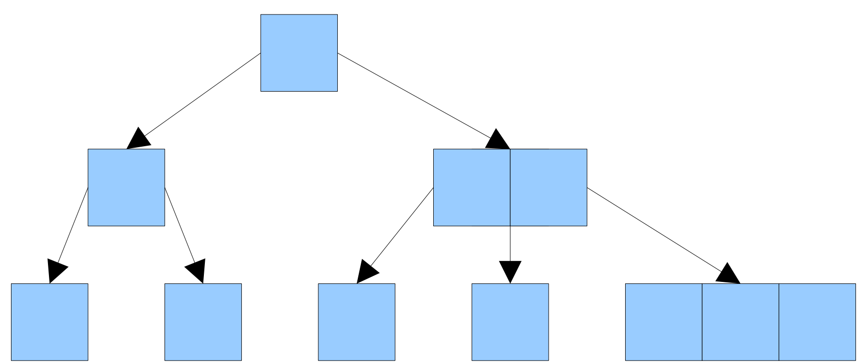
# Three Examples



## Two-Stack Queues

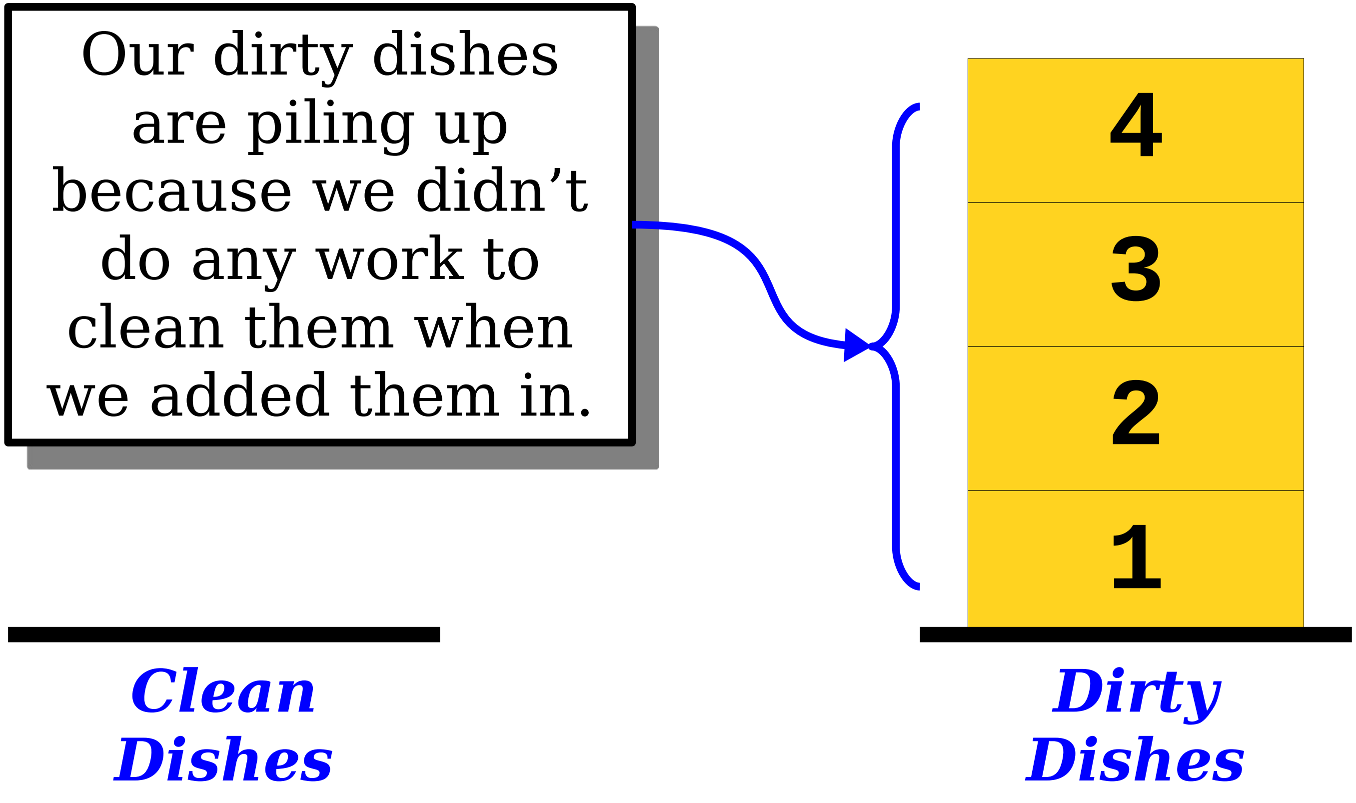



## Dynamic Arrays



## Building B-Trees

# The Two-Stack Queue



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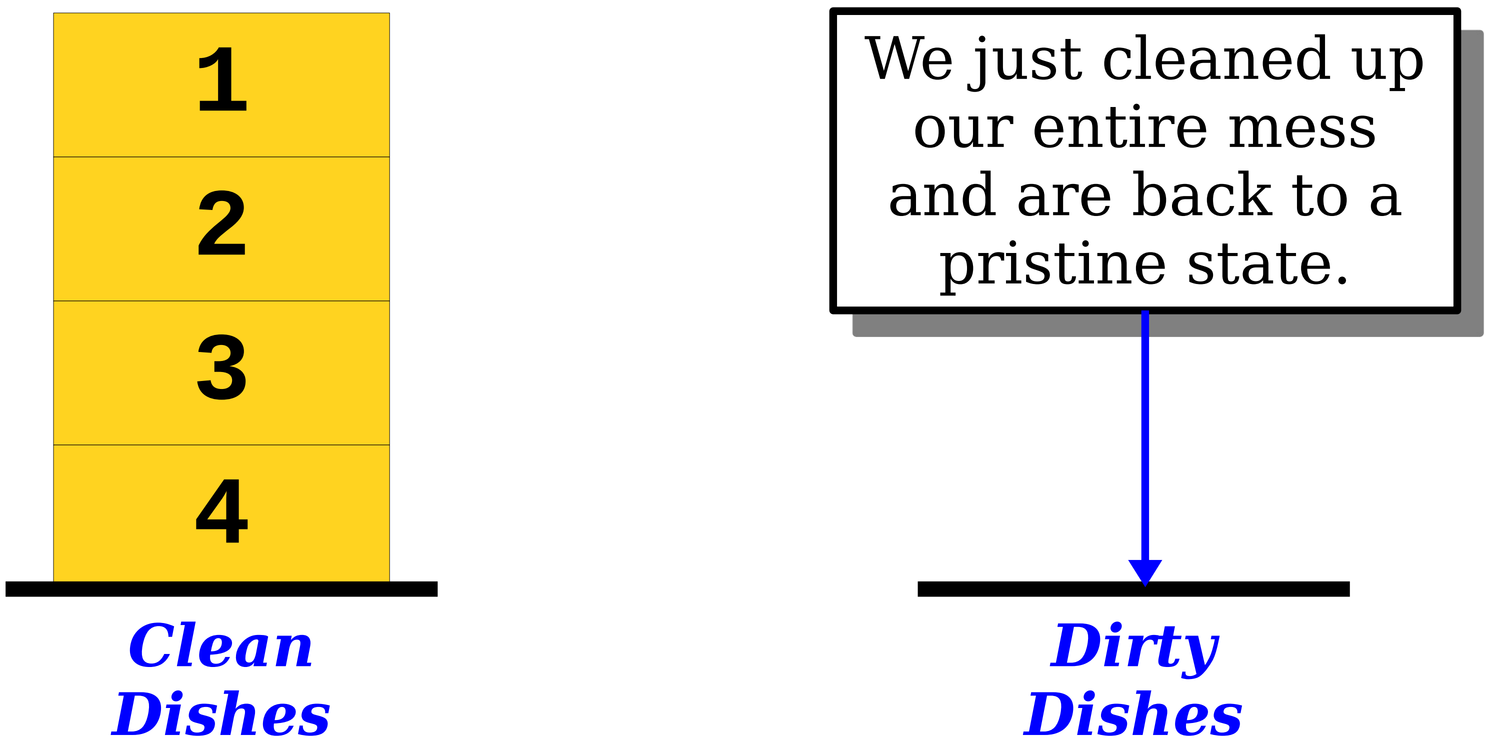
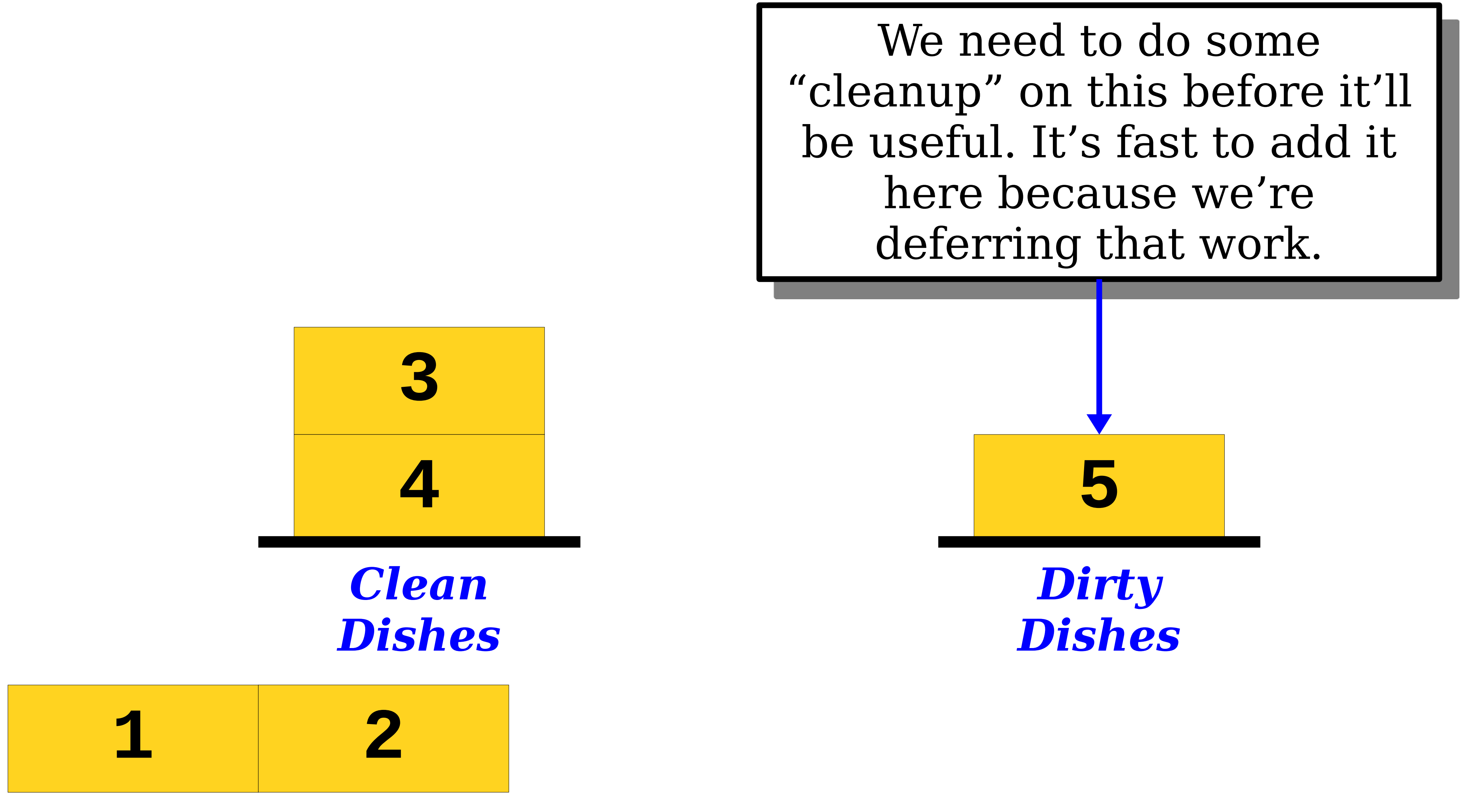


# The Two-Stack Queue



# The Two-Stack Queue

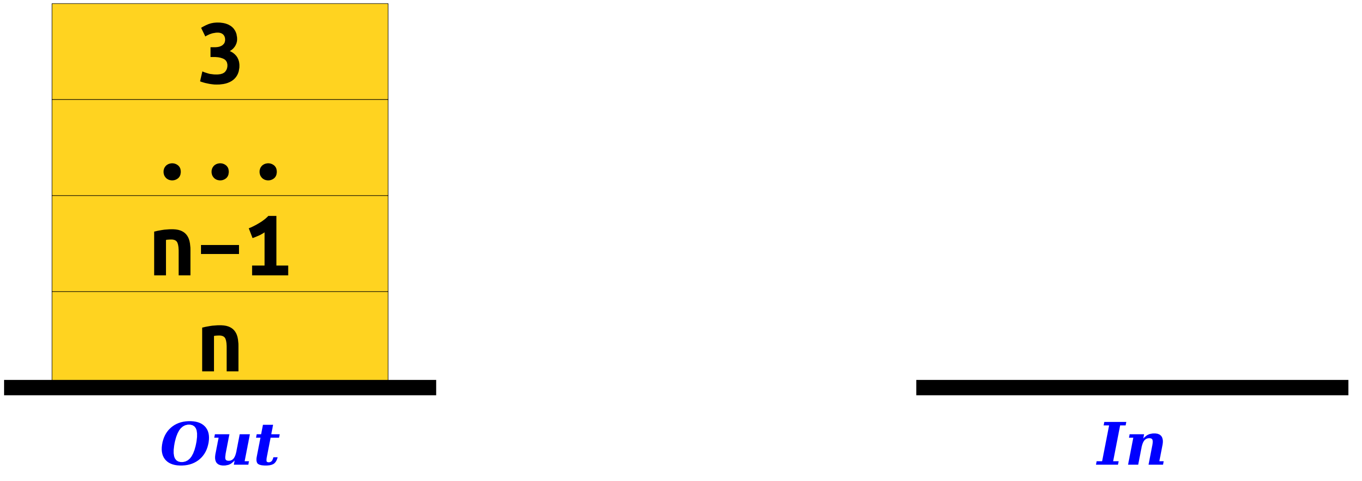
- Maintain an ***In*** stack and an ***Out*** stack.
- To enqueue an element, push it onto the ***In*** stack.
- To dequeue an element:
  - If the ***Out*** stack is nonempty, pop it.
  - If the ***Out*** stack is empty, pop elements from the ***In*** stack, pushing them into the ***Out*** stack. Then dequeue as usual.

# The Two-Stack Queue

- Each enqueue takes time O(1).
  - Just push an item onto the ***In*** stack.
- Dequeues can vary in their runtime.
  - Could be O(1) if the ***Out*** stack isn't empty.
  - Could be Θ($n$) if the ***Out*** stack is empty.

# The Two-Stack Queue

- ***Intuition:*** We only do expensive dequeues after a long run of cheap enqueues.
- Think "dishwasher:" we very slowly introduce a lot of dirty dishes to get cleaned up all at once.
- Provided we clean up all the dirty dishes at once, and provided that dirty dishes accumulate slowly, this is a fast strategy!



# The Two-Stack Queue

- ***Key Fact:*** Any series of $n$ operations on an (initially empty) two-stack queue will take time $O(n)$.
- ***Why?***

Formulate a hypothesis!

# The Two-Stack Queue

- ***Key Fact:*** Any series of *n* operations on an (initially empty) two-stack queue will take time O(*n*).
- ***Why?***

Discuss with your neighbors!

# The Two-Stack Queue

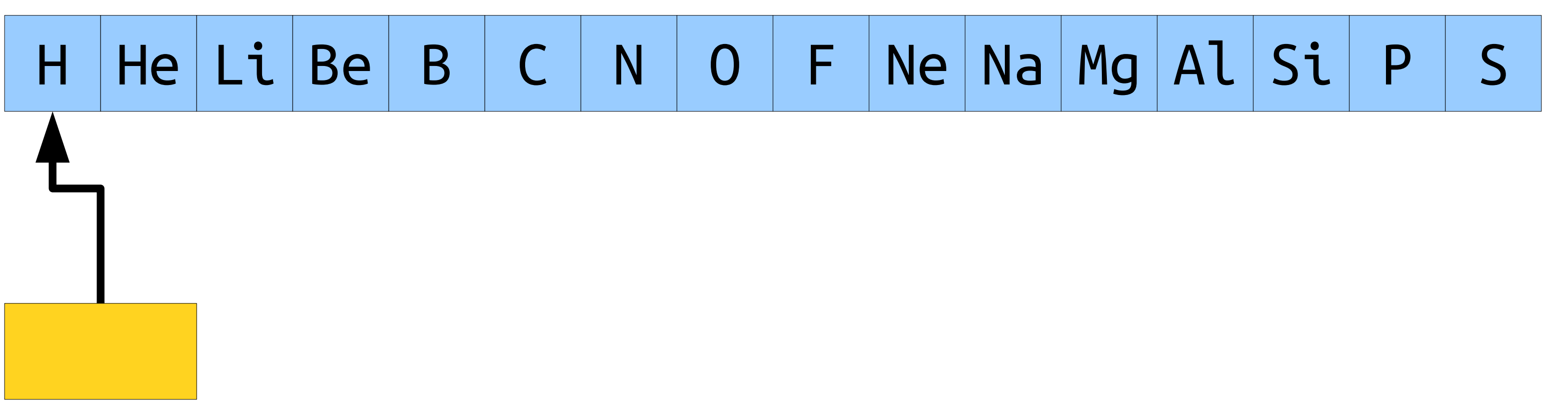
- ***Key Fact:*** Any series of *n* operations on an (initially empty) two-stack queue will take time O(*n*).
- ***Why?***
- Each item is pushed into at most two stacks and popped from at most two stacks.
- Adding up the work done per element across all *n* operations, we can do at most O(*n*) work.

# The Two-Stack Queue

- It's correct but misleading to say the cost of a dequeue is O($n$).
  - This is comparatively rare.
- It's wrong, but useful, to pretend the cost of a dequeue is O(1).
  - Some operations take more time than this.
  - However, if we pretend each operation takes time O(1), then the sum of all the costs never underestimates the total.
- ***Question:*** What's an honest, accurate way to describe the runtime of the two-stack queue?

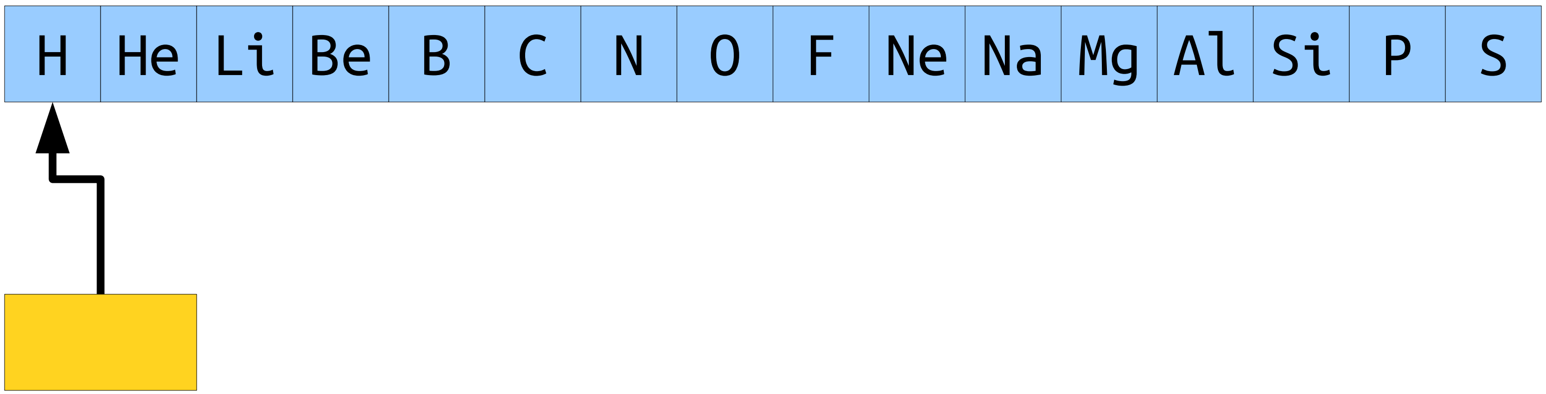
# Dynamic Arrays

- A ***dynamic array*** is the most common way to implement a list of values.
- Maintain an array slightly bigger than the one you need. When you run out of space, double the array size and copy the elements over.



# Dynamic Arrays

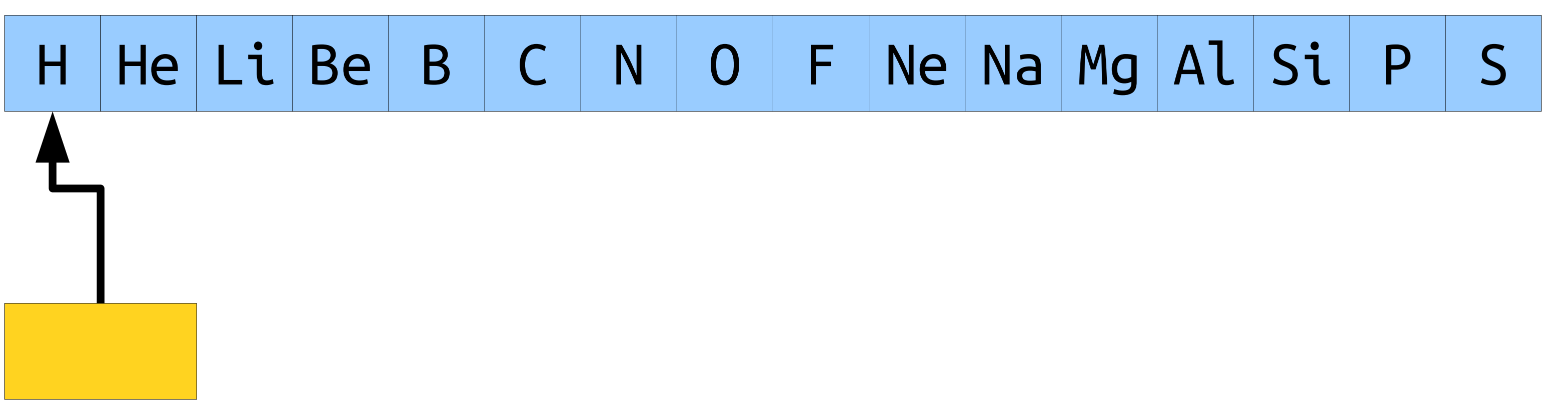
- Most appends to a dynamic array take time O(1).
- Infrequently, we do Θ($n$) work to copy all $n$ elements from the old array to a new one.
- Think "dishwasher:"
  - We slowly accumulate "messes" (filled slots).
  - We periodically do a large "cleanup" (copying the array).
- ***Claim:*** The cost of doing $n$ appends to an initially empty dynamic array is always O($n$).



# Dynamic Arrays

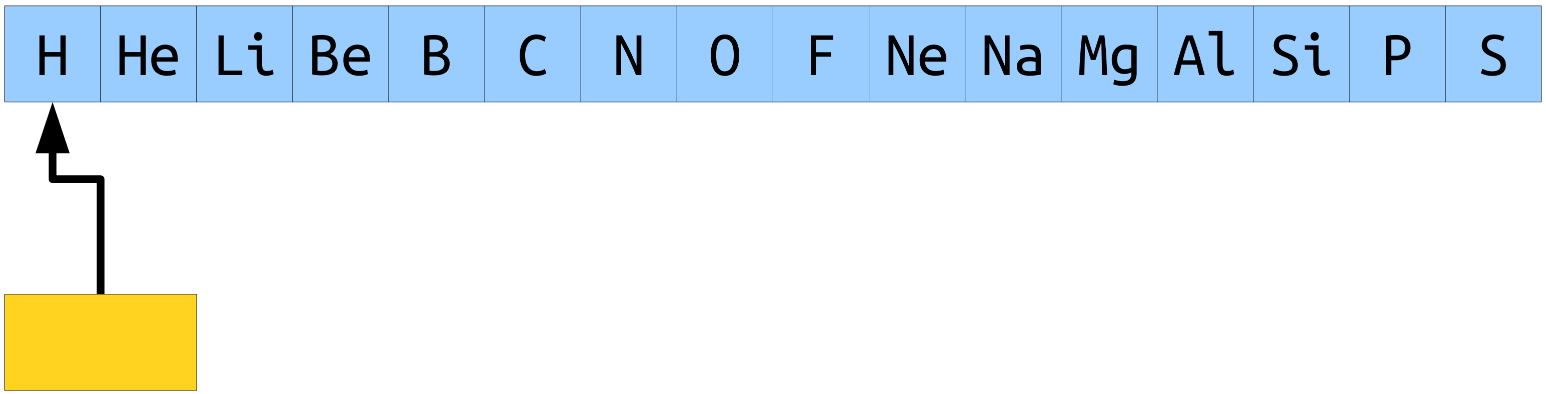
- ***Claim:*** Appending $n$ elements always takes time $O(n)$.
- The array doubles at sizes $2^0, 2^1, 2^2, \ldots$, etc.
- The very last doubling is at the largest power of two less than $n$. This is at most $2^{\lfloor \log_2 n \rfloor}$. (Do you see why?)
- Total work done across all doubling is at most

$$
\begin{aligned}
2^0 + 2^1 + \ldots + 2^{\lfloor \log_2 n \rfloor} &= 2^{\lfloor \log_2 n \rfloor + 1} - 1 \\
&\leq 2^{\log_2 n + 1} \\
&= 2n.
\end{aligned}
$$



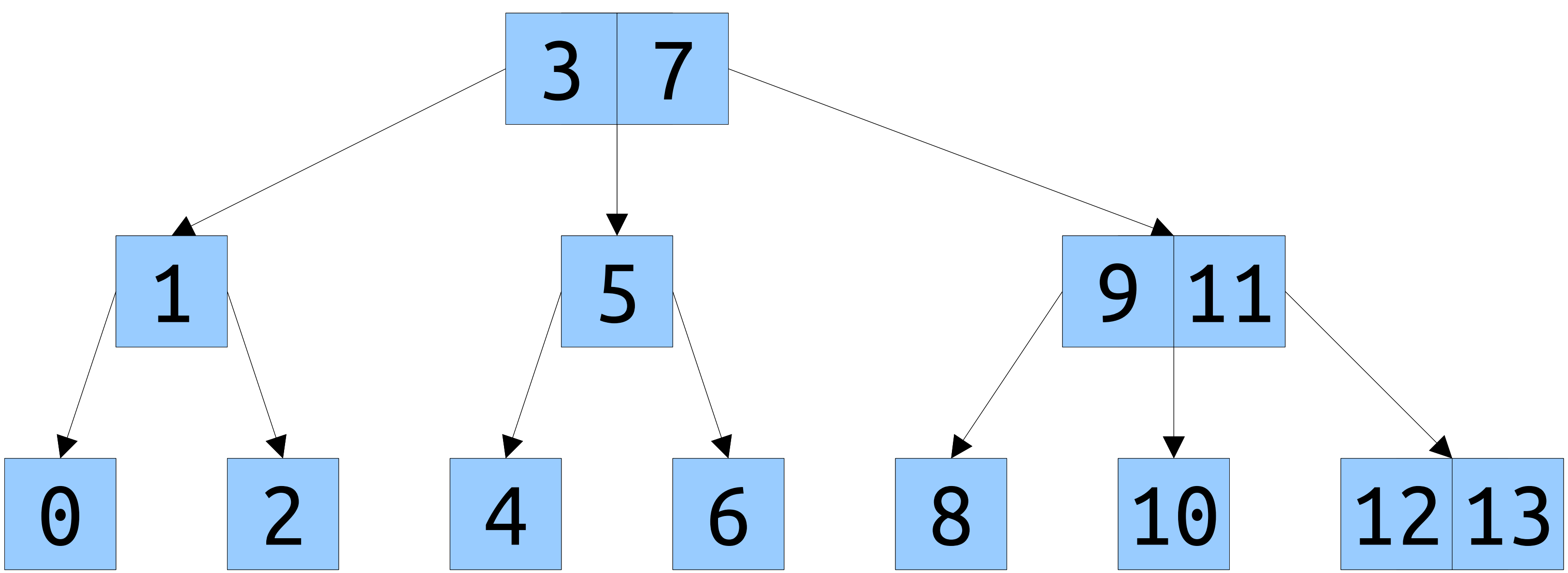
# Dynamic Arrays

- It's correct but misleading to say the cost of an append is O($n$).
  - This is comparatively rare.
- It's wrong, but useful, to pretend that the cost of an append is O(1).
  - Some operations take more time than this.
  - However, pretending each operation takes O(1) time never underestimates the true runtime.
- ***Question:*** What's an honest, accurate way to describe the runtime of the dynamic array?



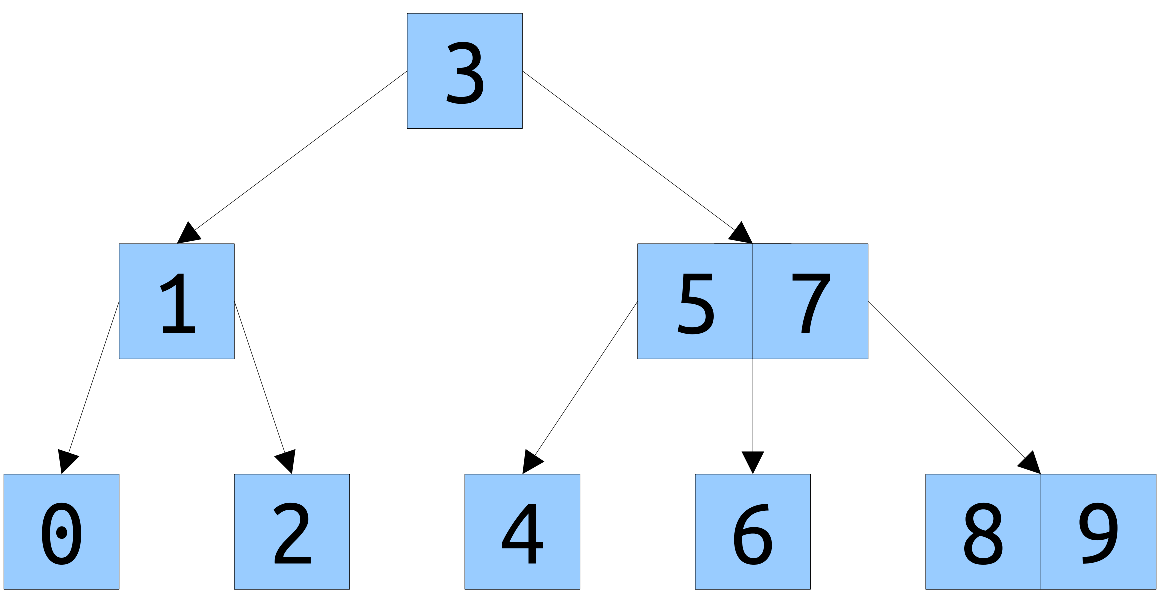
# Building B-Trees

- You're given a sorted list of $n$ values and a value of $b$.
- What's the most efficient way to construct a B-tree of order $b$ holding these $n$ values?
- ***One Option:*** Think really hard, calculate the shape of a B-tree of order $b$ with $n$ elements in it, then place the items into that B-tree in sorted order.
- Is there an easier option?



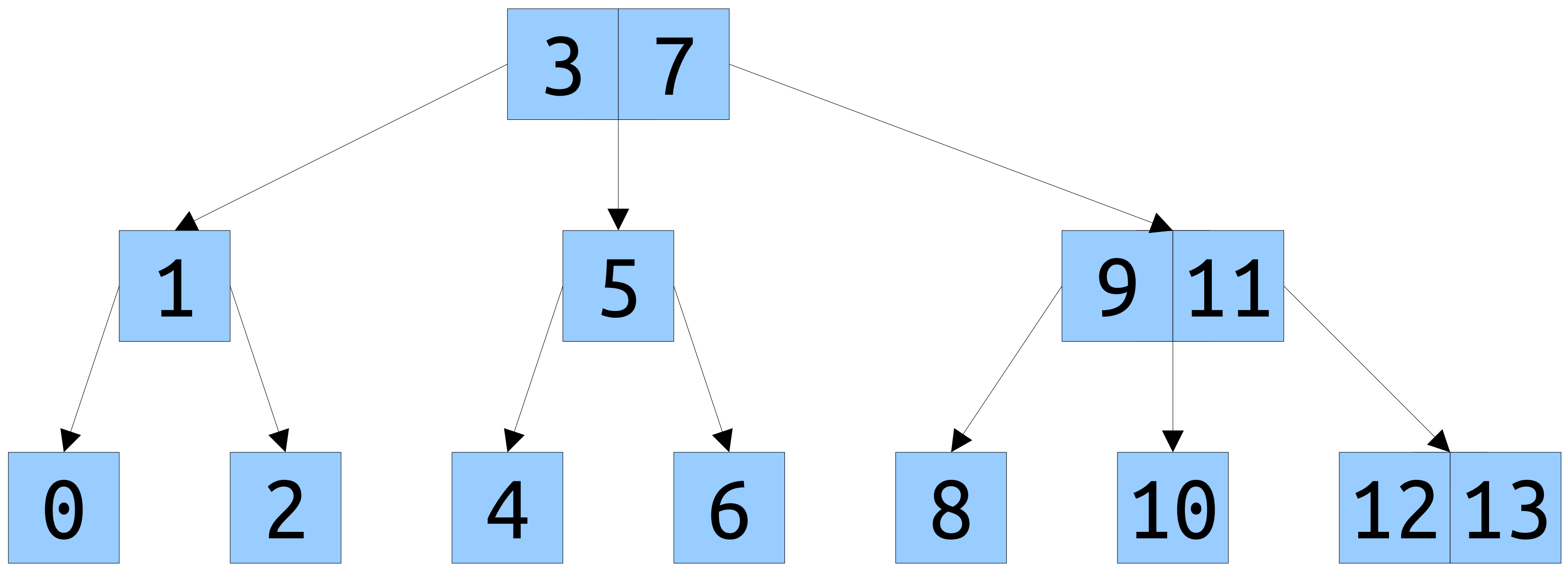
# Building B-Trees

- ***Idea 1:*** Insert the items into an empty B-tree in sorted order.
- Cost: $\Omega(n \log_b n)$, due to the top-down search.
- ***Can we do better?***



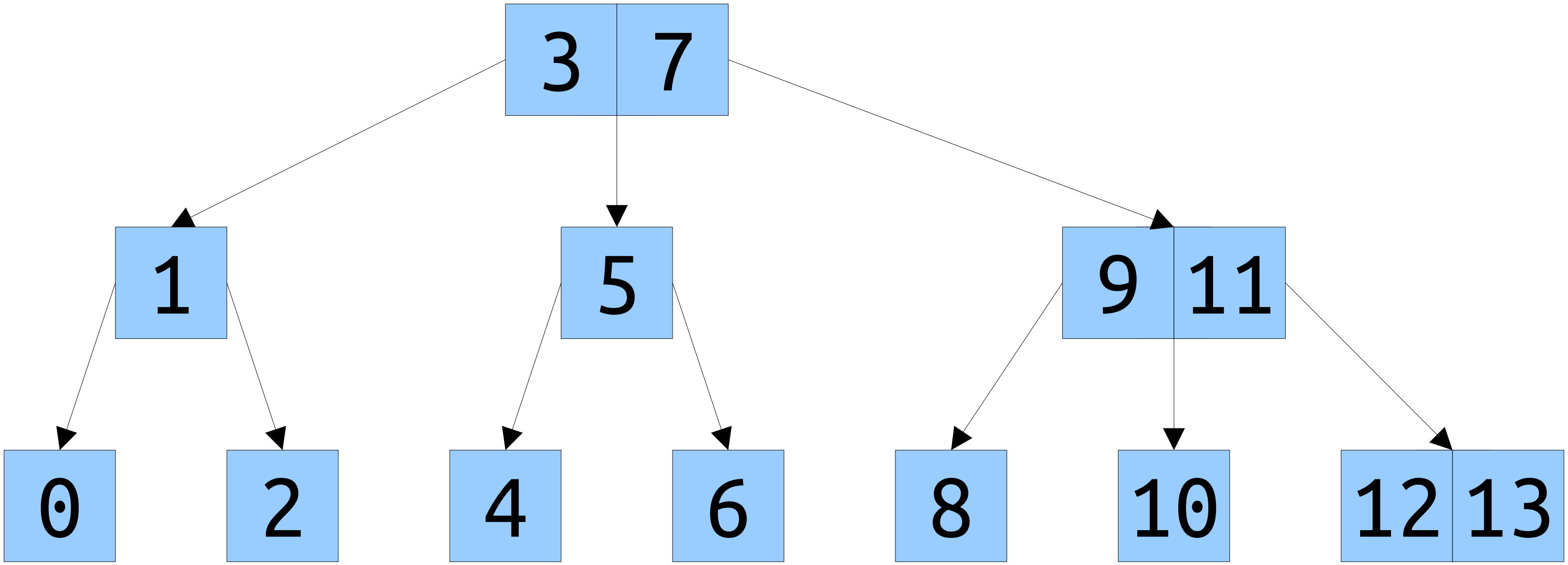
# Building B-Trees

- ***Idea 2:*** Since all insertions will happen at the rightmost leaf, store a pointer to that leaf. Add new values by appending to this leaf, then doing any necessary splits.
- ***Question:*** How fast is this?



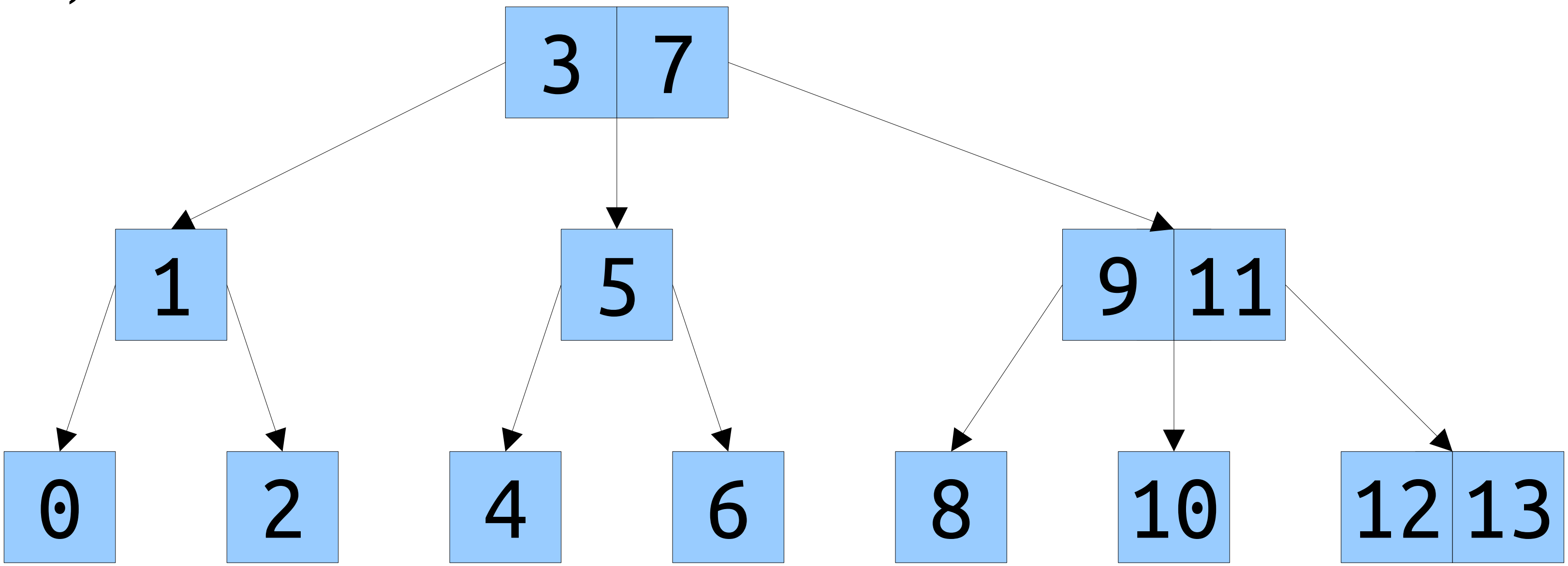
# Building B-Trees

- The cost of an insert varies based on the shape of the tree.
  - If no splits are required, the cost is O(1).
  - If one split is required, the cost is O(*b*).
  - If we have to split all the way up, the cost is $O(b \log_b n)$.
- Using our worst-case cost across *n* inserts gives a runtime bound of $O(nb \log_b \text{n})$
- ***Claim:*** The cost of *n* inserts is always O(*n*).



# Building B-Trees

- Of all the $n$ insertions into the tree, a roughly $^1/_b$ fraction will split a node in the bottom layer of the tree (a leaf).
- Of those, roughly a $^1/_b$ fraction will split a node in the layer above that.
- Of those, roughly a $^1/_b$ fraction will split a node in the layer above that.
- (etc.)



# Building B-Trees

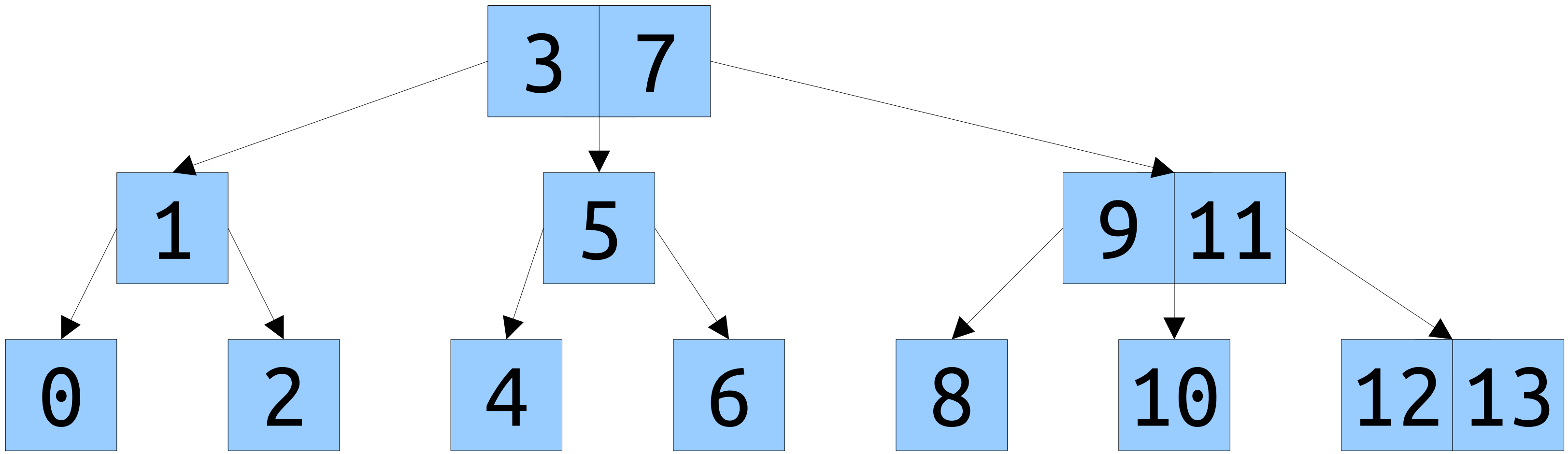
- Total number of splits:

$$
\begin{aligned}
& \frac{n}{b} \cdot \left(1 + \frac{1}{b} \cdot \left(1 + \frac{1}{b} \cdot \left(1 + \frac{1}{b} \cdot (\ldots)\right)\right)\right) \\
= \quad & \frac{n}{b} \cdot \left(1 + \frac{1}{b} + \frac{1}{b^2} + \frac{1}{b^3} + \frac{1}{b^4} + \ldots\right) \\
= \quad & \frac{n}{b} \cdot \Theta(1) \\
= \quad & \Theta\left(\frac{n}{b}\right)
\end{aligned}
$$

- Total cost of those splits: $\boldsymbol{\Theta(n)}$.

# Building B-Trees

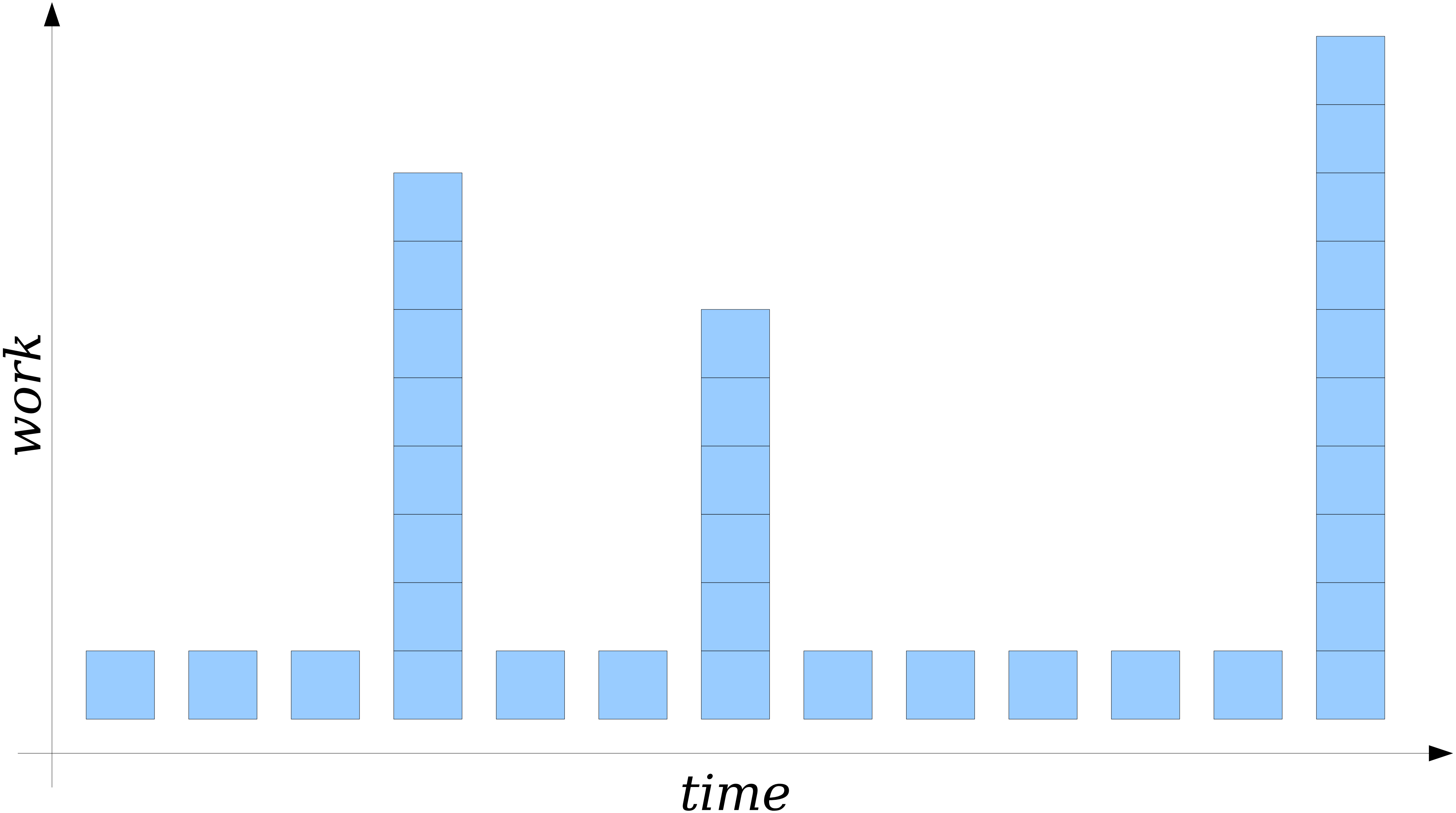
- It is correct but misleading to say the cost of an insert is O($b \log_b n$).
  - This is comparatively rare.
- It is wrong, but useful, to pretend that the cost of an insert is O(1).
  - Some operations take more time than this.
  - However, pretending each insert takes time O(1) never underestimates the total amount of work done across all operations.
- ***Question:*** What's an honest, accurate way to describe the cost of inserting one more value?



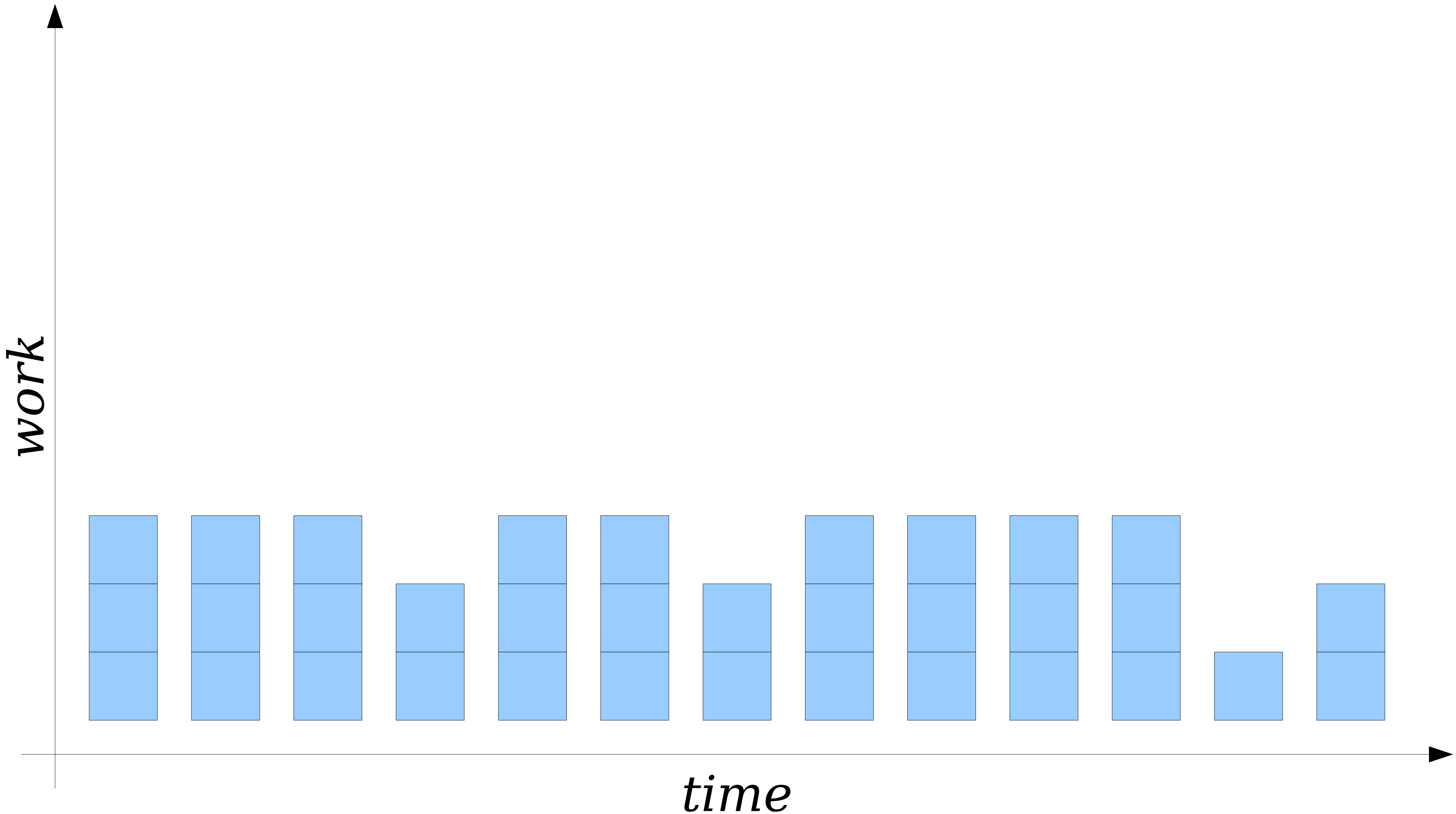
# Amortized Analysis

# The Setup

- We now have three examples of data structures where
  - ***individual operations may be slow***, but
  - ***any series of operations is fast***.
- Giving weak upper bounds on the cost of each operation is not useful for making predictions.
- How can we clearly communicate when a situation like this one exists?




***Key Idea:*** Backcharge expensive operations to cheaper ones.

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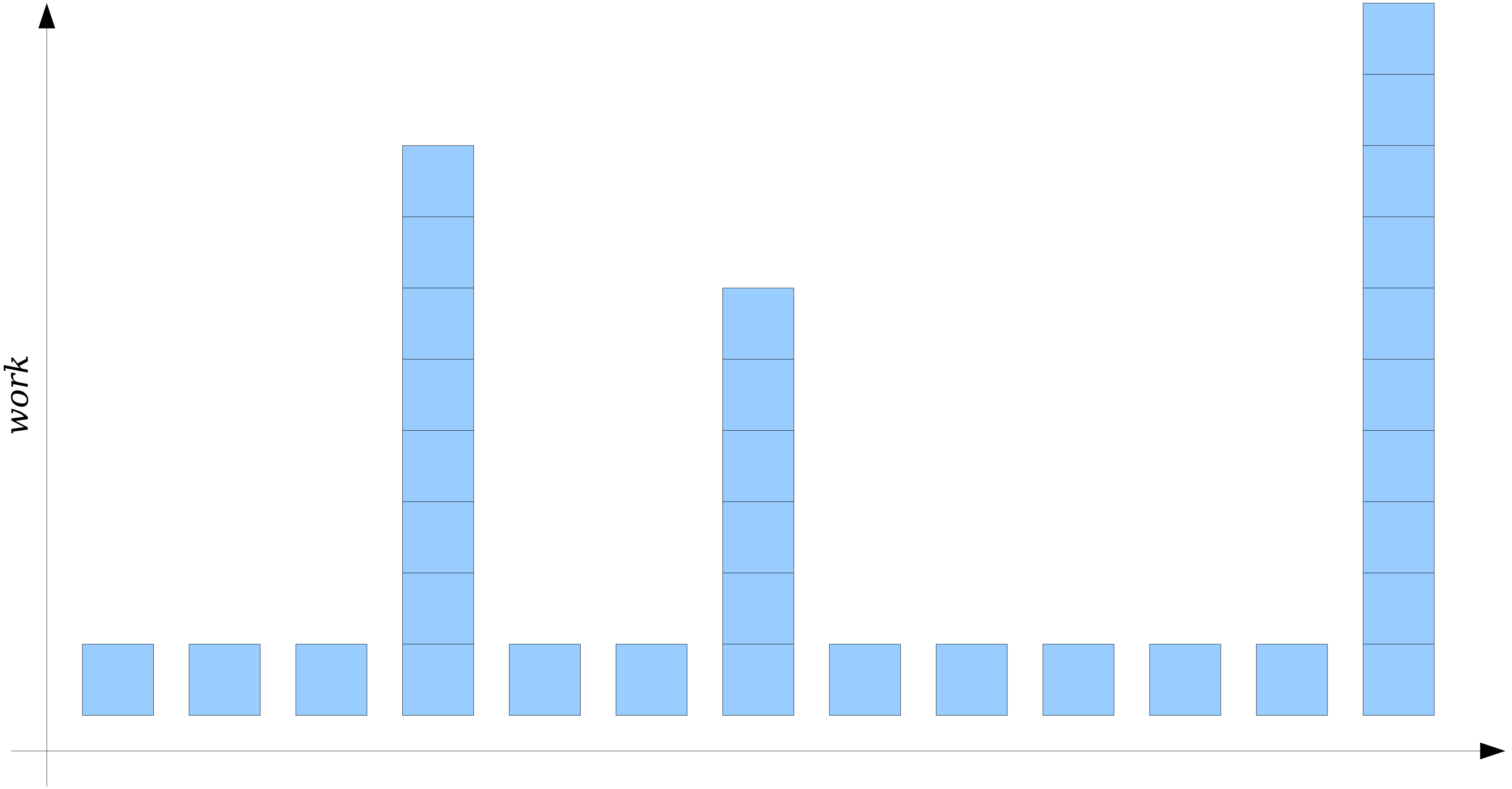
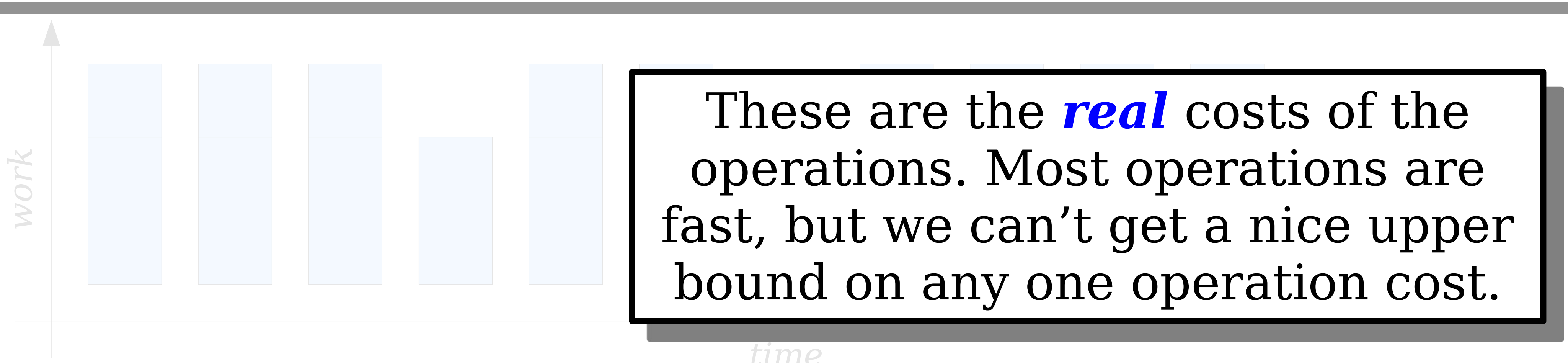
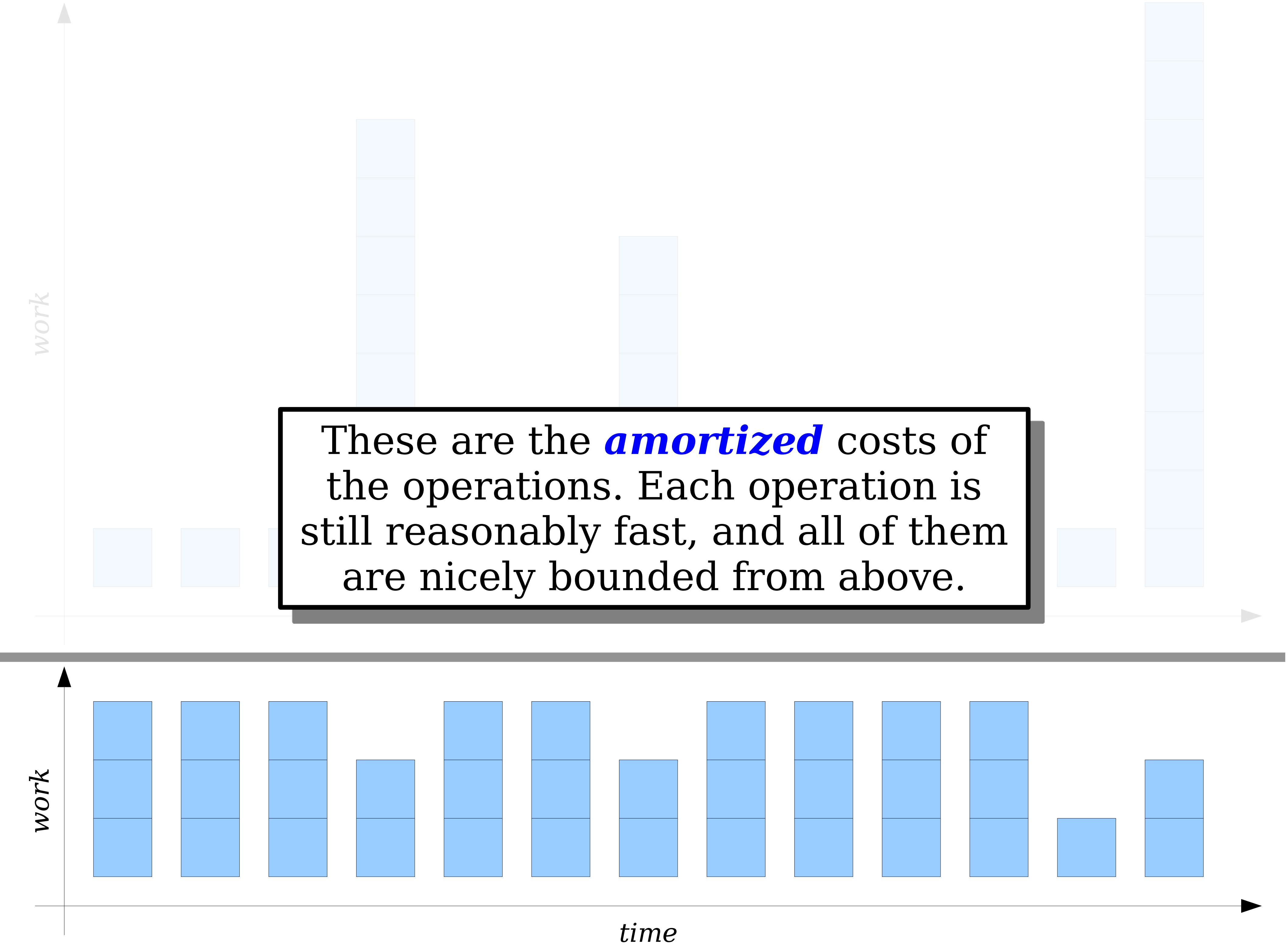



These are the ***real*** costs of the operations. Most operations are fast, but we can't get a nice upper bound on any one operation cost.



work
These are the ***amortized*** costs of the operations. Each operation is still reasonably fast, and all of them are nicely bounded from above.
work
time

# Amortized Analysis

- ***Key Idea:*** Assign each operation a (fake!) cost called its ***amortized cost*** such that*, for any series of operations performed,* the following is true:

$$\sum \textit{amortized-cost} \geq \sum \textit{real-cost}$$

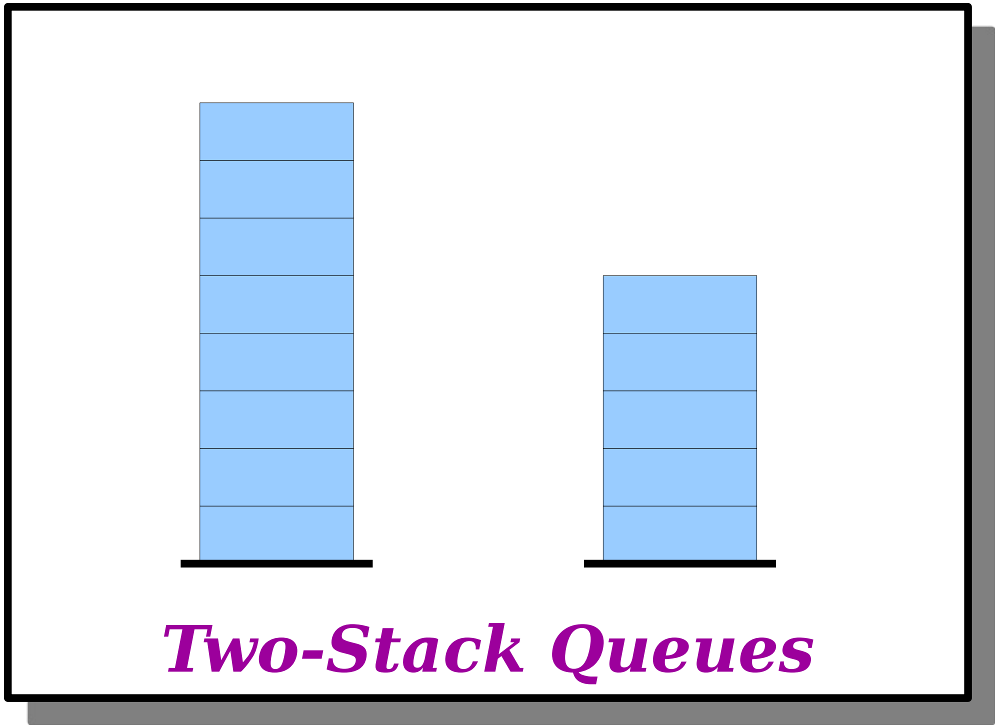
- Amortized costs shift work backwards from expensive operations onto cheaper ones.
  - Cheap operations are artificially made more expensive to pay for future cleanup work.
  - Expensive operations are artificially made cheaper by shifting the work backwards.

# Where We’re Going

- The ***amortized*** cost of an enqueue or dequeue into a two-stack queue is O(1).
- Any sequence of $n$ operations on a two-stack queue will take time

$$n \cdot O(1) = O(n).$$

- However, each individual operation may take more than O(1) time to complete.



# Where We're Going

- The ***amortized*** cost of appending to a dynamic array is O(1).
- Any sequence of $n$ appends to a dynamic array will take time

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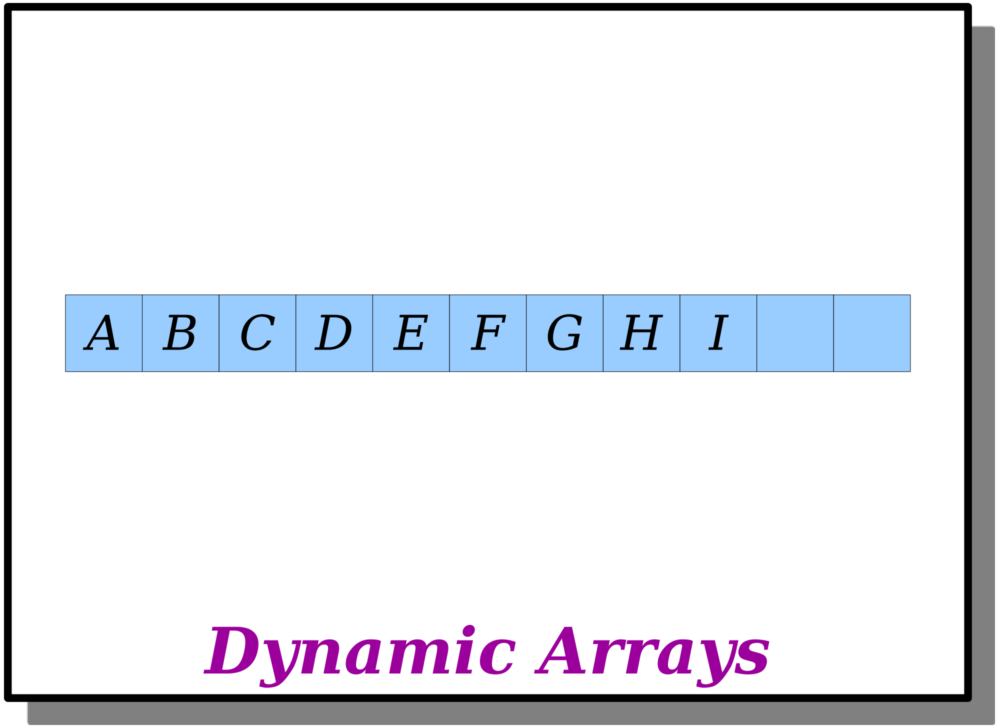
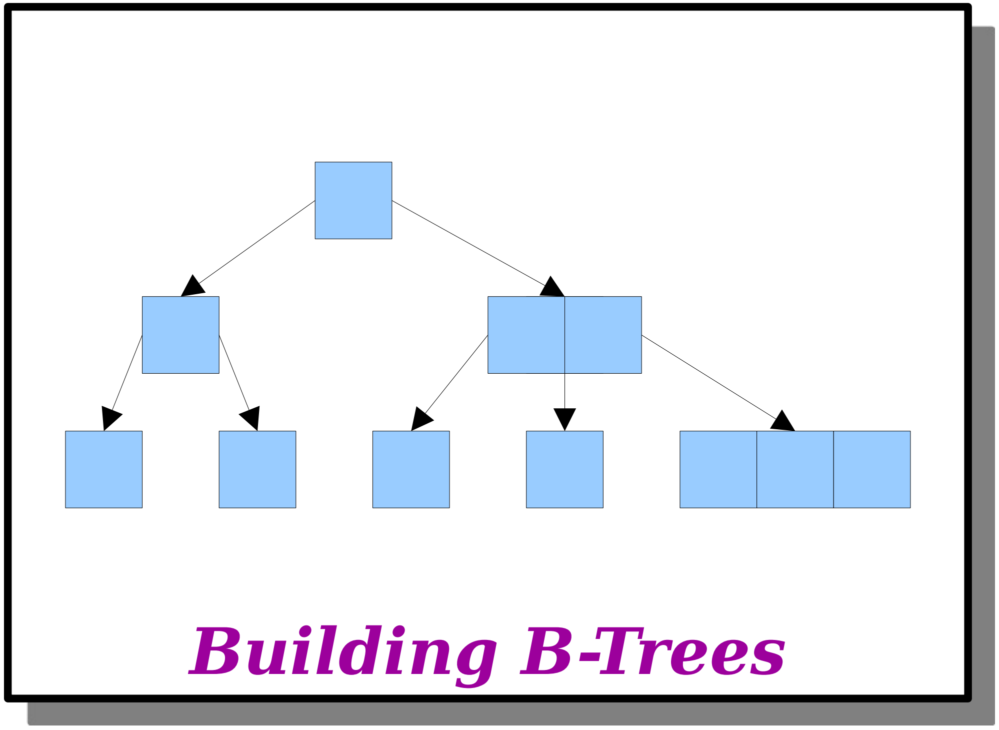


# Where We're Going

- The ***amortized*** cost of inserting a new element at the end of a B-tree, assuming we have a pointer to the rightmost leaf, is O(1).
- Any sequence of $n$ appends will take time

$$n \cdot O(1) = O(n).$$

- However, each individual operation may take more than O(1) time to complete.



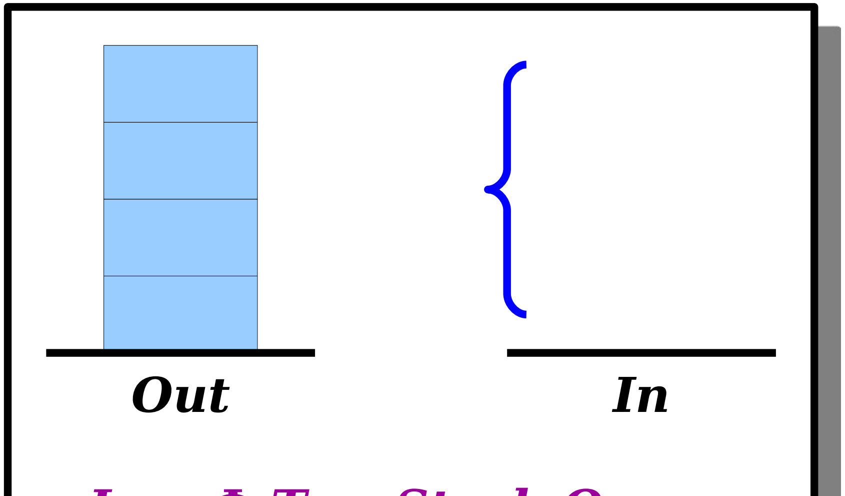
# Formalizing This Idea

# Assigning Amortized Costs

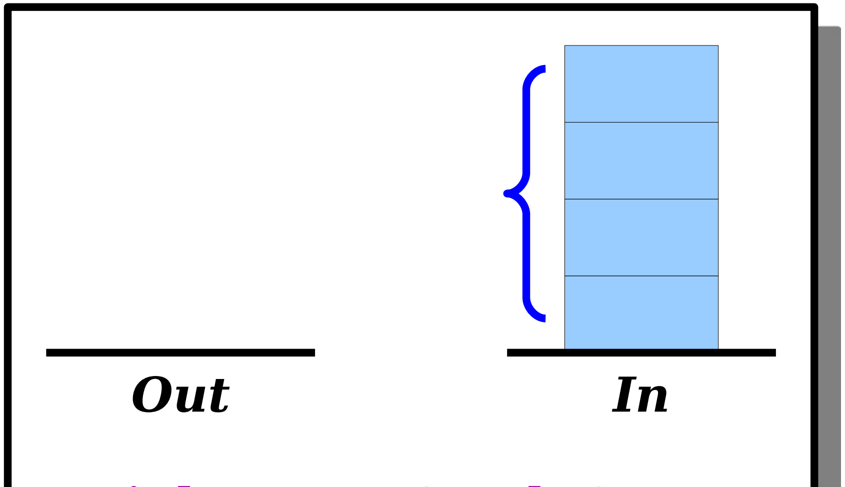
- The approach we've taken so far for assigning amortized costs is called an ***aggregate analysis***.
  - Directly compute the maximum possible work done across any sequence of operations, then divide that by the number of operations.
- This approach works well here, but it doesn't scale well to more complex data structures.
  - What if different operations contribute to / clean up messes in different ways?
  - What if it's not clear what sequence is the worst-case sequence of operations?
- In practice, we tend to use a different strategy called the ***potential method*** to assign amortized costs.

# Potential Functions

- To assign amortized costs, we'll need to measure how "messy" the data structure is.
- For each data structure, we define a ***potential function*** Φ such that
  - Φ is small when the data structure is "clean," and
  - Φ is large when the data structure is "messy."




***Low Φ Two-Stack Queue***

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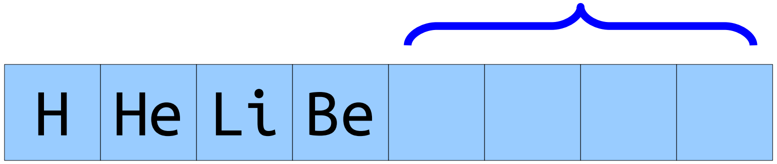
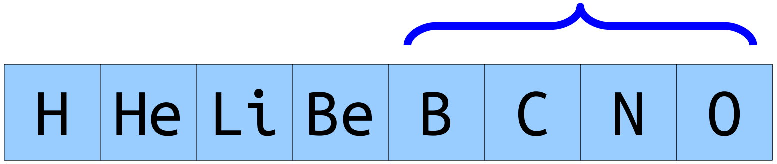



***Low Φ Dynamic Array***




***High Φ Dynamic Array***

# Potential Functions

- Once we've chosen a potential function Φ, we define the amortized cost of an operation to be

$$\textit{amortized-cost} = \textit{real-cost} + k \cdot \Delta\Phi$$

  where $k$ is a constant under our control and ΔΦ is the difference between Φ just after the operation finishes and Φ just before the operation started:

$$\Delta\Phi = \Phi_{after} - \Phi_{before}$$

- Intuitively:
  - If Φ increases, the data structure got "messier," and the amortized cost is *higher* than the real cost.
  - If Φ decreases, the data structure got "cleaner," and the amortized cost is *lower* than the real cost.

# Why This Works

$$\sum amortized\text{-}cost = \sum \left(real\text{-}cost + k \cdot \Delta\Phi\right)$$

$$= \sum real\text{-}cost + k \cdot \sum \Delta\Phi$$

$$= \sum real\text{-}cost + k \cdot \left(\Phi_{end} - \Phi_{start}\right)$$

Think "fundamental theorem of calculus,"
but for discrete derivatives!

$$\int_a^b f'(x)dx = f(b) - f(a)$$

$$\sum_{x=a}^{b} \Delta f(x) = f(b+1) - f(a)$$

Look up ***finite calculus*** if you're curious to learn more!

# Why This Works

$$\begin{aligned}
\sum \textit{amortized-cost} &= \sum \left(\textit{real-cost} + k \cdot \Delta\Phi\right) \\
&= \sum \textit{real-cost} + k \cdot \sum \Delta\Phi \\
&= \sum \textit{real-cost} + k \cdot \left(\Phi_{end} - \Phi_{start}\right) \\
&\geq \sum \textit{real-cost}
\end{aligned}$$

Let's make two assumptions:

$$\Phi \geq 0.$$

$$\Phi_{start} = 0.$$

# Why This Works

$$
\begin{aligned}
\sum amortized\text{-}cost &= \sum \left(real\text{-}cost + k \cdot \Delta\Phi\right) \\
&= \sum real\text{-}cost + k \cdot \sum \Delta\Phi \\
&= \sum real\text{-}cost + k \cdot \left(\Phi_{end} - \Phi_{start}\right) \\
&\geq \sum real\text{-}cost
\end{aligned}
$$

Assigning costs this way will never, in any circumstance, overestimate the total amount of work done.

# The Story So Far

- We will assign amortized costs to each operation such that

$$\sum \textit{\textbf{amortized-cost}} \geq \sum \textit{\textbf{real-cost}}$$

- To do so, define a ***potential function*** $\Phi$ such that
  - $\Phi$ measures how “messy” the data structure is,
  - $\Phi_{start} = 0$, and
  - $\Phi \geq 0$.
- Then, define amortized costs of operations as

$$\textit{\textbf{amortized-cost}} = \textit{\textbf{real-cost}} + k \cdot \Delta\Phi$$

  for a choice of $k$ under our control.

# The Two-Stack Queue

$\Phi$ = height of ***In*** stack

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$\Phi$ = height of ***In*** stack


$$\begin{aligned} \textit{amortized-cost} &= \textit{real-cost} + k \cdot \Delta\Phi \\ &= O(1) + k \cdot 1 \\ &= \mathbf{O(1)} \end{aligned}$$

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$\Phi$ = height of ***In*** stack


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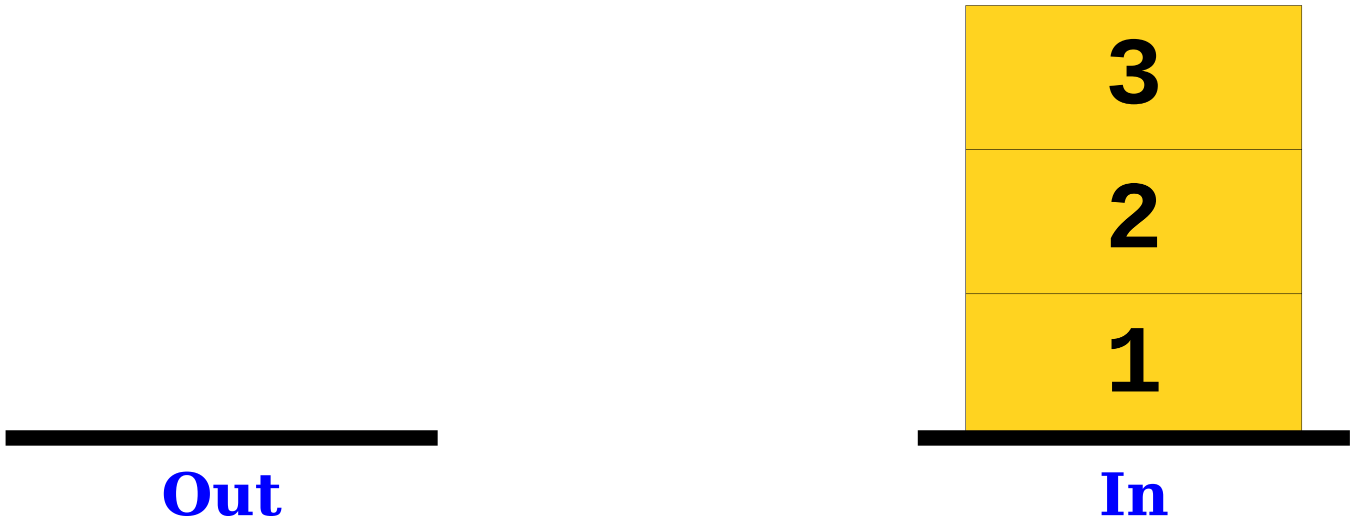
Φ = height of ***In*** stack

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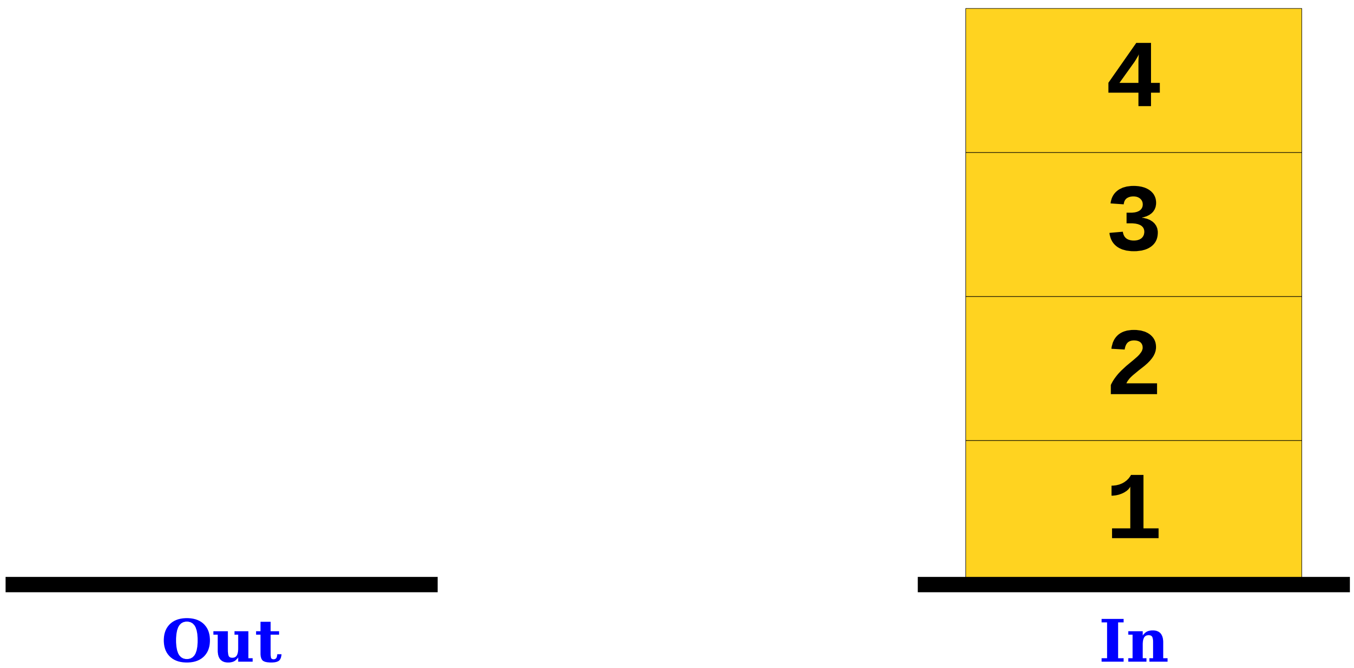


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# The Two-Stack Queue

$\Phi$ = height of ***In*** stack


*amortized-cost* = *real-cost* + $k \cdot \Delta\Phi$
= $O(h) + k \cdot -h$ // $h$ = height of ***In*** stack
= **O(1)** // Choose $k$ strategically

# The Two-Stack Queue

$\Phi$ = height of ***In*** stack


$$\begin{aligned} \textit{amortized-cost} &= \textit{real-cost} + k \cdot \Delta\Phi \\ &= O(1) + k \cdot 0 \\ &= \mathbf{O(1)} \end{aligned}$$

***Theorem:*** The amortized cost of any enqueue or dequeue operation on a two-stack queue is O(1).

***Proof:*** Let Φ be the height of the *In* stack in the two-stack queue. Each enqueue operation does a single push and increases the height of the *In* stack by one. Therefore, its amortized cost is

$$O(1) + k \cdot \Delta\Phi = O(1) + k \cdot 1 = O(1).$$

Now, consider a dequeue operation. If the *Out* stack is nonempty, then the dequeue does O(1) work and does not change Φ. Its cost is therefore

$$O(1) + k \cdot \Delta\Phi = O(1) + k \cdot 0 = O(1).$$

Otherwise, the *Out* stack is empty. Suppose the *In* stack has height $h$. The dequeue does $O(h)$ work to pop the elements from the *In* stack and push them onto the *Out* stack, followed by one additional pop for the dequeue. This is $O(h)$ total work.

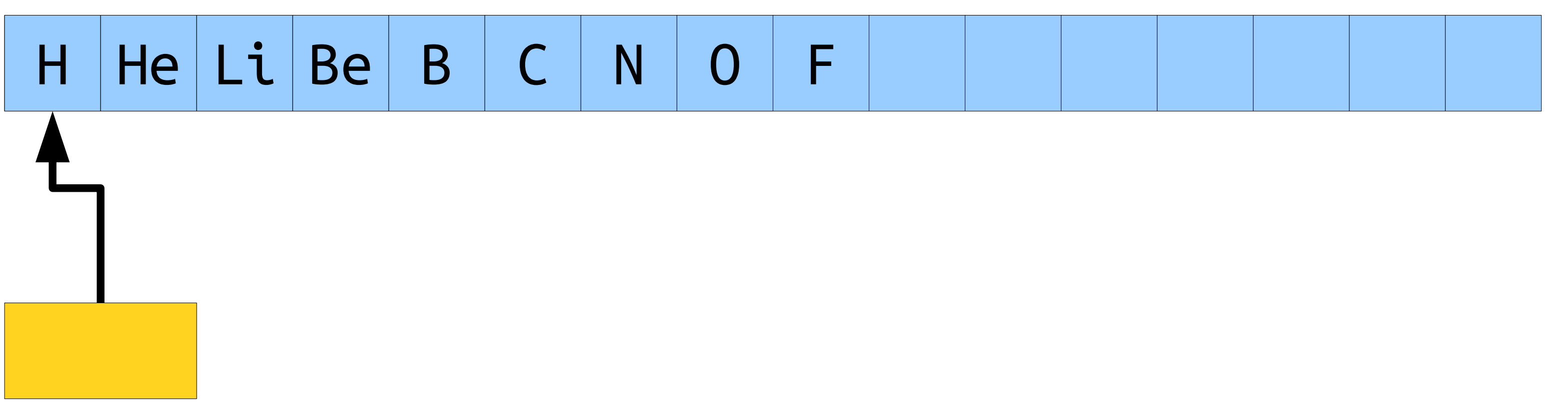
At the beginning of this operation, we have $\Phi = h$. At the end of this operation, we have $\Phi = 0$. Therefore, $\Delta\Phi = -h$, so the amortized cost of the operation is

$$O(h) + k \cdot -h = O(1),$$

assuming we pick $k$ to cancel out the constant factor hidden in the $O(h)$ term. ■

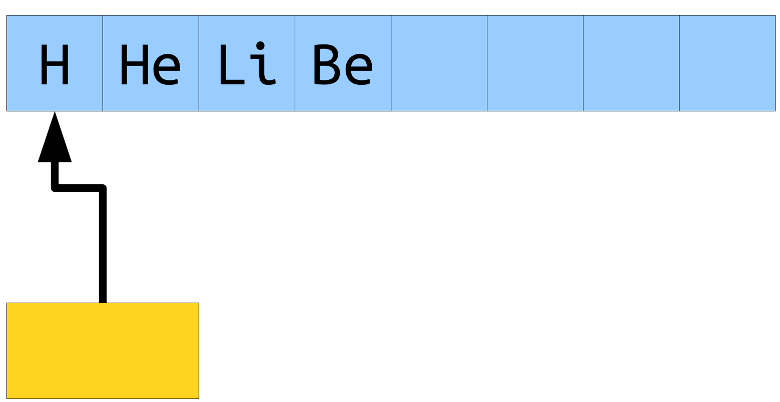
# Analyzing Dynamic Arrays

- ***Goal:*** Choose a potential function Φ such that the amortized cost of an append is O(1).
- ***Initial (wrong!) guess:*** Set Φ to be the number of free slots left in the array.



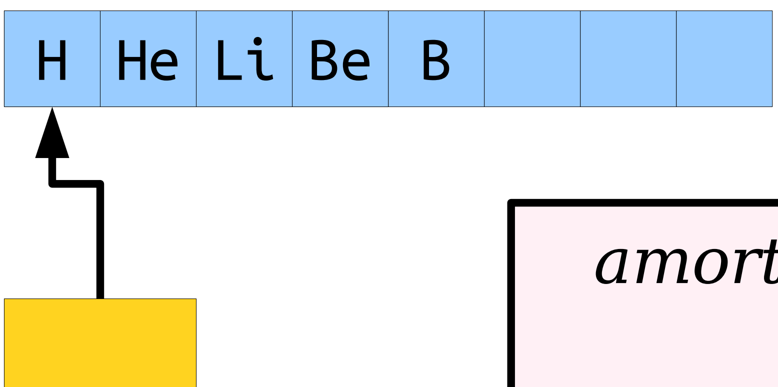
# Dynamic Arrays

$\Phi$ = number of free slots



# Dynamic Arrays

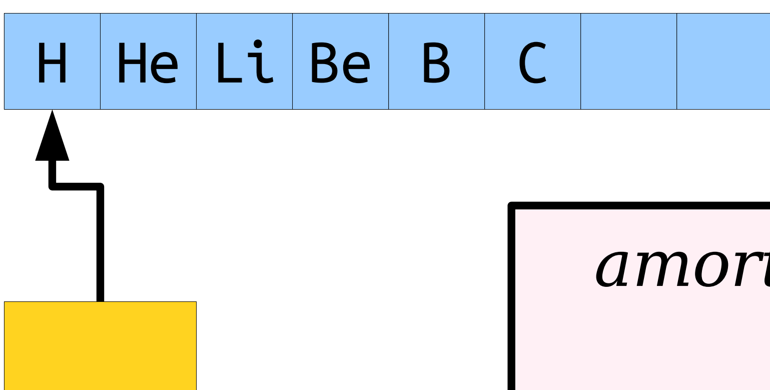
$\Phi$ = number of free slots




$$
\begin{aligned}
\textit{amortized-cost} &= \textit{real-cost} + k \cdot \Delta\Phi \\
&= O(1) + k \cdot -1 \\
&= \mathbf{O(1)}
\end{aligned}
$$

# Dynamic Arrays

$\Phi$ = number of free slots

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# Dynamic Arrays

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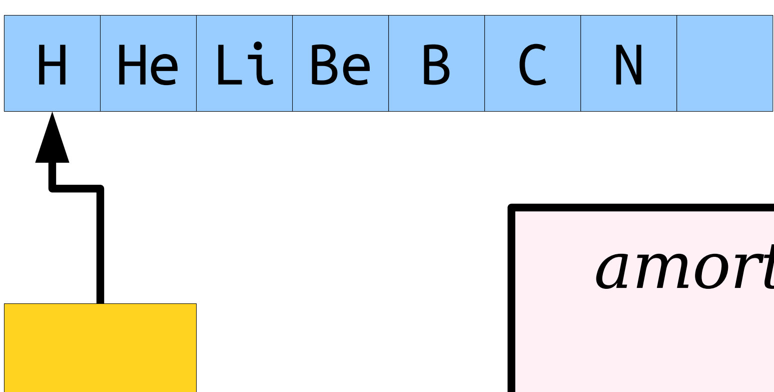


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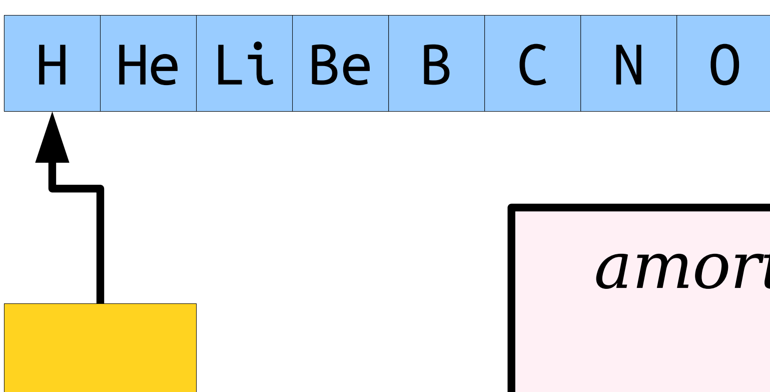
# Dynamic Arrays

$\Phi$ = number of free slots



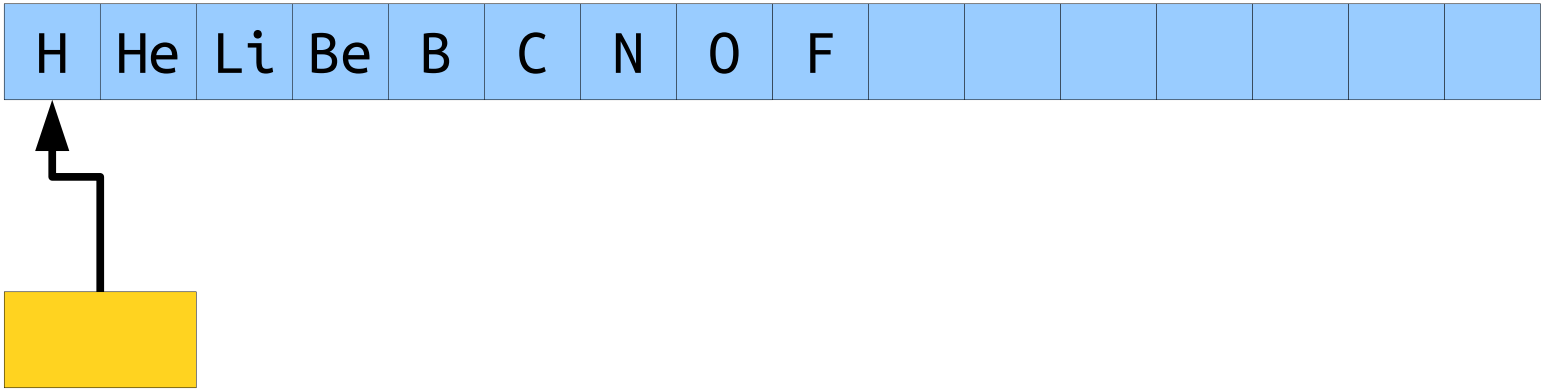

$$\begin{aligned}
\textit{amortized-cost} &= \textit{real-cost} + k \cdot \Delta\Phi \\
&= O(1) + k \cdot -1 \\
&= \mathbf{O(1)}
\end{aligned}$$

# Dynamic Arrays

$\Phi$ = number of free slots

With this choice of $\Phi$, what is the amortized cost of an append to an array of size *n* when no free slots are left?

Formulate a hypothesis!

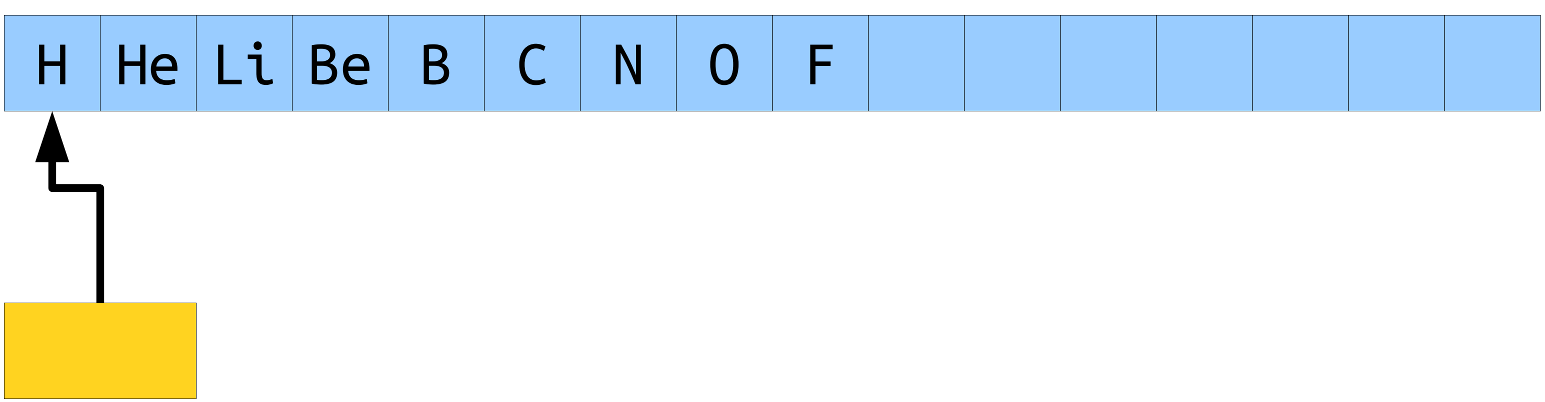


# Dynamic Arrays

$\Phi$ = number of free slots

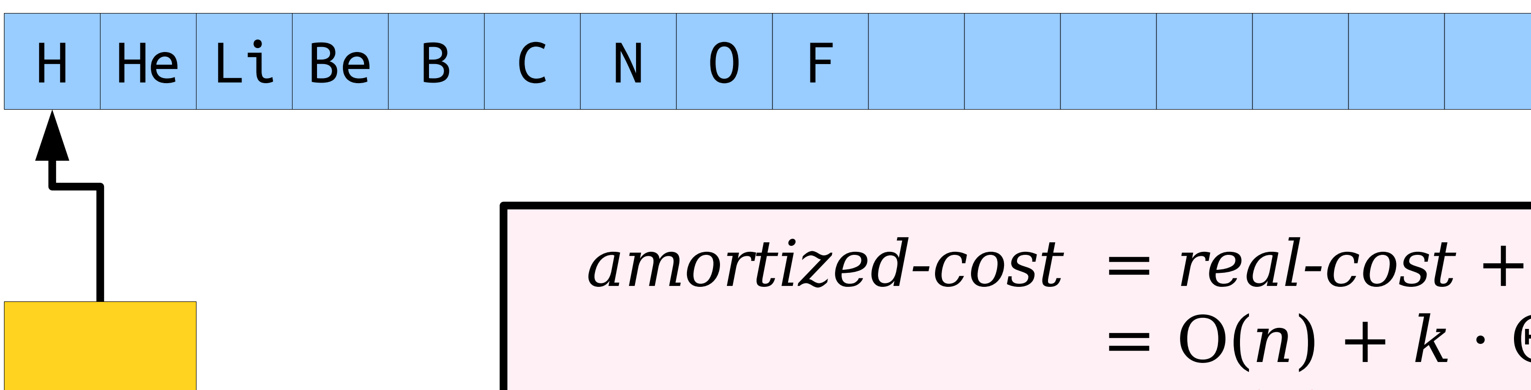
With this choice of $\Phi$, what is the amortized cost of an append to an array of size $n$ when no free slots are left?

Discuss with your neighbors!



# Dynamic Arrays

Φ = number of free slots




$$
\begin{aligned}
\textit{amortized-cost} &= \textit{real-cost} + k \cdot \Delta\Phi \\
&= O(n) + k \cdot \Theta(n) \\
&= \mathbf{O(\boldsymbol{n})}
\end{aligned}
$$

# Analyzing Dynamic Arrays

- ***Intuition:*** Φ should measure how "messy" the data structure is.
  - Having lots of free slots means there's very little mess.
  - Having few free slots means there's a lot of mess.
- We basically got our potential function backwards. Oops.
- ***Question:*** What should Φ be?

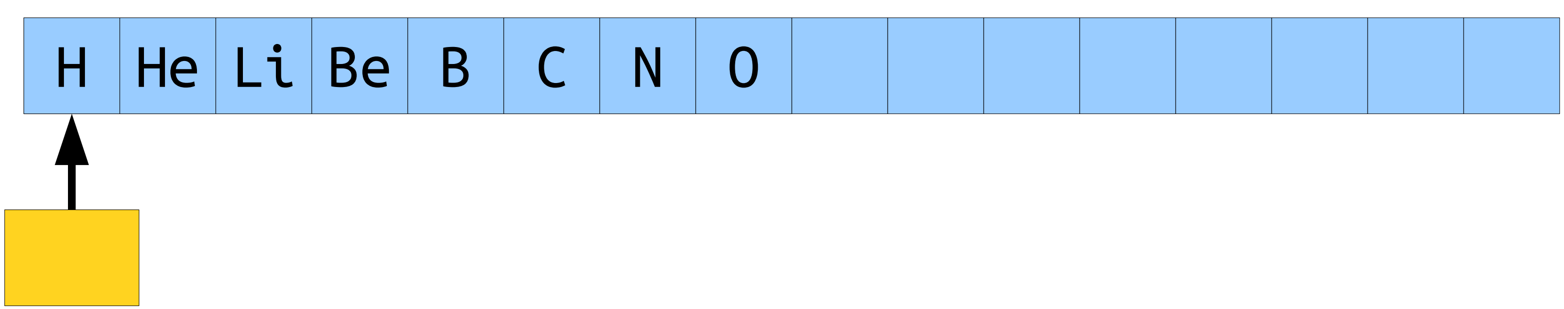
# Analyzing Dynamic Arrays

- The amortized cost of an append is

$$\textbf{\textit{amortized-cost = real-cost + k · ΔΦ}}.$$

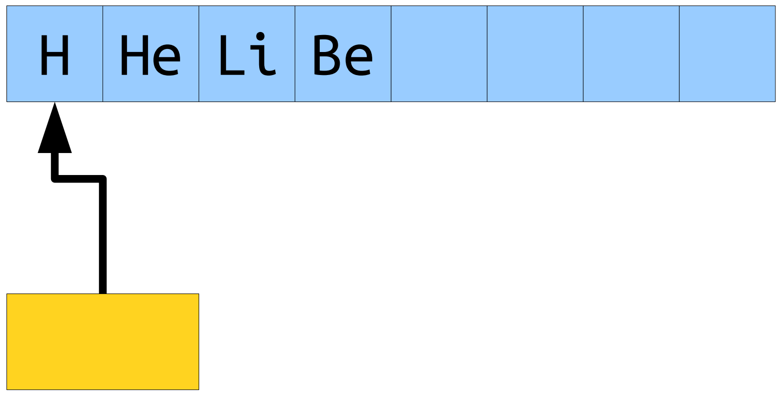
- When we double the array size, our real cost is $\Theta(n)$. We need $\Delta\Phi$ to be something like $-n$.
- ***Goal:*** Pick $\Phi$ so that
  - when there are no slots left, $\Phi \approx n$, and
  - right after we double the array size, $\Phi \approx 0$.
- With some trial and error, we can come up with

$$\boldsymbol{\Phi} = \textbf{\textit{\#elems - \#free-slots}}$$



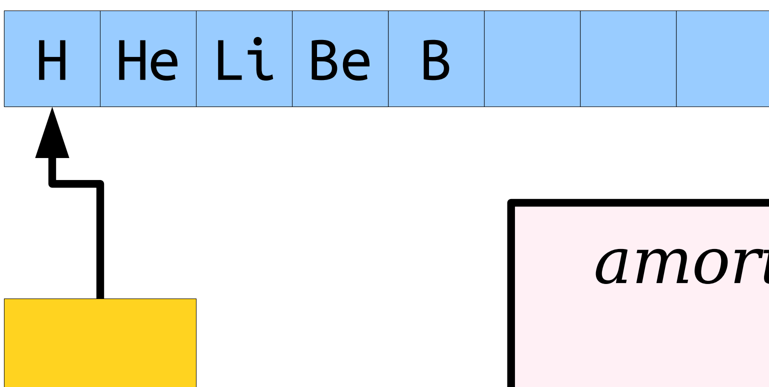
# Dynamic Arrays

$\Phi$ = *#elems – #free-slots*



# Dynamic Arrays

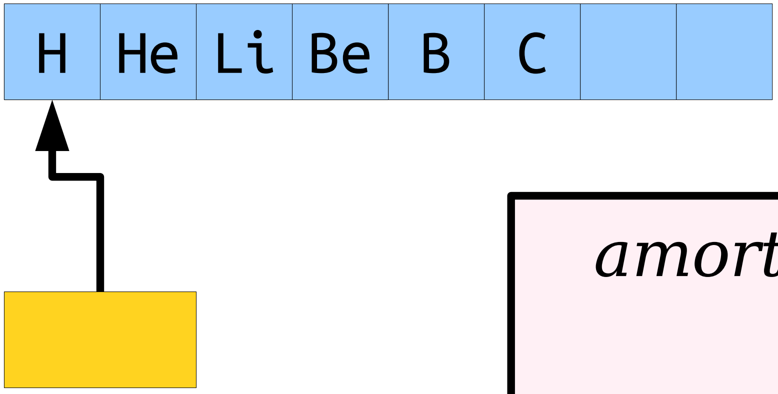
$$\Phi = \#elems - \#free\text{-}slots$$




$$\begin{aligned} amortized\text{-}cost &= real\text{-}cost + k \cdot \Delta\Phi \\ &= O(1) + k \cdot 2 \\ &= \mathbf{O(1)} \end{aligned}$$

# Dynamic Arrays

$\Phi = \#elems - \#free\text{-}slots$

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\end{aligned}
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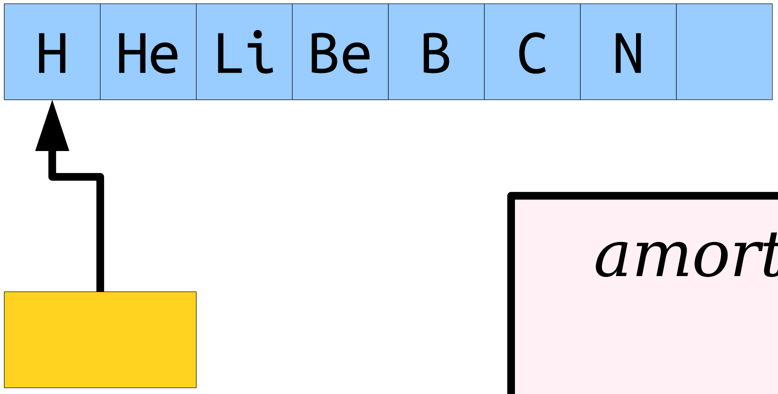
# Dynamic Arrays

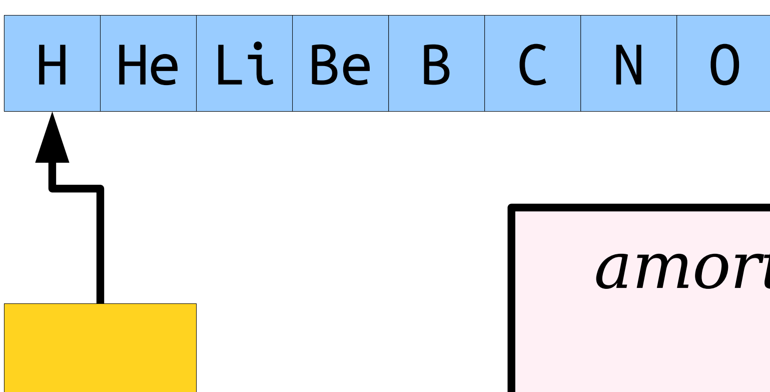
$$\Phi = \#elems - \#free\text{-}slots$$




$$\begin{aligned} amortized\text{-}cost &= real\text{-}cost + k \cdot \Delta\Phi \\ &= O(1) + k \cdot 2 \\ &= \mathbf{O(1)} \end{aligned}$$

# Dynamic Arrays

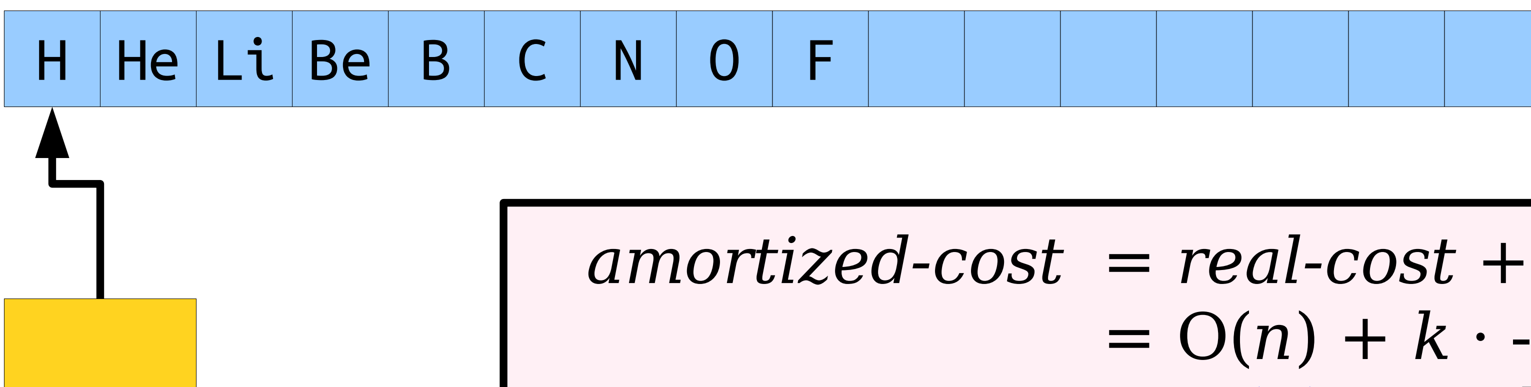
$\Phi = \#elems - \#free\text{-}slots$




$$
\begin{aligned}
amortized\text{-}cost &= real\text{-}cost + k \cdot \Delta\Phi \\
&= O(1) + k \cdot 2 \\
&= \mathbf{O(1)}
\end{aligned}
$$

# Dynamic Arrays

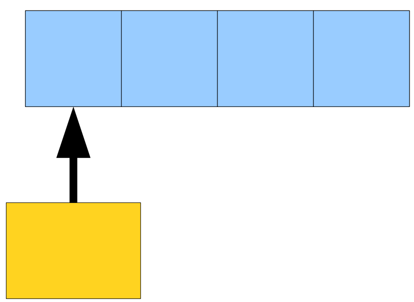
$\Phi$ = *#elems – #free-slots*




$$
\begin{aligned}
\textit{amortized-cost} &= \textit{real-cost} + k \cdot \Delta\Phi \\
&= O(n) + k \cdot -\Theta(n) \\
&= \mathbf{O(1)} \text{ // Pick } k \text{ well}
\end{aligned}
$$

# A Caveat

- We require that $\Phi_{start} = 0$ and that $\Phi \geq 0$.
- What happens when we have a newly-created dynamic array?



$\Phi = -4$

- ***Quick fix:*** This is an edge case, so set

$$\Phi = \max\{0, \#elems - \#free\text{-}slots\}$$

***Theorem:*** The amortized cost of an append to a dynamic array is O(1).

***Proof:*** Suppose the dynamic array has initial capacity $2C = O(1)$. Then, define $\Phi = \max\{\ 0, n - \#free\text{-}slots\ \}$, where $n$ is the number of elements stored in the dynamic array. Note that for $n < C$ that an append simply fills in a free slot and leaves $\Phi = 0$, so the amortized cost of such an append is O(1). Otherwise, we have $n > C$ and $\Phi = n - \#free\text{-}slots$.

Consider any append. If the append does not trigger a resize, it does O(1) work, increases $n$ by one, and decreases $\#free\text{-}slots$ by one, so the amortized cost is

$$O(1) + k \cdot \Delta\Phi = O(1) + k \cdot 2 = O(1).$$

Otherwise, the operation copies $n$ elements into a new array twice as large as before, increasing the number of free slots to $n$, then fills one of those slots. Just before the operation we had $\Phi = n$, and just after the operation we have $\Phi = 2$. Therefore, the amortized cost is

$$O(n) + k \cdot \Delta\Phi = O(n) + k \cdot (2 - n) = O(n) - nk + 2k,$$

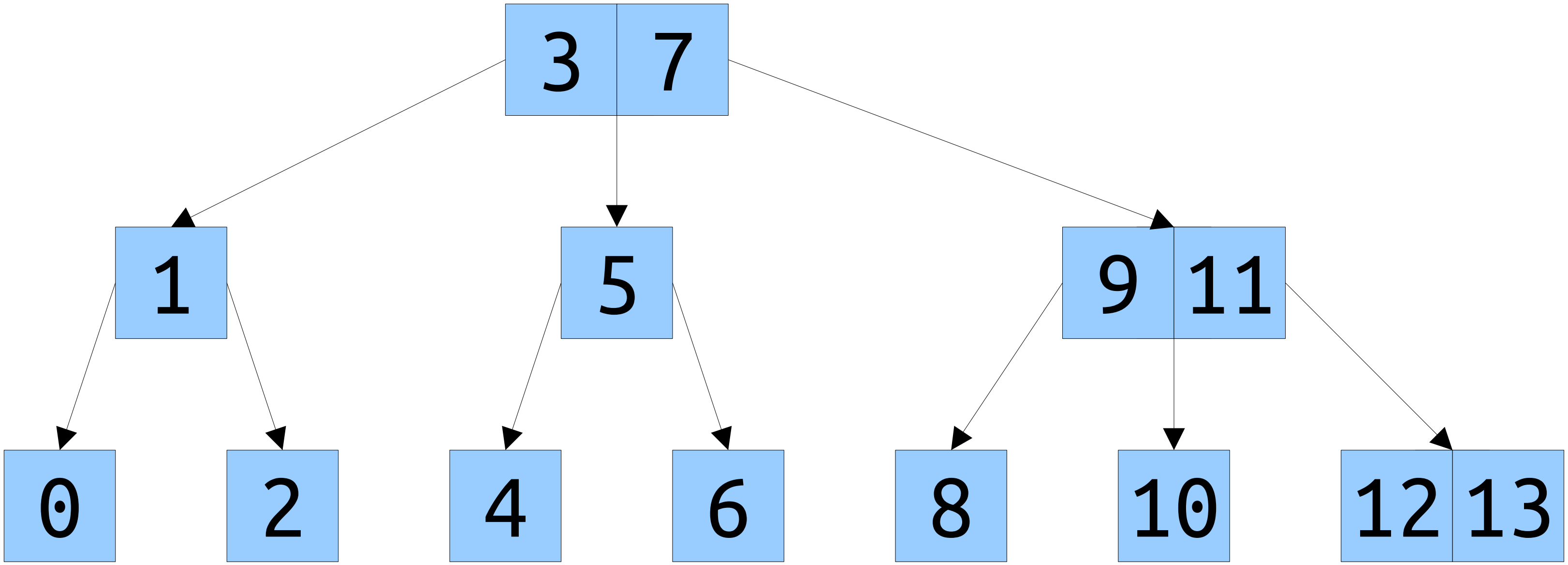
which can be made to equal O(1) by choosing the the $k$ term to match the constant hidden in the $O(n)$ term. ■

# Some Exercises

- Suppose we grow the array not by a factor of two, but by a fixed constant $\alpha > 1$. Find a choice of $\Phi$ so that the amortized cost of an append is O(1).
- Suppose we also allow elements to be removed from the array, and when it's ¼ full we shrink it by a factor of two. Find a choice of $\Phi$ so the amortized cost of appending or removing the last element is O(1).

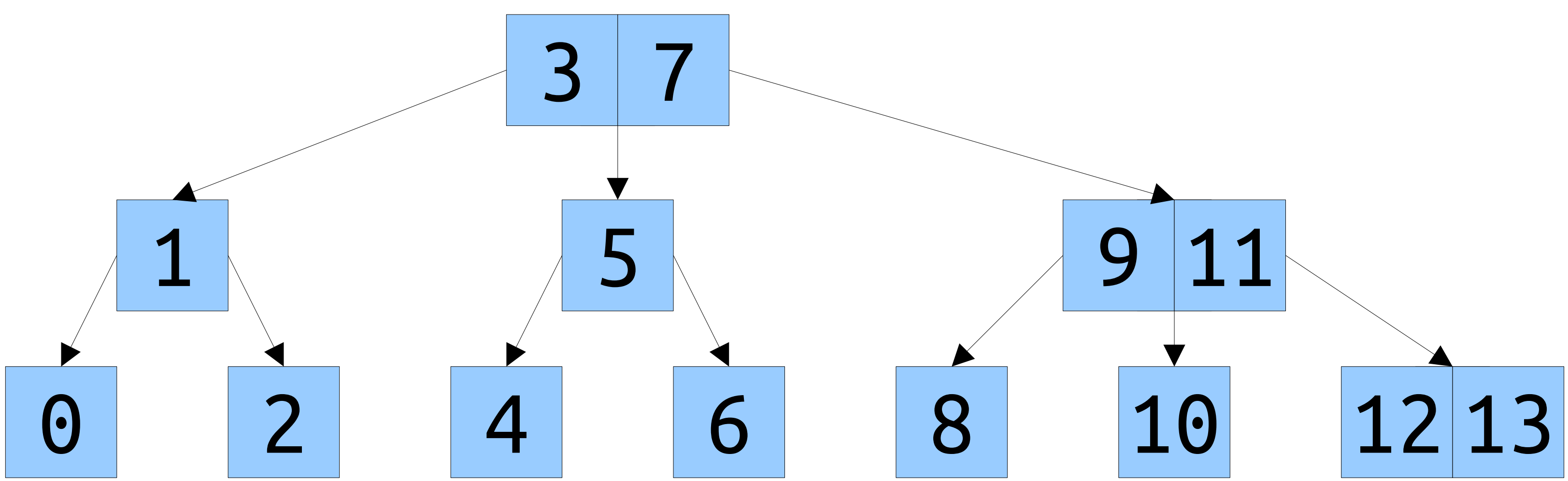
# Building B-Trees

- ***Algorithm:*** Store a pointer to the rightmost leaf. To add an item, append it to the rightmost leaf, splitting and kicking the median key up if we are out of space.



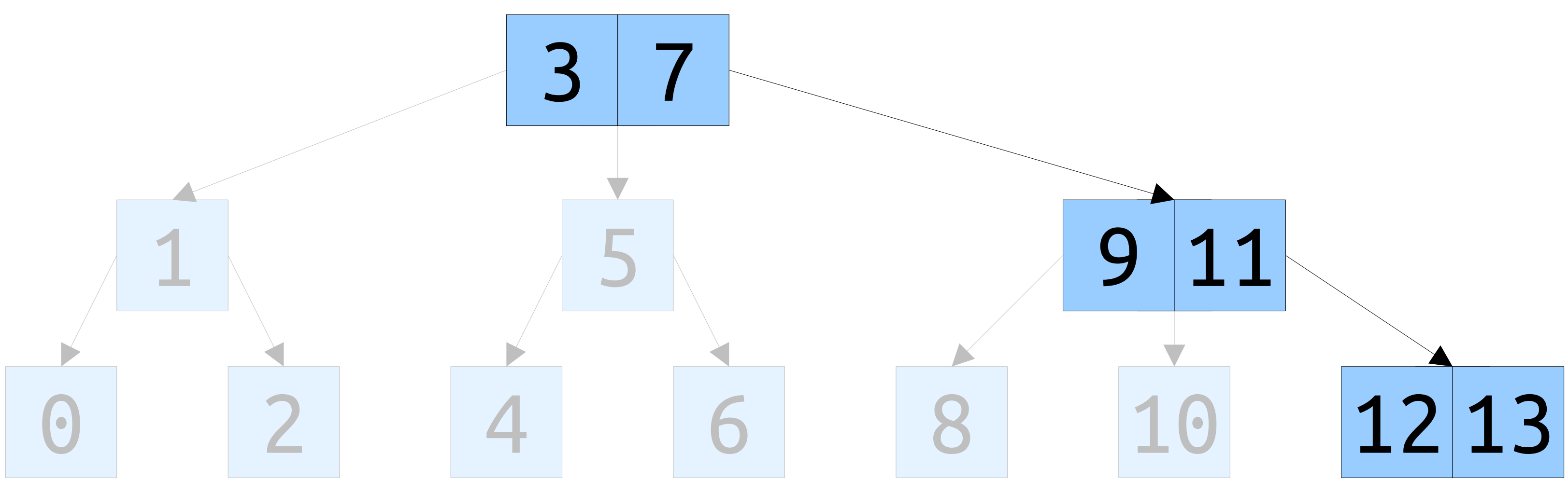
# Building B-Trees

- What is the actual cost of appending an element?
  - Suppose that we perform splits at $L$ layers in the tree.
  - Each split takes time $\Theta(b)$ to copy and move keys around.
  - Total cost: **$\Theta(bL)$**.
- ***Goal:*** Pick a potential function $\Phi$ so that we can offset this cost and make each append cost amortized O(1).



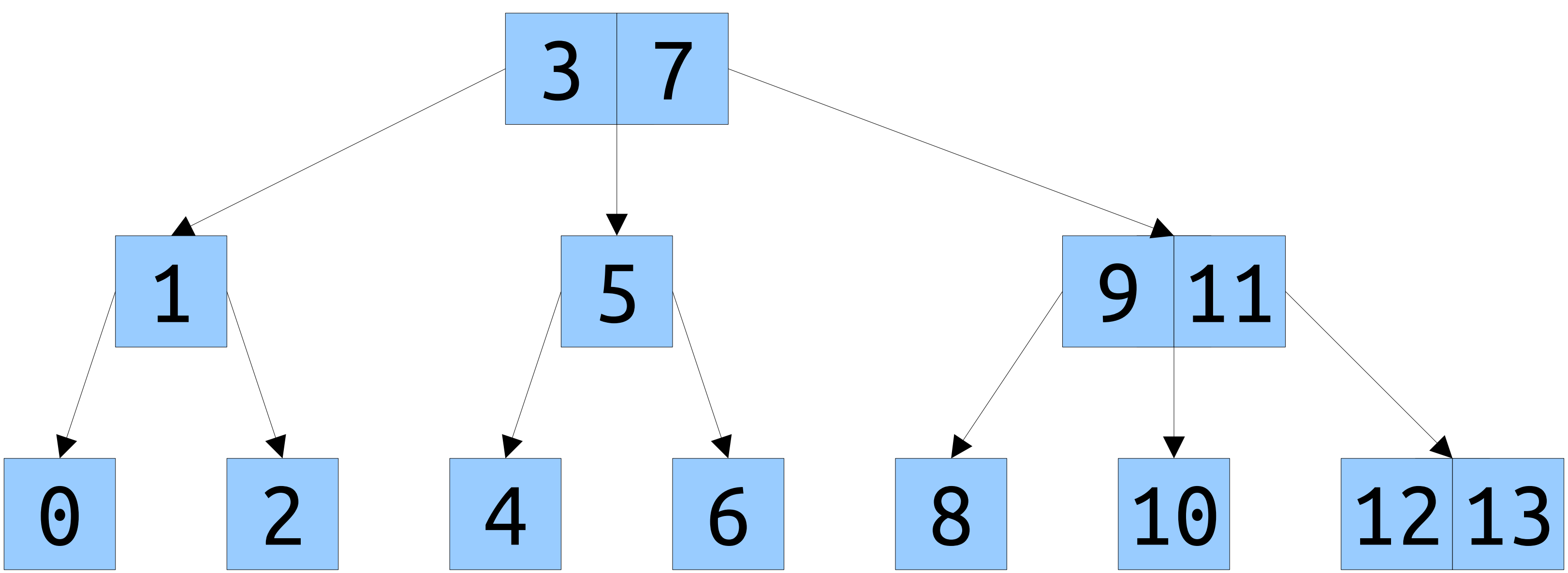
# Building B-Trees

- Our potential function should, intuitively, quantify how "messy" our data structure is.
- Some observations:
  - We only care about nodes in the right spine of the tree.
  - Nodes in the right spine slowly have keys added to them. When they split, they lose (about) half of their keys.
- ***Idea:*** Set Φ to be the number of keys in the right spine of the tree.



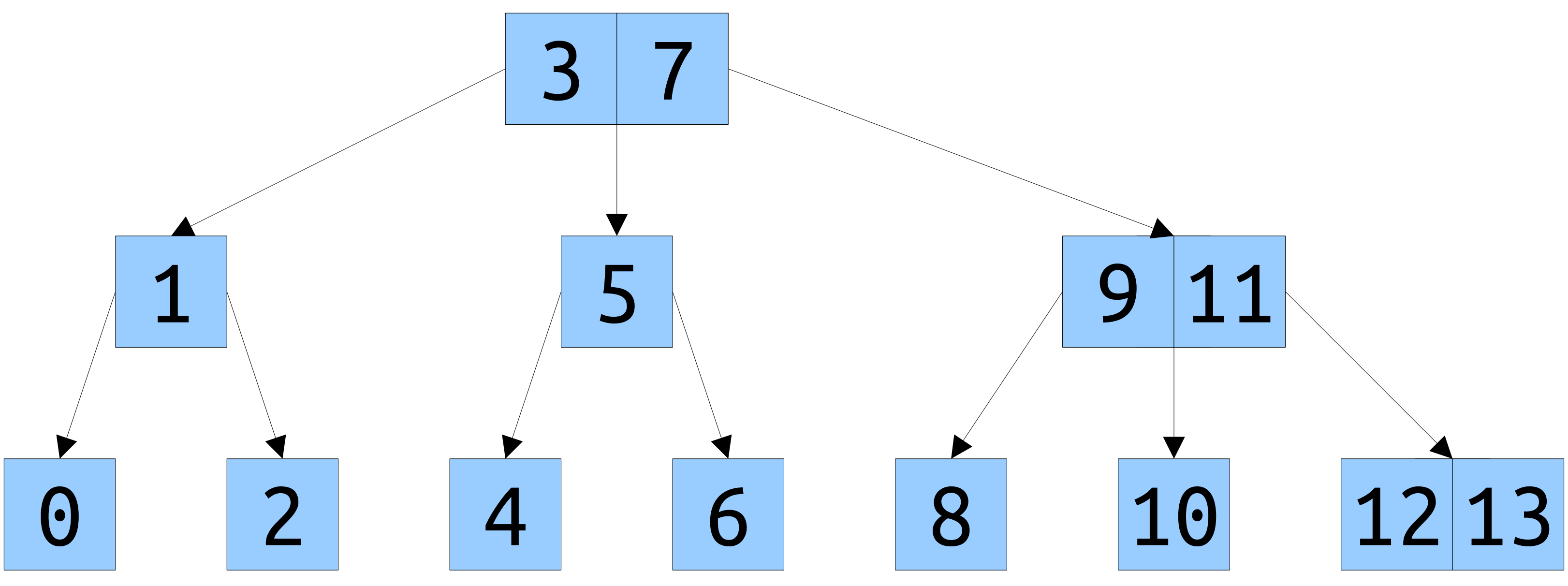
# Building B-Trees

- Let $\Phi$ be the number of keys on the right spine.
- Each split moves (roughly) half the keys from the split node into a node off the right spine.
- Change in potential per split: $-\Theta(b)$.
- Net $\Delta\Phi$: **$-\Theta(bL)$**.



# Building B-Trees

- Actual cost of an append that does $L$ splits: $O(bL)$.
- $\Delta\Phi$ for that operation: $-\Theta(bL)$.
- Amortized cost: **O(1)**.



***Theorem:*** The amortized cost of appending to a B-tree by inserting it into the rightmost leaf node and applying fixup rules is O(1).

***Proof:*** Assume we are working with a B-tree of order $b$. Let Φ be the number of nodes on the right spine of the B-tree.

Suppose we insert a value into the tree using the algorithm described above. Suppose this causes $L$ nodes to be split. Each of those splits requires $\Theta(b)$ work for a net total of $\Theta(bL)$ work.

Each of those $L$ splits moves $\Theta(b)$ keys off of the right spine of the tree, decreasing Φ by $\Theta(b)$ for a net drop in potential of $-\Theta(bL)$. In the layer just above the last split, we add one more key into a node, increasing Φ by one. Therefore, $\Delta\Phi = -\Theta(bL)$.

Overall, this tells us that the amortized cost of inserting a key this way is

$$\Theta(bL) + k \cdot \Delta\Phi = \Theta(bL) - k \cdot \Theta(bL),$$

which can be made to be O(1) by choosing k to equate the constants hidden in the O and Θ terms. ■

# More to Explore

- You can implement a ***deque*** (a doubly-ended queue) using a B-tree with pointers to the first and last leaves.
  - This is sometimes called a ***finger tree***.
  - Finger trees are used extensively in purely functional programming languages.
  - By extending the analysis from here, you can show the amortized cost of appending or removing from each end of the finger tree is O(1).
- Red/black trees are modeled on 2-3-4 trees. You can build a red/black tree from $n$ sorted keys in time O($n$) this way.
  - ***Great exercise:*** Explore how to do this, and work out what choice of $\Phi$ to make.

# To Summarize

# Amortized Analysis

- Some data structures accumulate messes slowly, then clean up those messes in single, large steps.
- We can assign ***amortized*** costs to operations. These are fake costs such that summing up the amortized costs never underestimates the sum of the real costs.
- To do so, we define a potential function Φ that, intuitively, measures how "messy" the data structure is. We then set

$$\textbf{\textit{amortized-cost}} = \textbf{\textit{real-cost}} + k \cdot \Delta\Phi.$$

- For simplicity, we assume that Φ is nonnegative and that Φ for an empty data structure is zero.

# Next Time

- ***Scapegoat Trees***
  - Building a balanced BST, lazily.