Scapegoat Trees

Outline for Today

- ***Recap from Last Time***
  - What is amortization, again?
- ***Lazy Balanced Trees***
  - Messes are okay, up to a point.
- ***Lazy Tree Insertions***
  - Deferring updates until they're needed.
- ***Lazy Tree Deletions***
  - And some associated subtleties.

Recap from Last Time

Amortized Analysis

- We will assign amortized costs to each operation such that

$$\sum \textit{amortized-cost} \geq \sum \textit{real-cost}$$

- To do so, define a ***potential function*** Φ such that
  - Φ measures how “messy” the data structure is,
  - $\Phi_{start} = 0$, and
  - $\Phi \geq 0$.
- Then, define amortized costs of operations as

$$\textit{amortized-cost} = \textit{real-cost} + k \cdot \Delta\Phi$$

  for a choice of k under our control.
- Intuitively:
  - If an operation makes a mess that needs to be cleaned up later, its amortized cost will be higher than its original cost.
  - If an operation cleans up a mess, its amortized cost will be lower than its real cost.

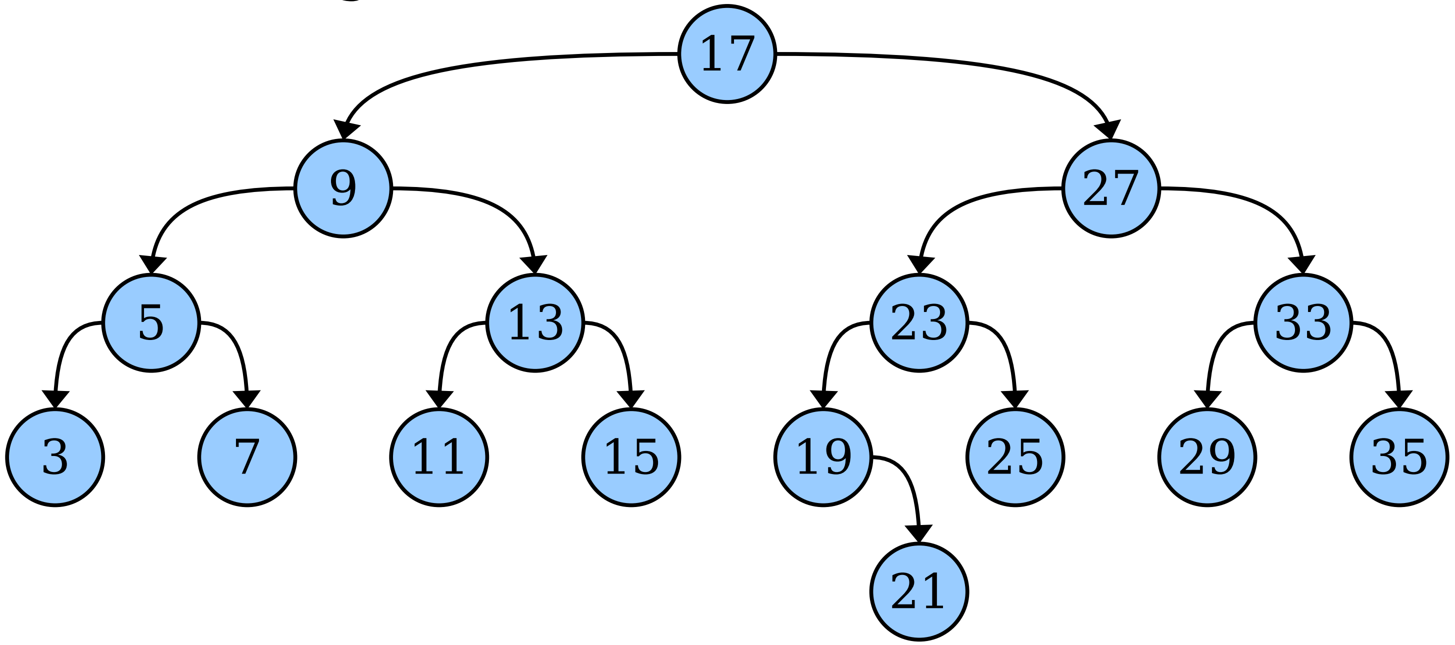
New Stuff!

Balanced Trees

- The red/black trees we explored earlier are worst-case efficient and guaranteed to have a height of O(log *n*).
- However, explaining how they work and deriving the basic insertion rules took two lectures – and we still didn't finish covering all cases.
- ***Goal for today:*** Find a simpler way to keep a tree balanced, under the assumption we're okay with amortized-efficient rather than worst-case efficient lookups.

On O(log *n*) Height

- A perfectly-balanced binary search tree with $n > 0$ nodes has height at most lg n.
  - (lg n denotes $\log_2 n$.)
- However, this tree shape is difficult to maintain: a single insertion or deletion might require a lot of node reshuffling.



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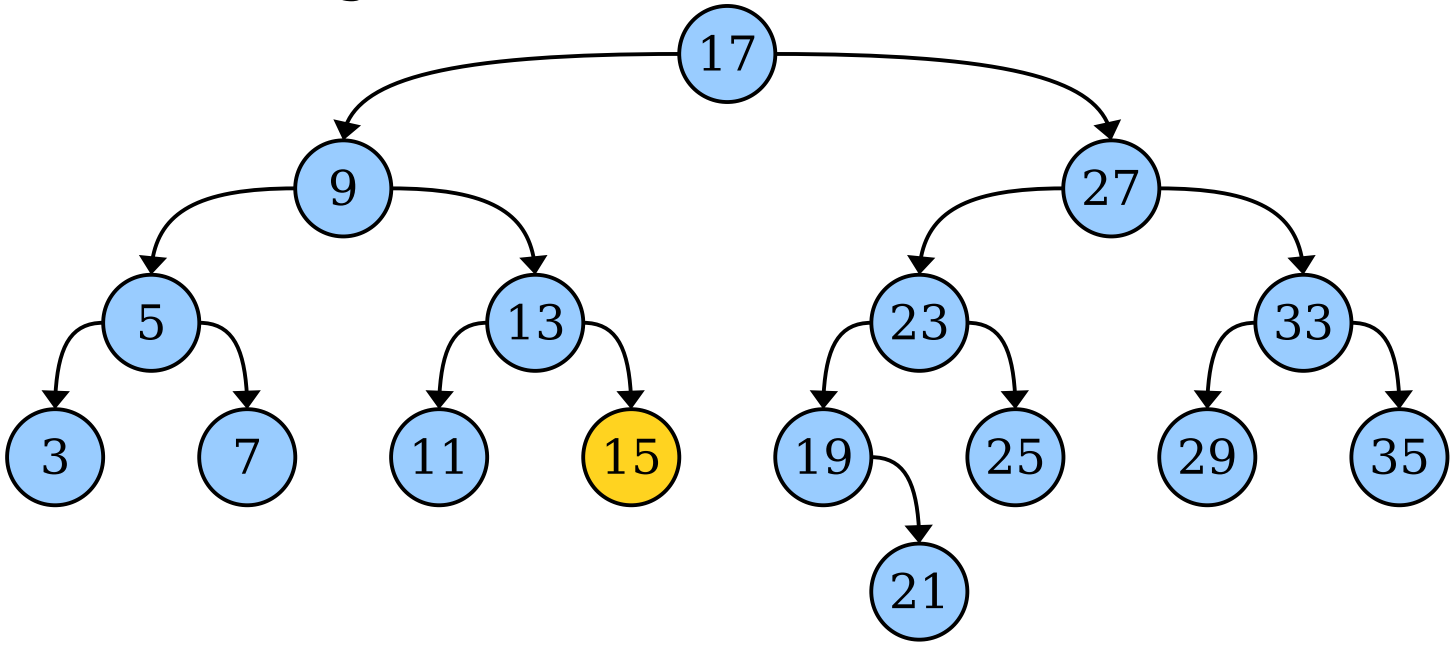

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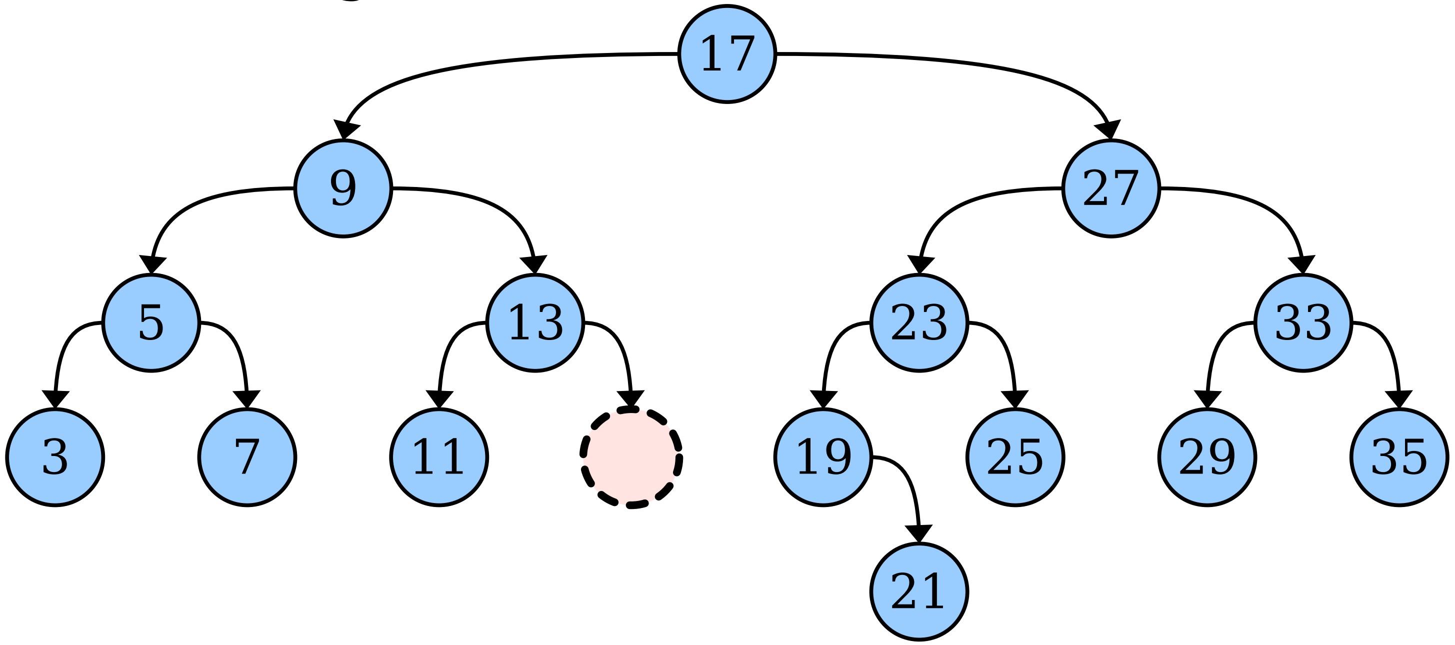

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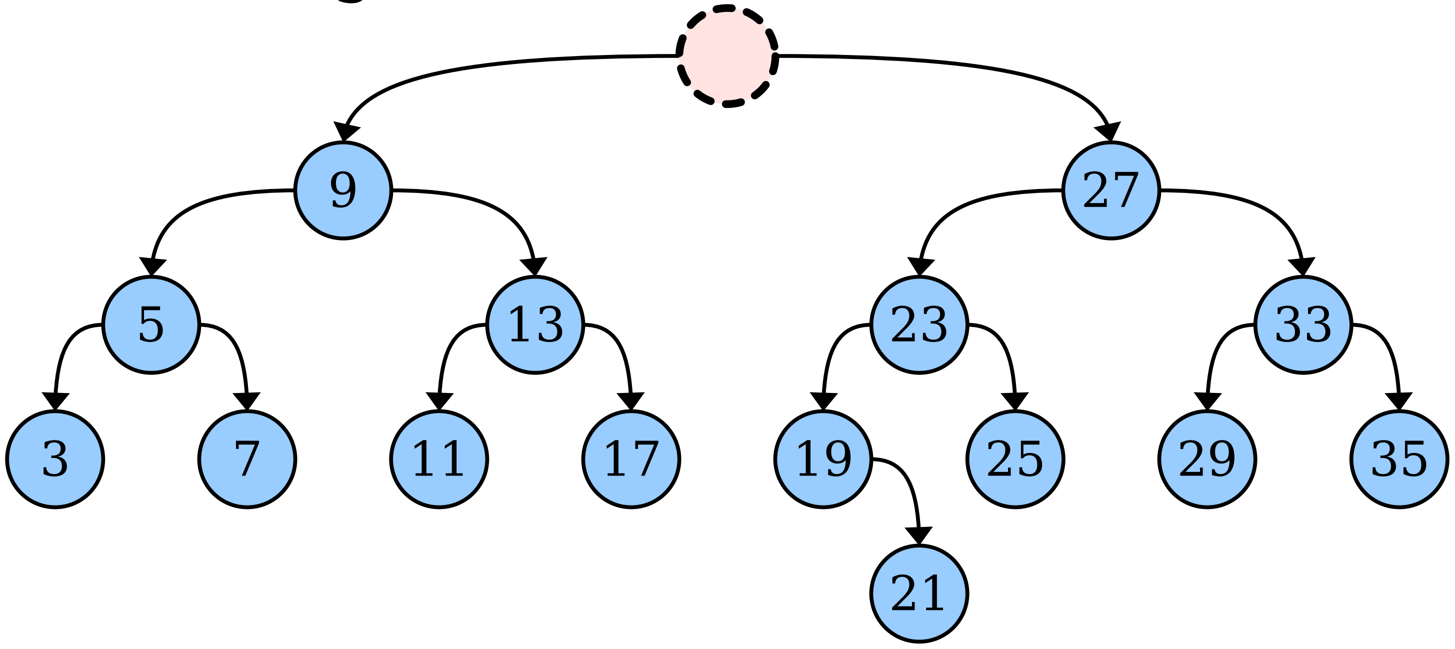

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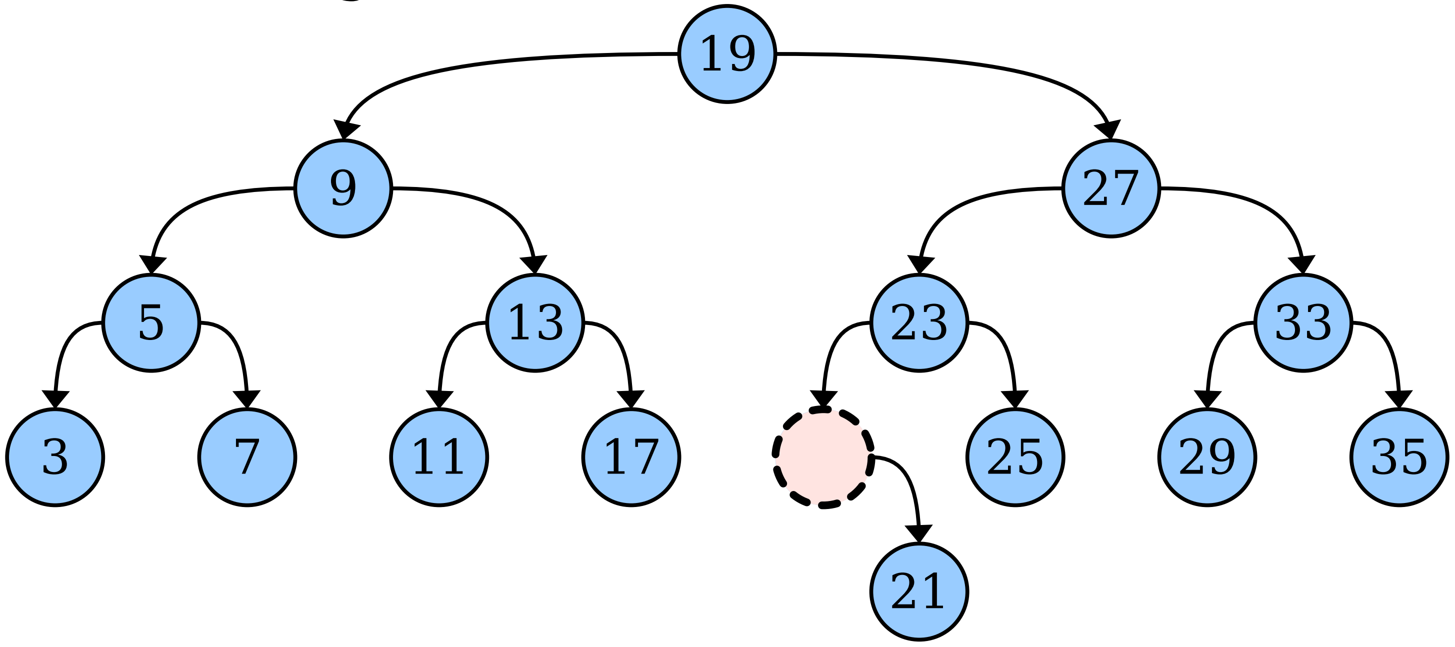

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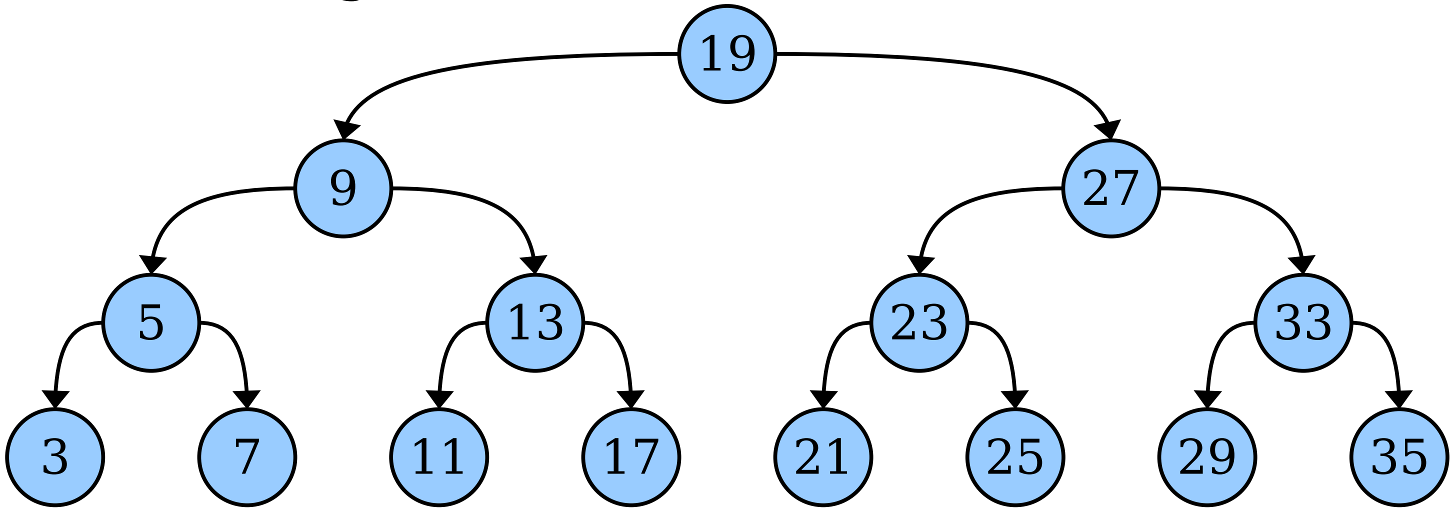

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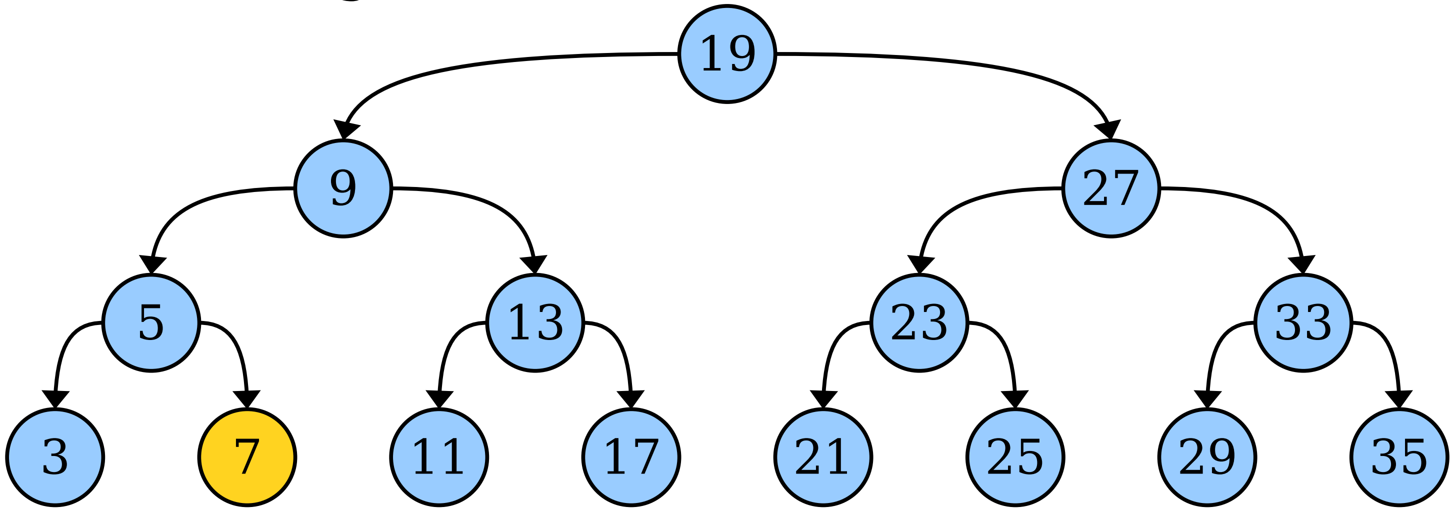

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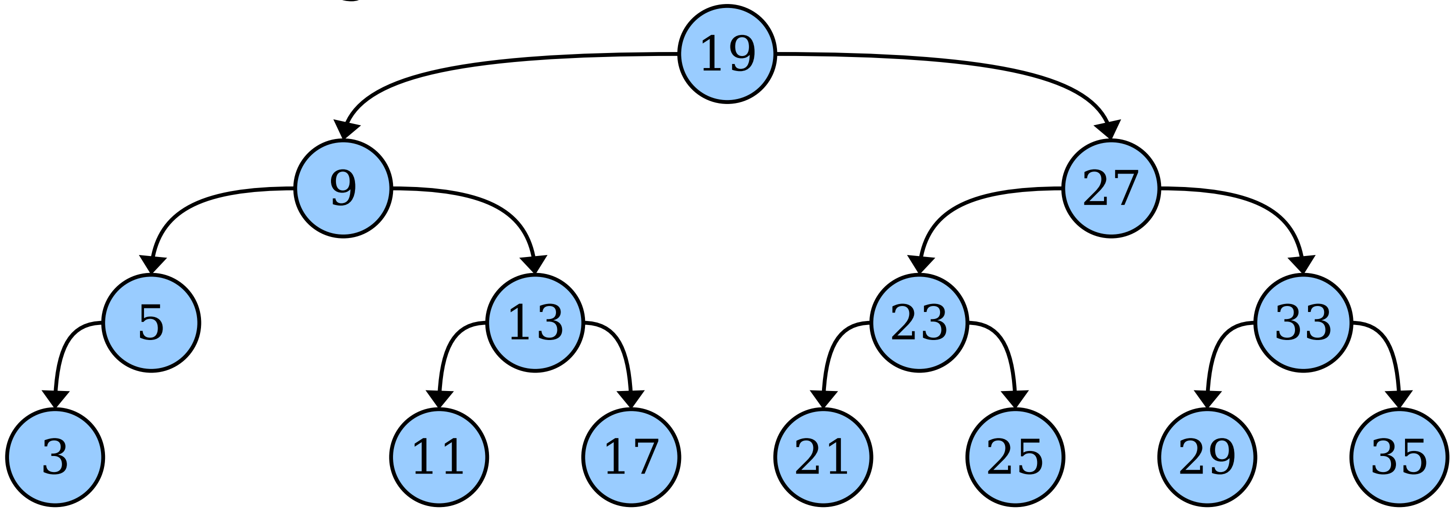

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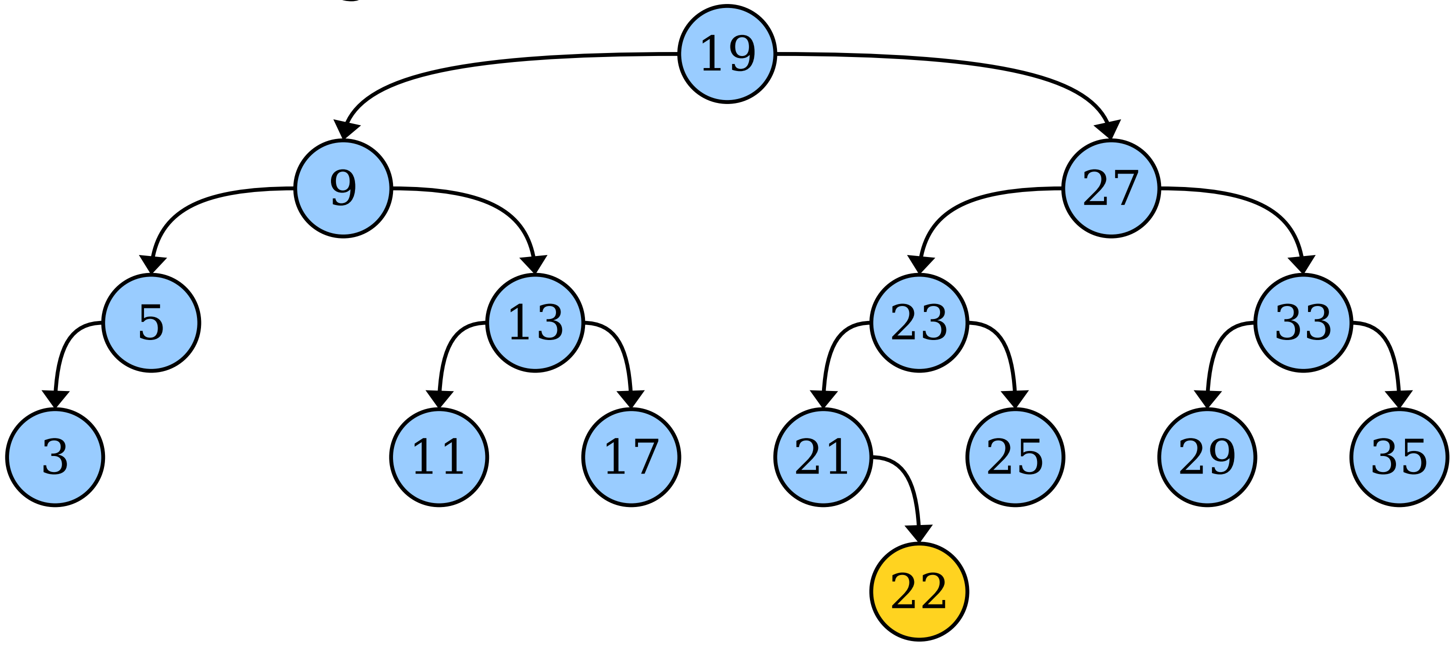

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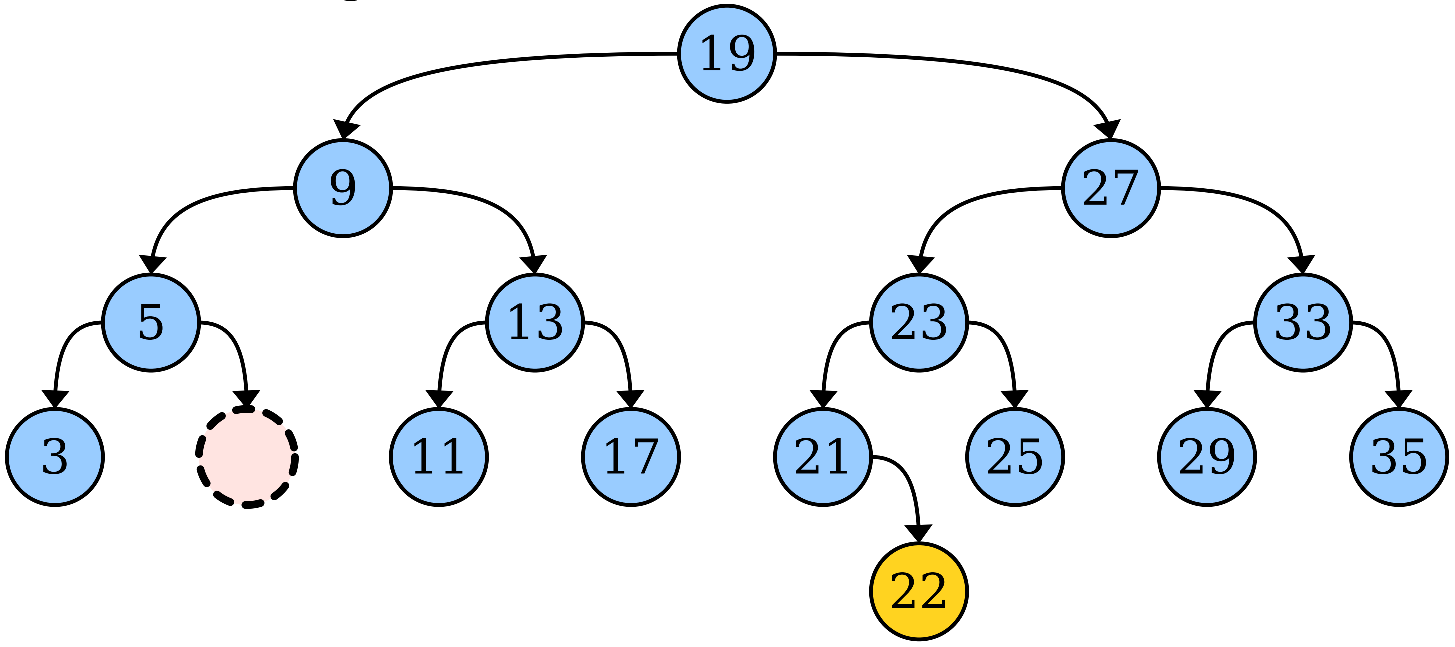

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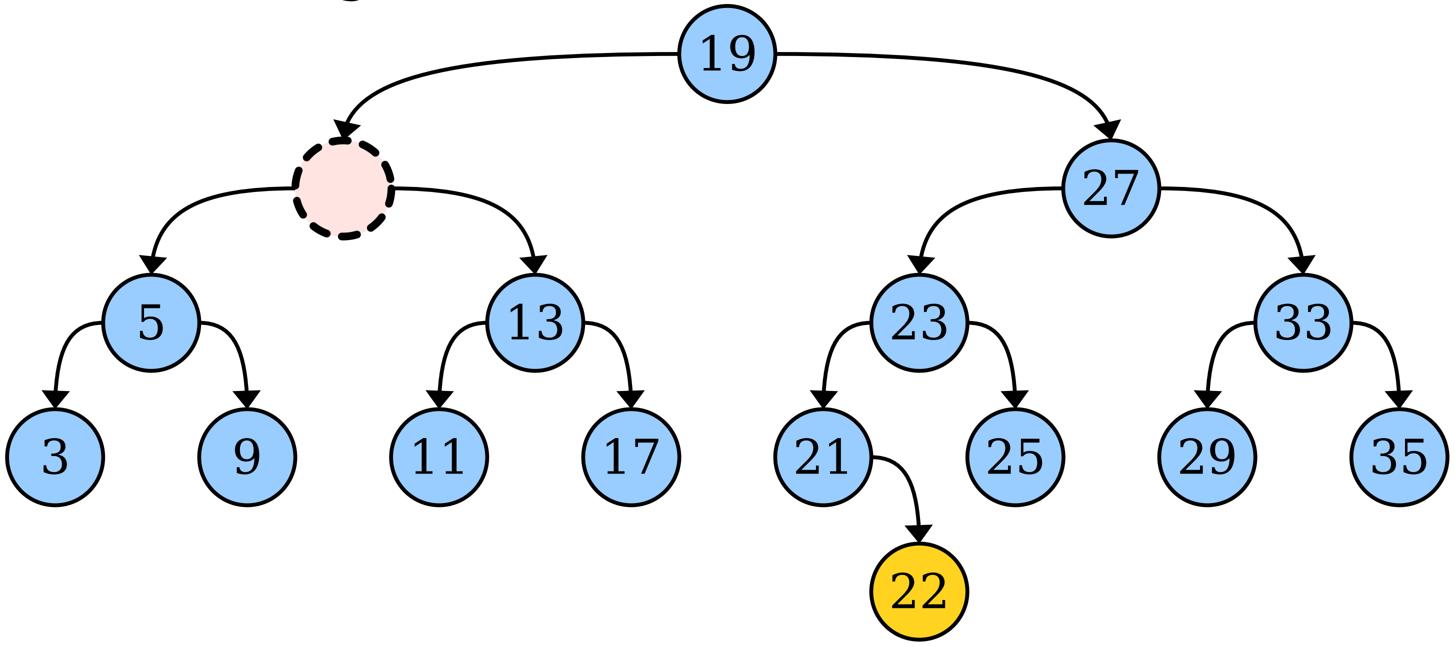

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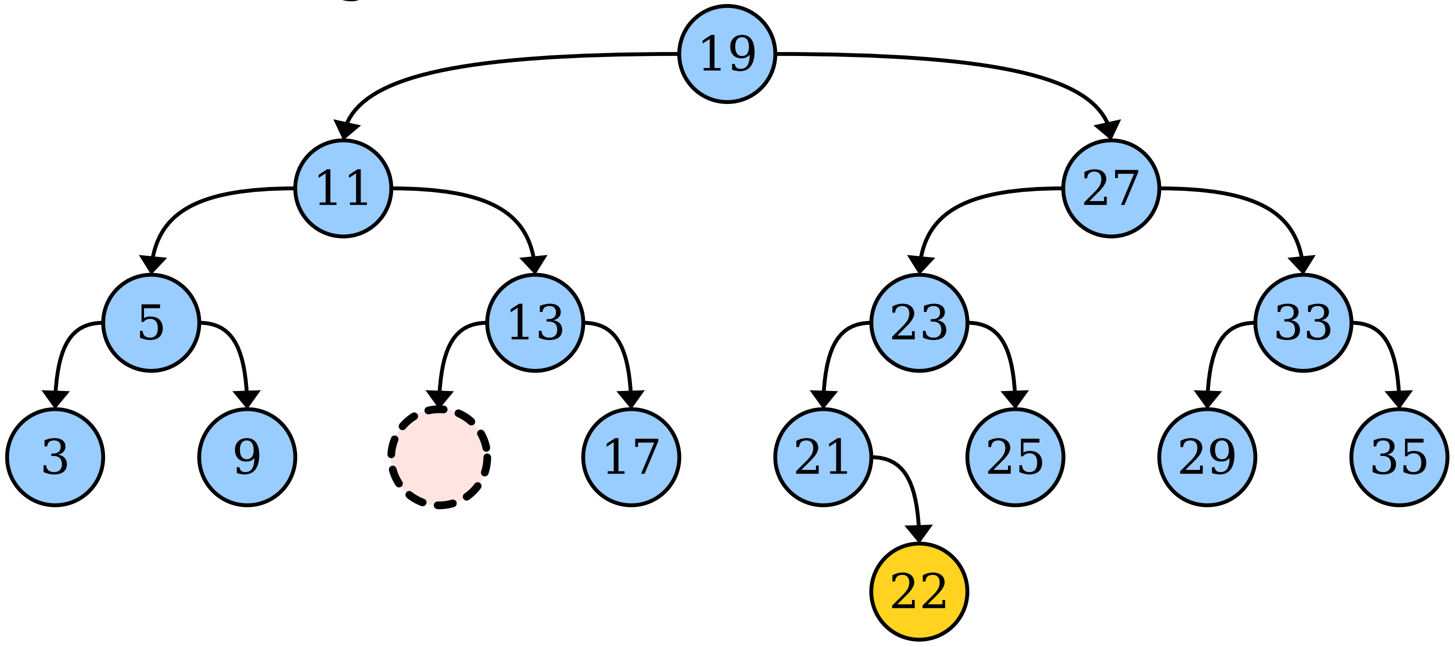

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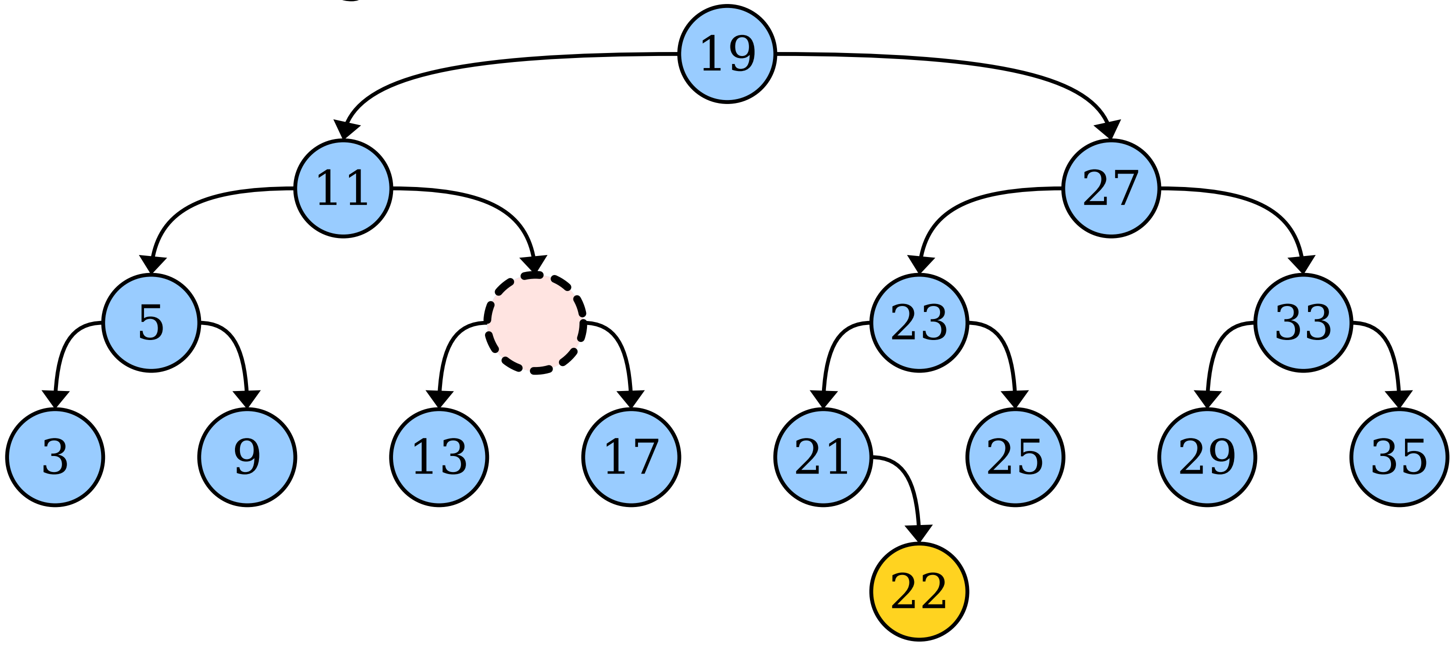

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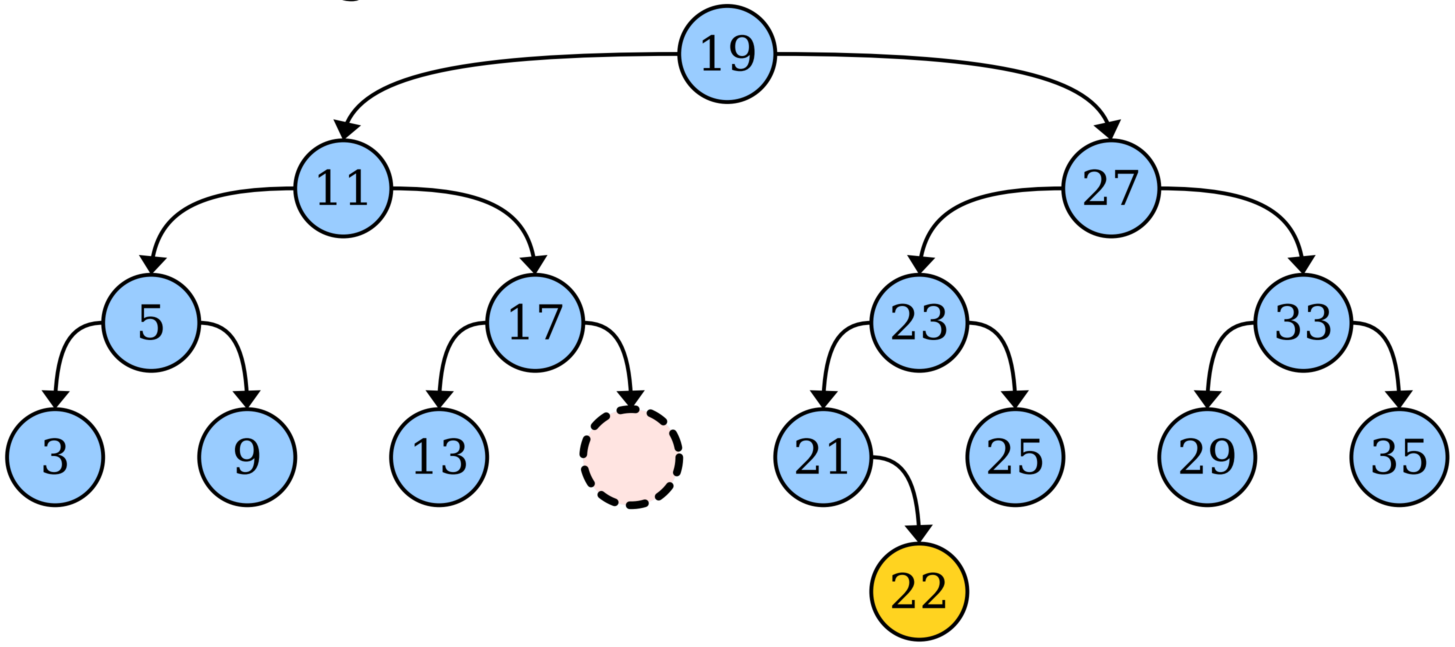

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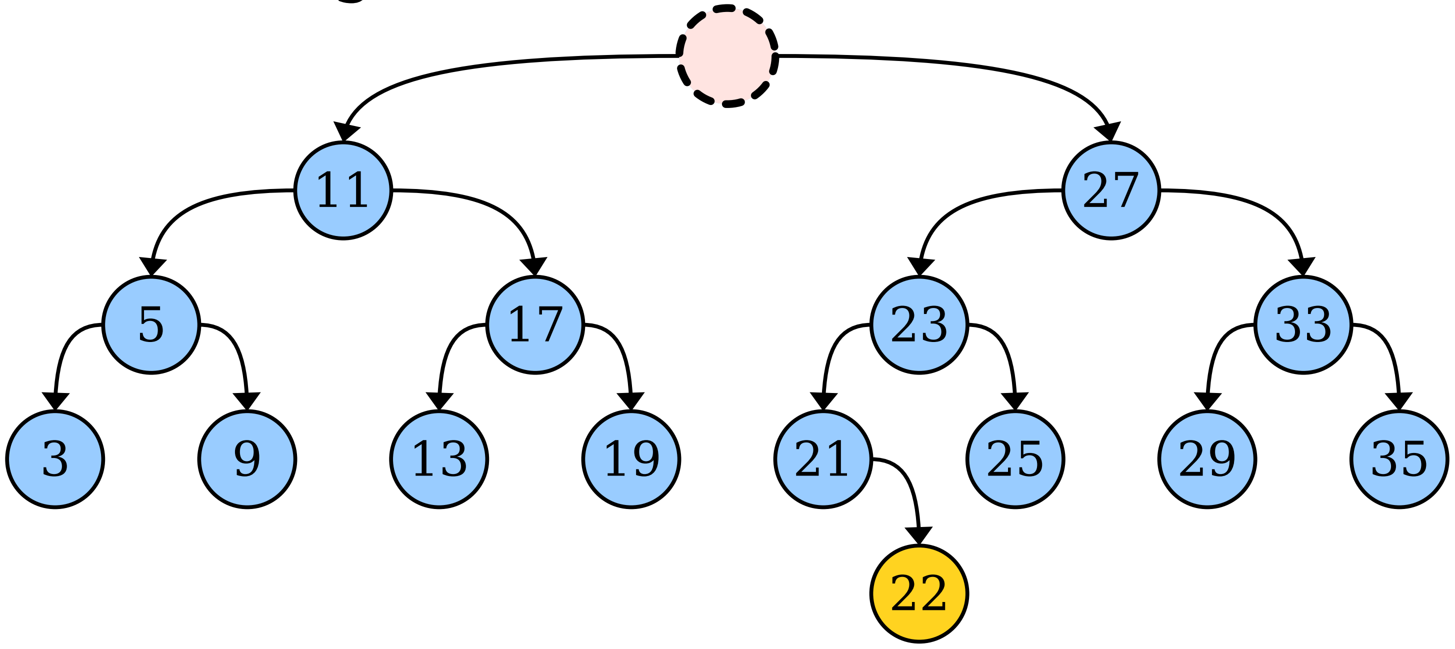

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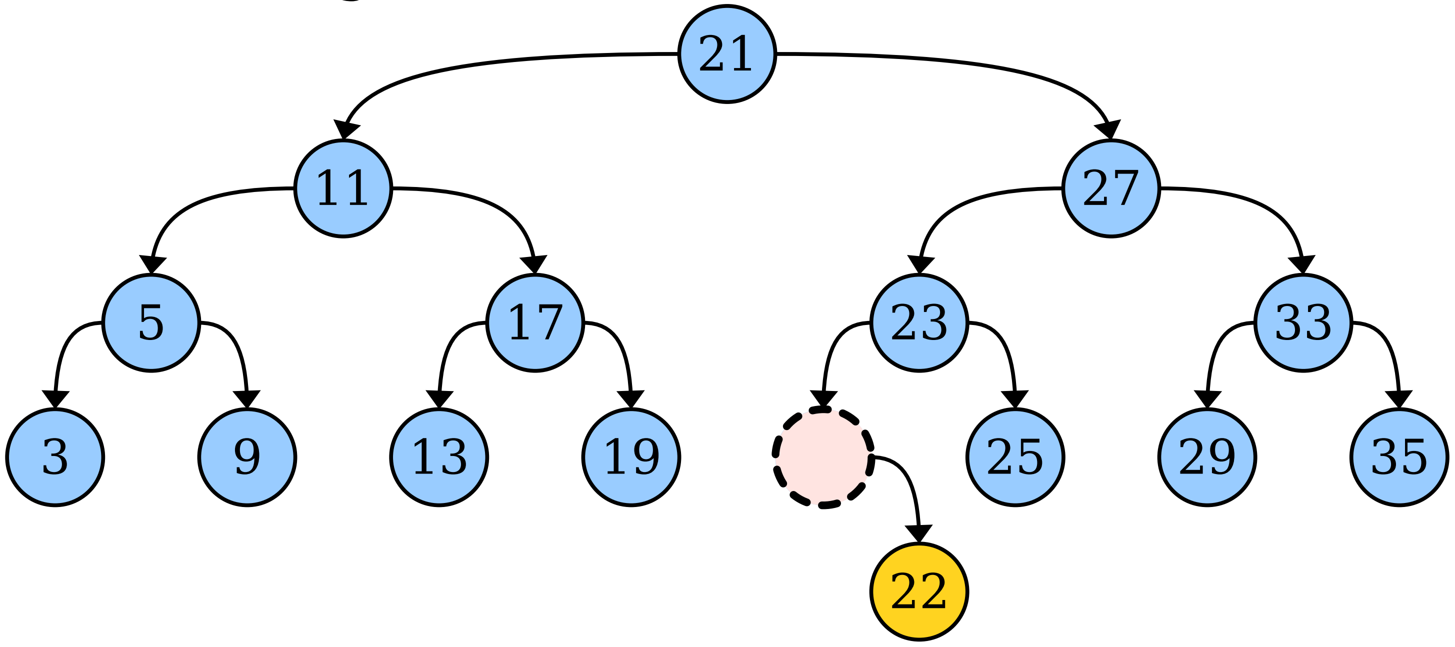

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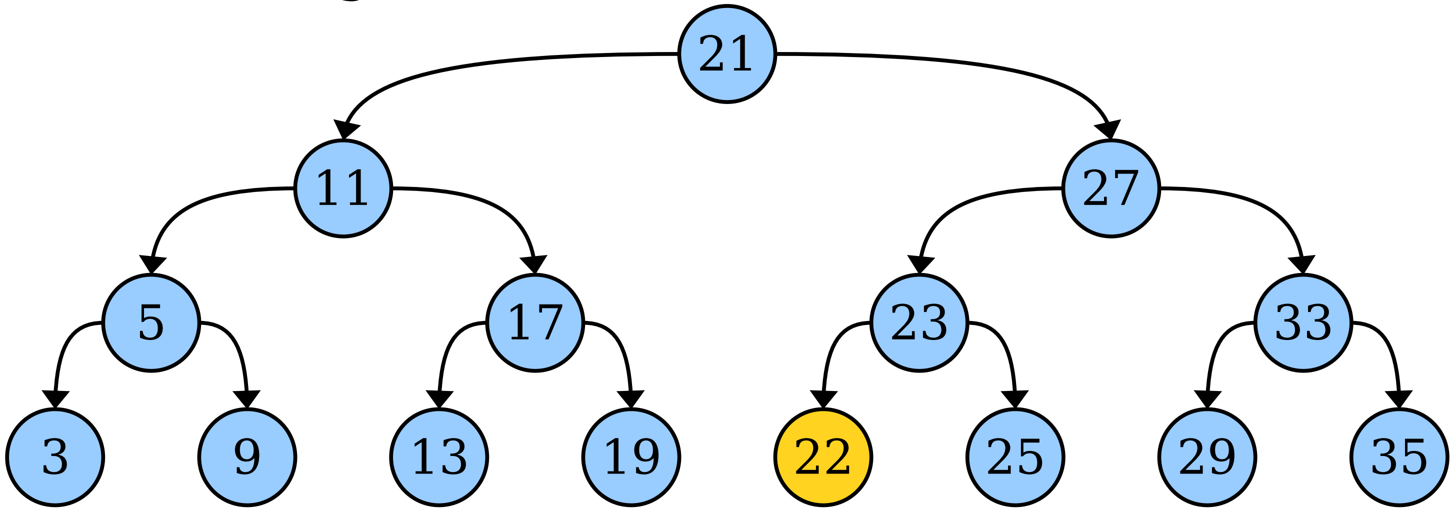

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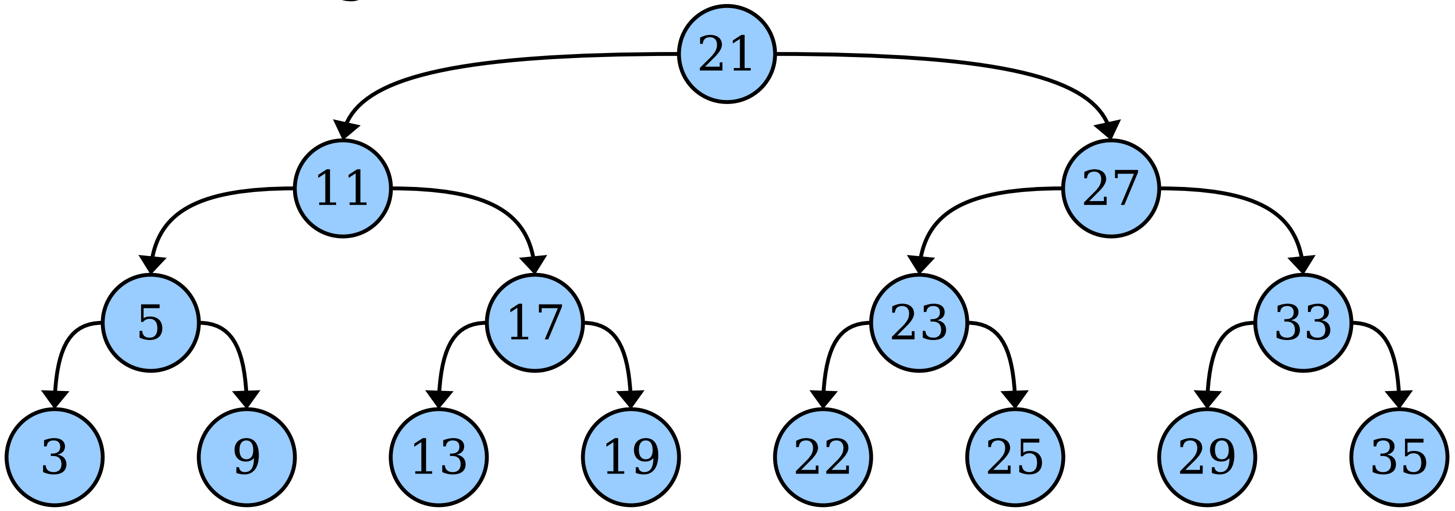
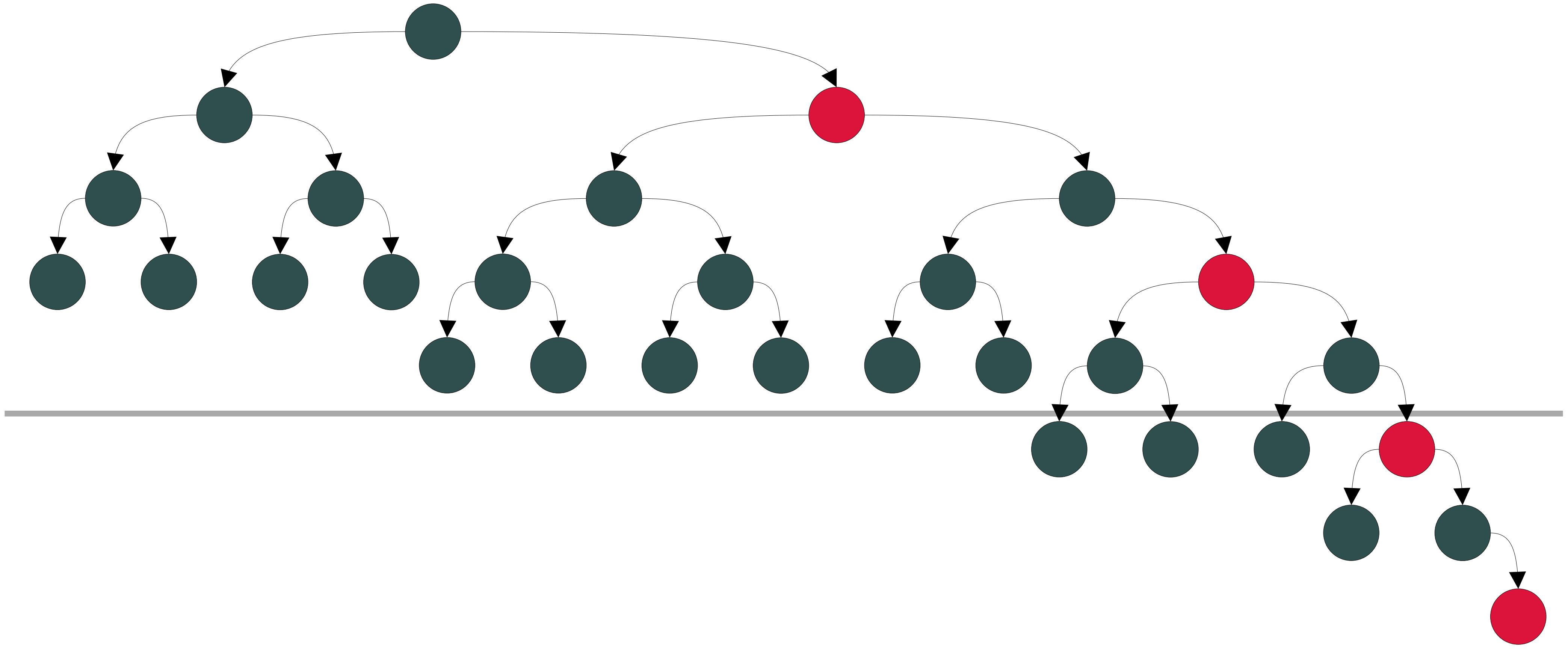


On O(log n) Height

- To speed up logic after insertions or deletions, most balanced BSTs only guarantee height of multiple of lg n.
- For example, red/black trees have height at most (roughly) 2 lg n in the worst case.

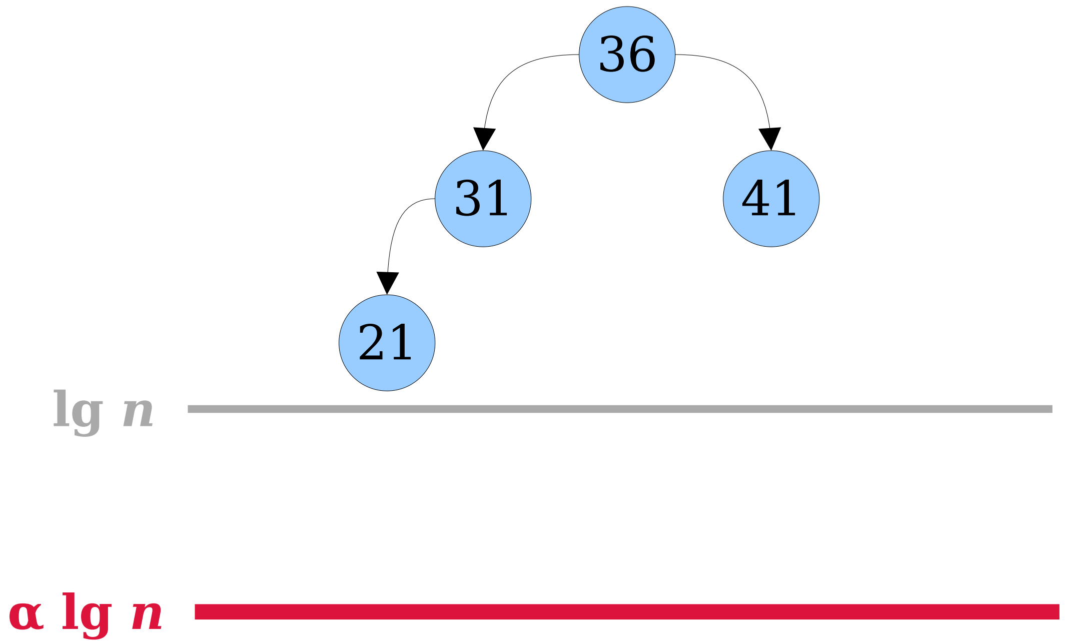


On O(log *n*) Height

- We’re already comfortable with trees whose heights are $\alpha \lg n$ for some α > 1.
- ***Question:*** Can we design a balanced tree purely based on this restriction, without any other structural constraints?

Adding Slack Space

- Pick a fixed constant $\alpha > 1$.
- Set the maximum height on our tree to $\alpha \lg n$.
- As long as we don't exceed this maximum height, all operations on our BST will run in time $O(\log n)$, and we don't really care about the shape of the tree.



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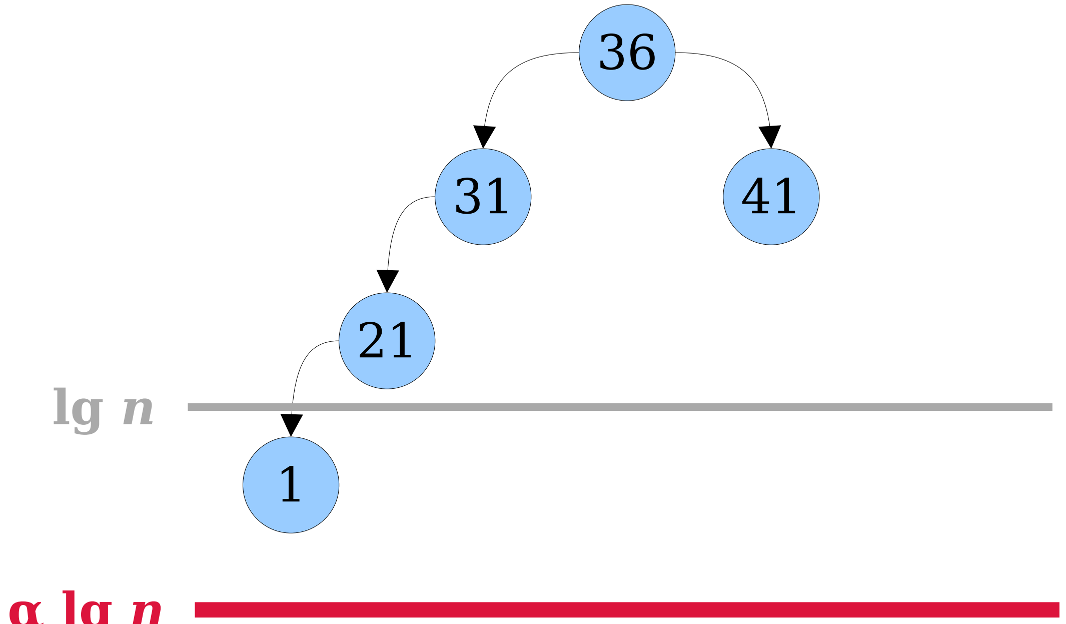

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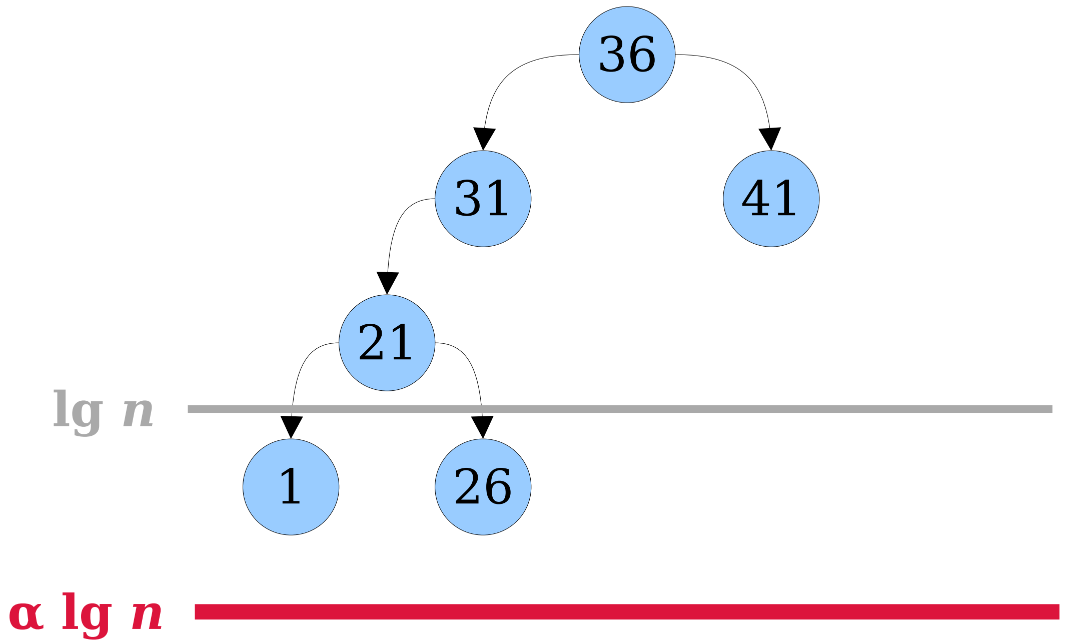

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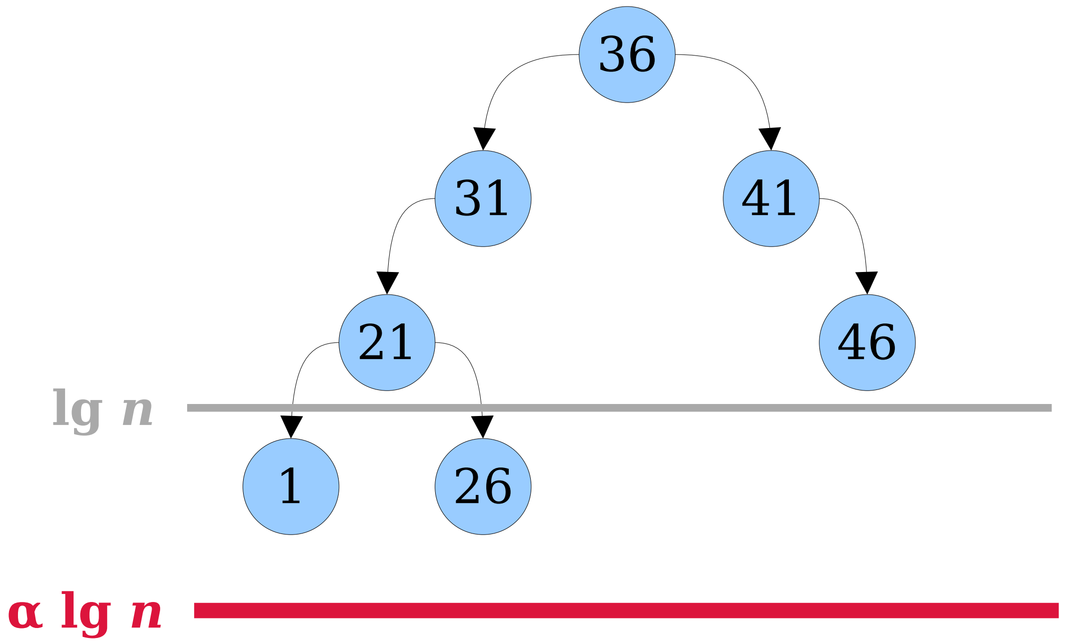

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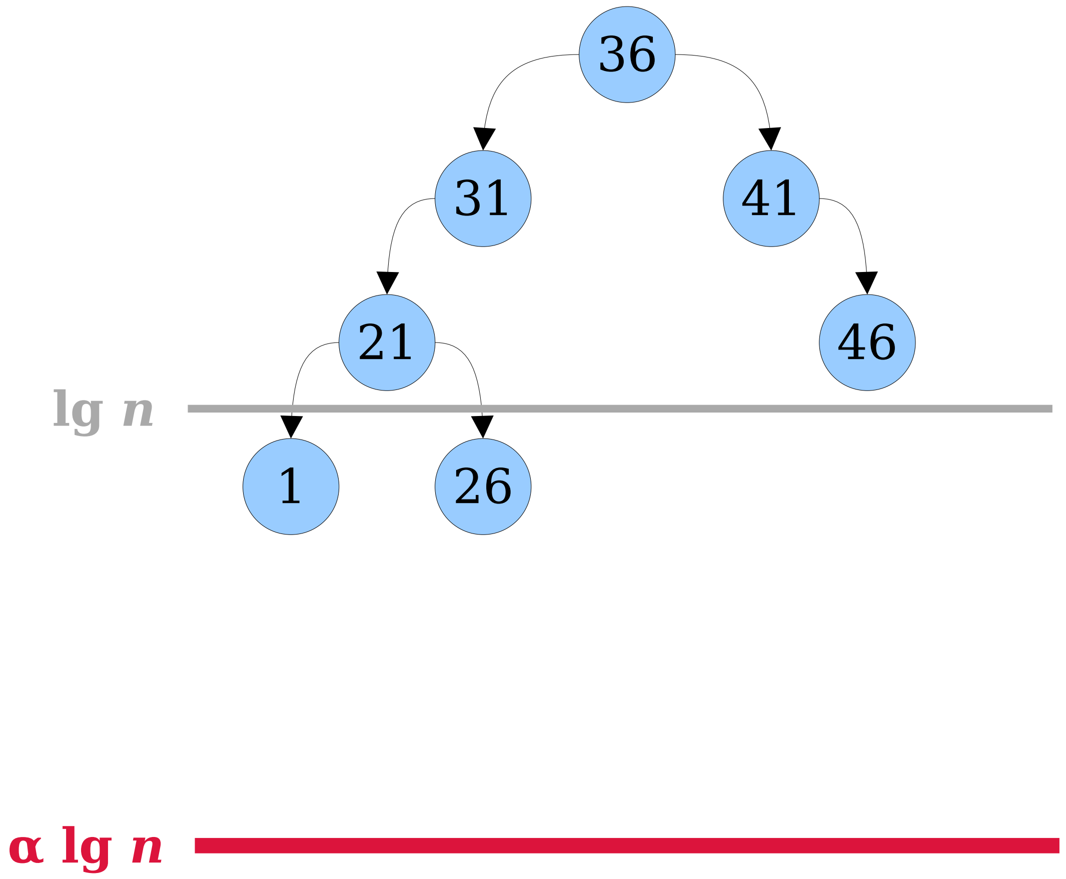

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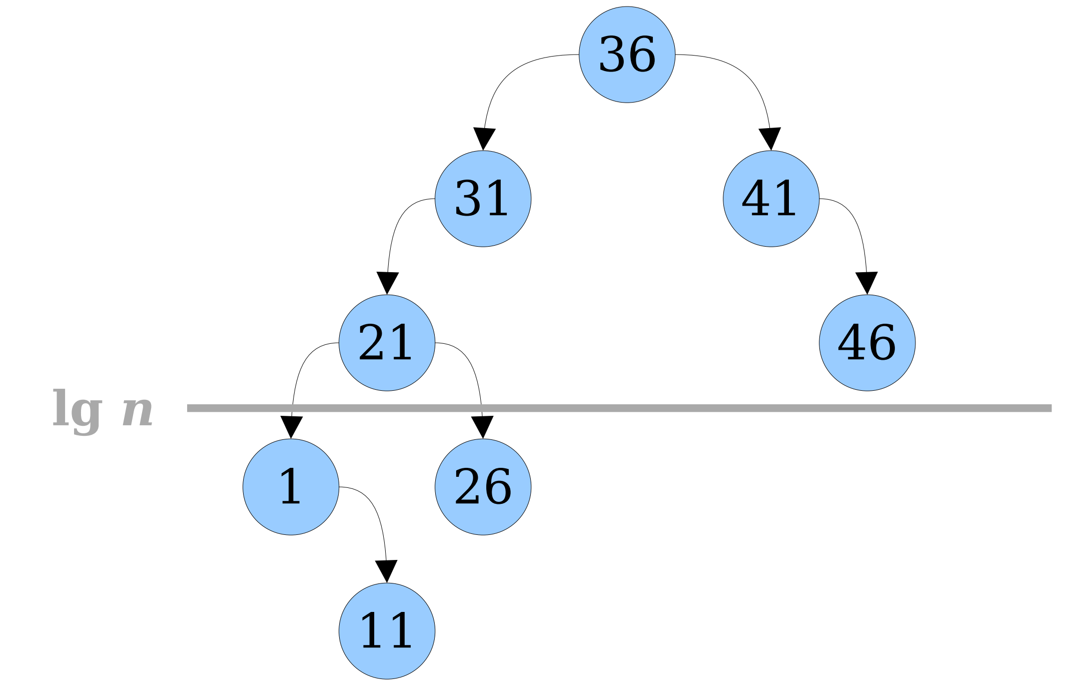

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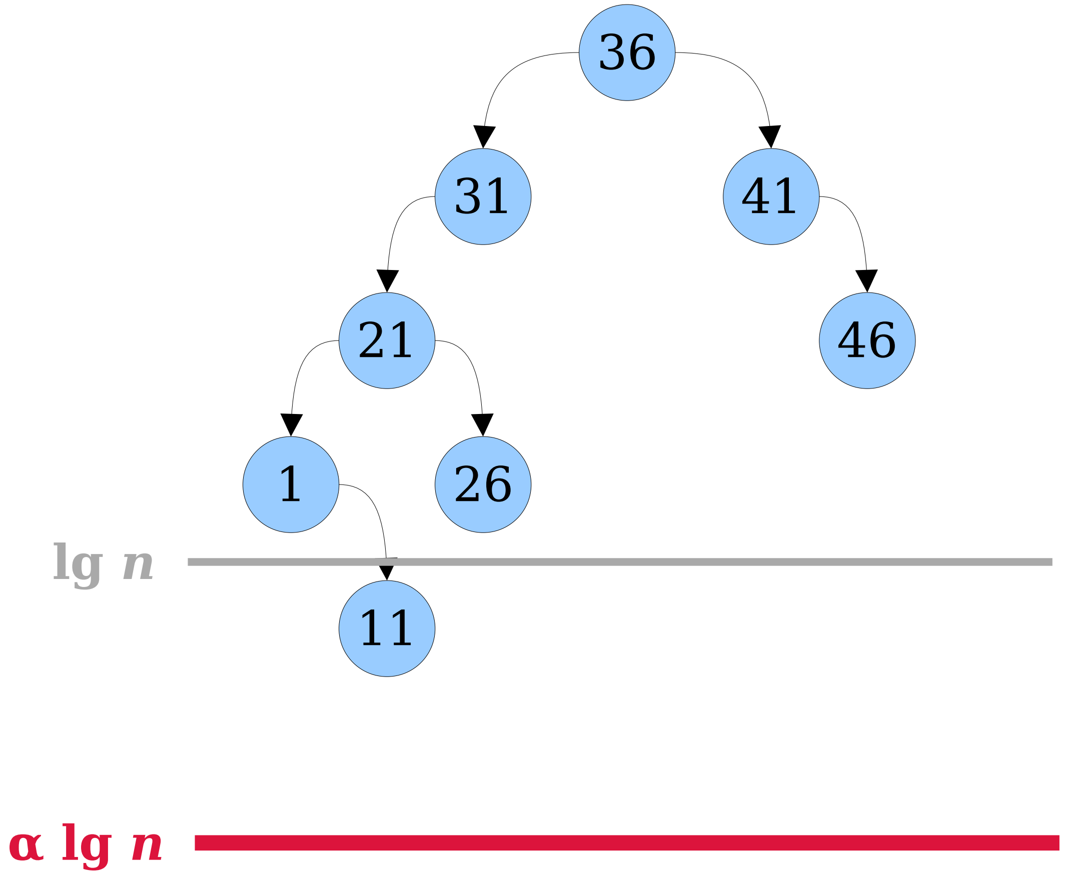

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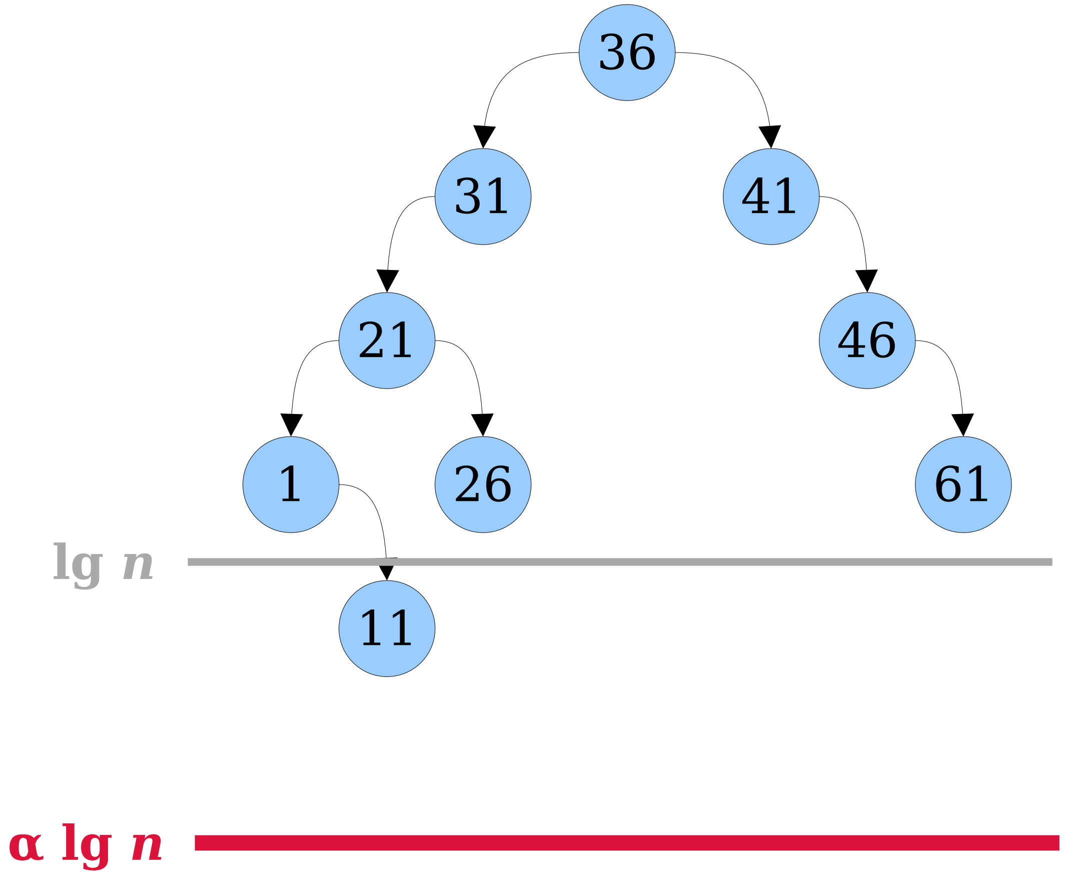

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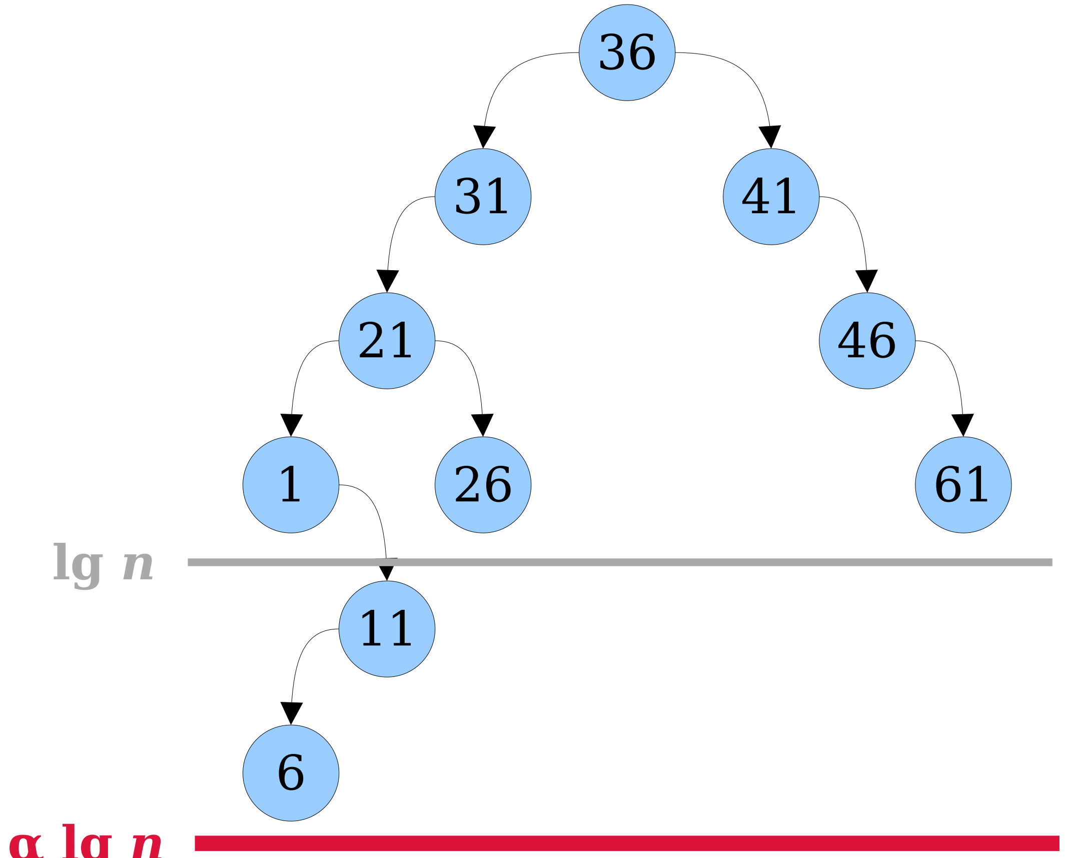

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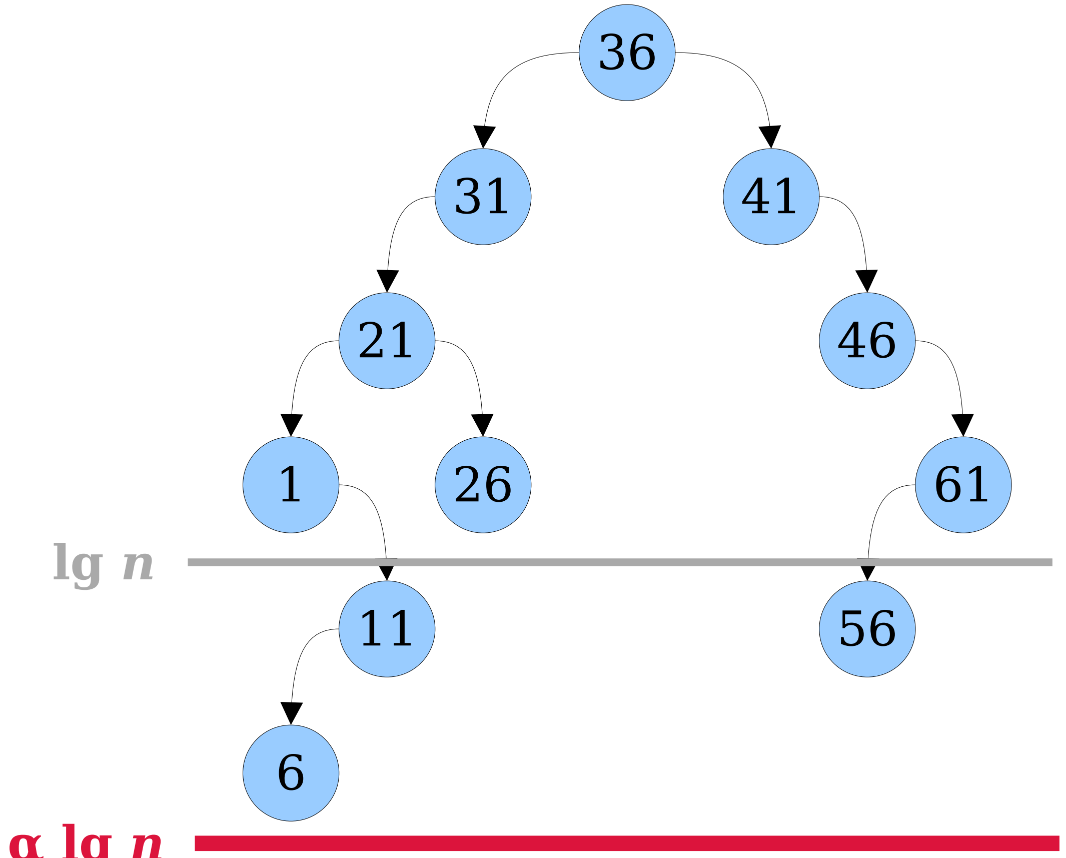

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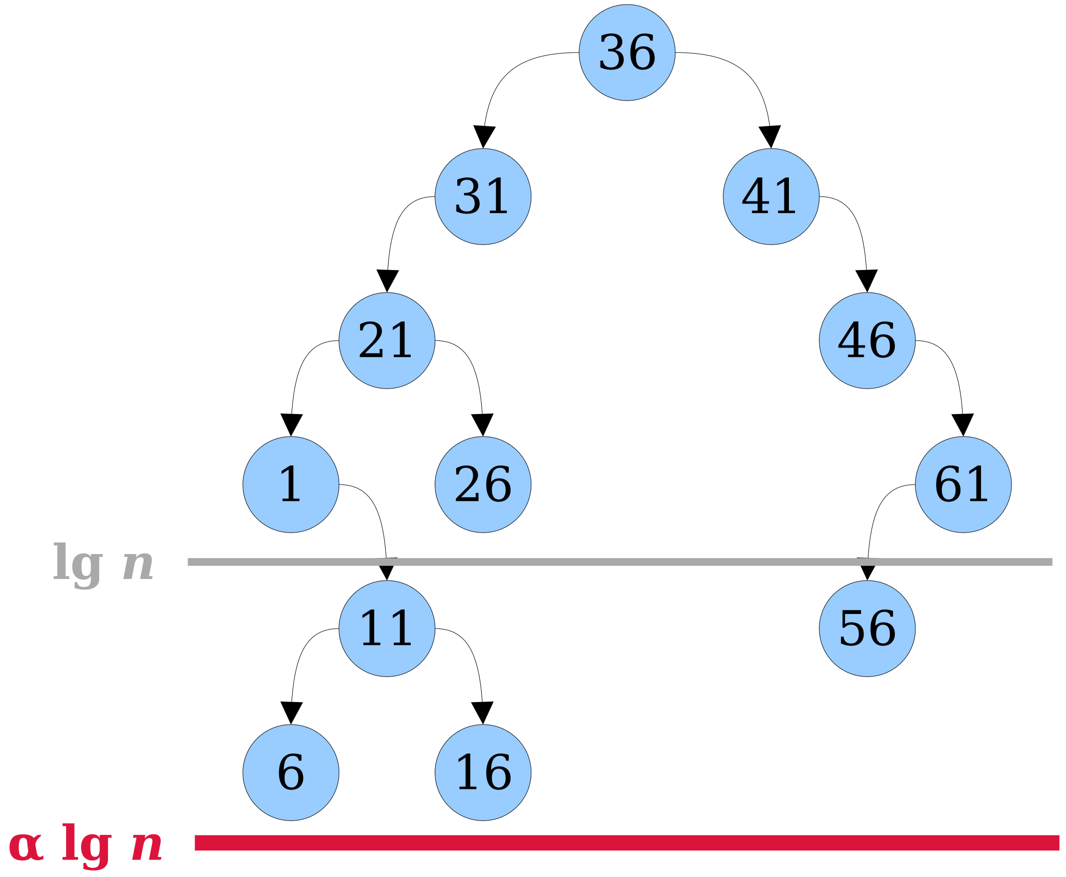

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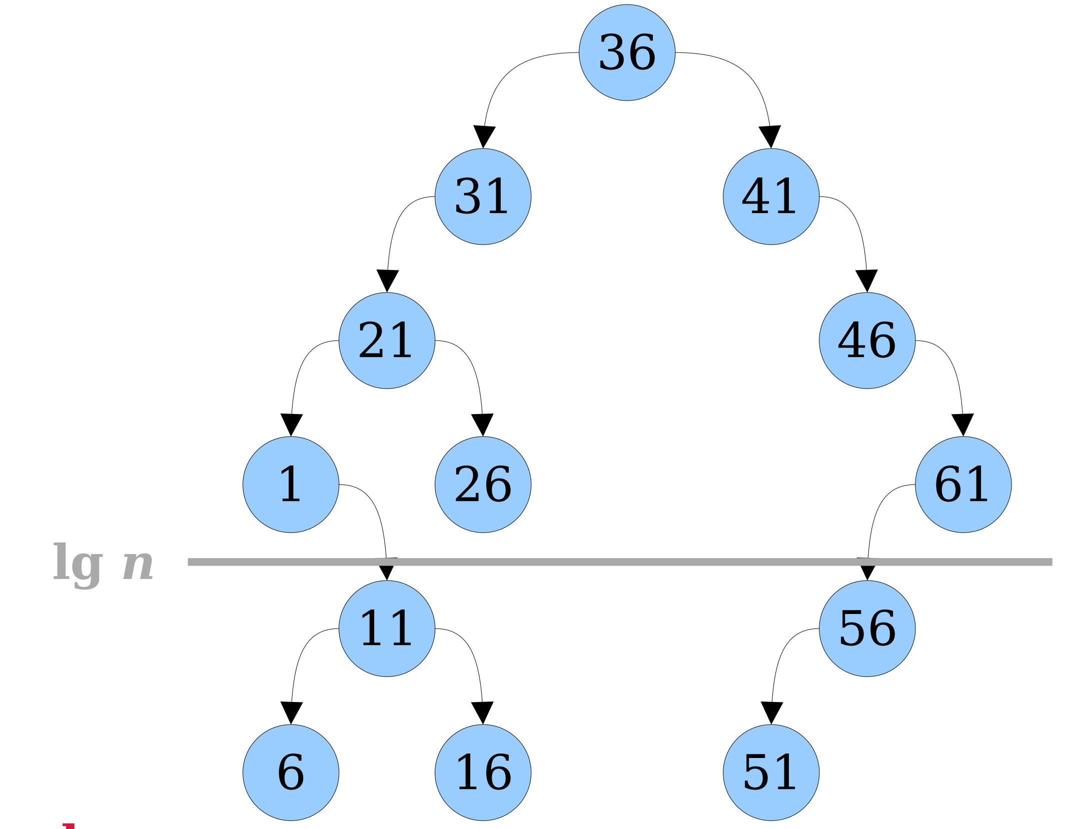
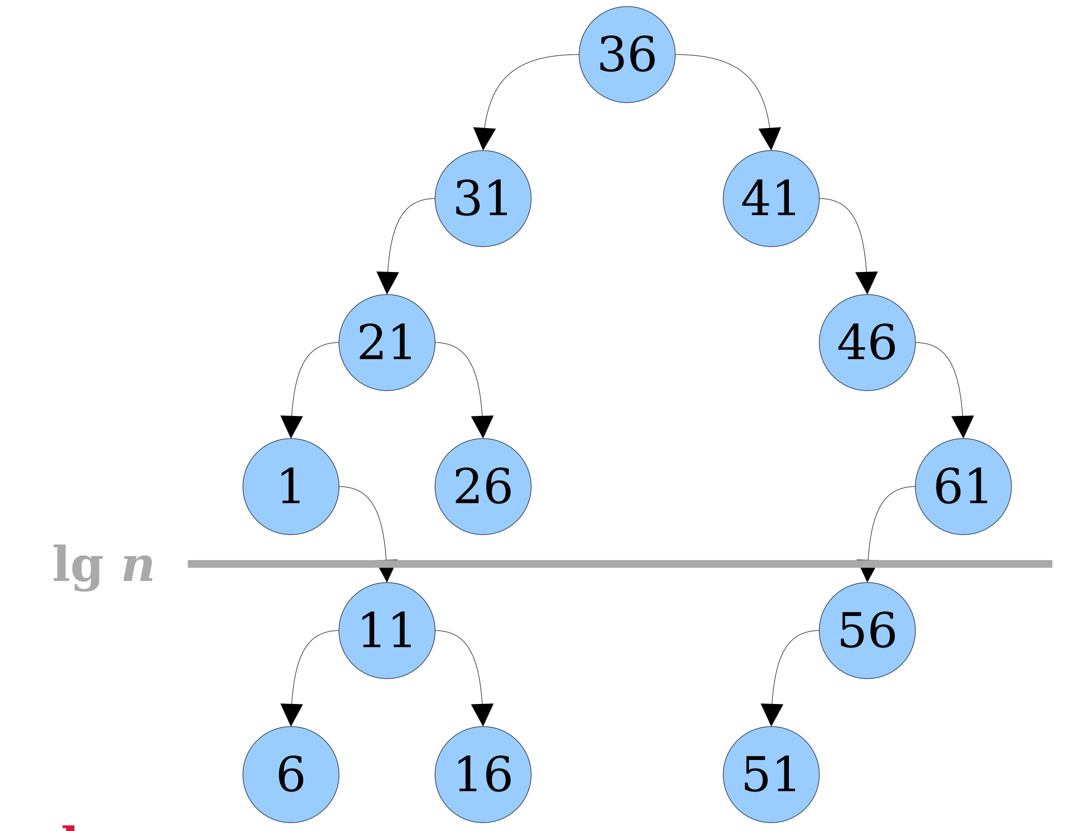


Adding Slack Space

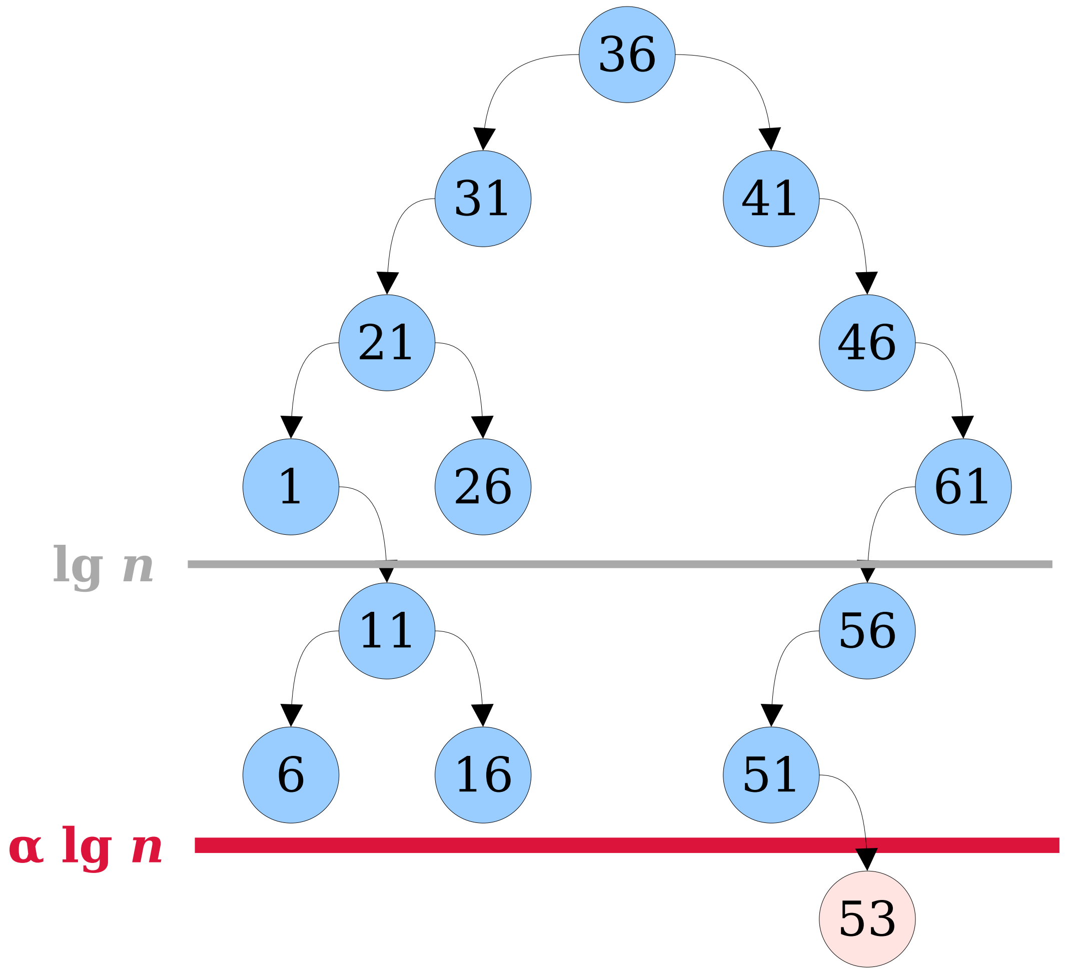
- For each node v in our BST, let **size(v)** denote the number of nodes in the subtree rooted at v and **height(v)** denote the height of the subtree rooted at v.
- We'll say that a node v is ***α-balanced*** if

  height(v) ≤ α lg size(v).
- Intuitively, a α-balanced node is the root of a subtree whose height is within a factor of α of optimal.



Adding Slack Space

- Suppose, however, that after doing an insertion, our tree exceeds $\alpha \lg n$.
- At this point, we need to do some sort of "cleanup" on the tree to pull it back to a reasonable height.
- Ideally, we'll want to minimize the amount of cleanup we need to do so that this step will run quickly.



Scapegoat Nodes

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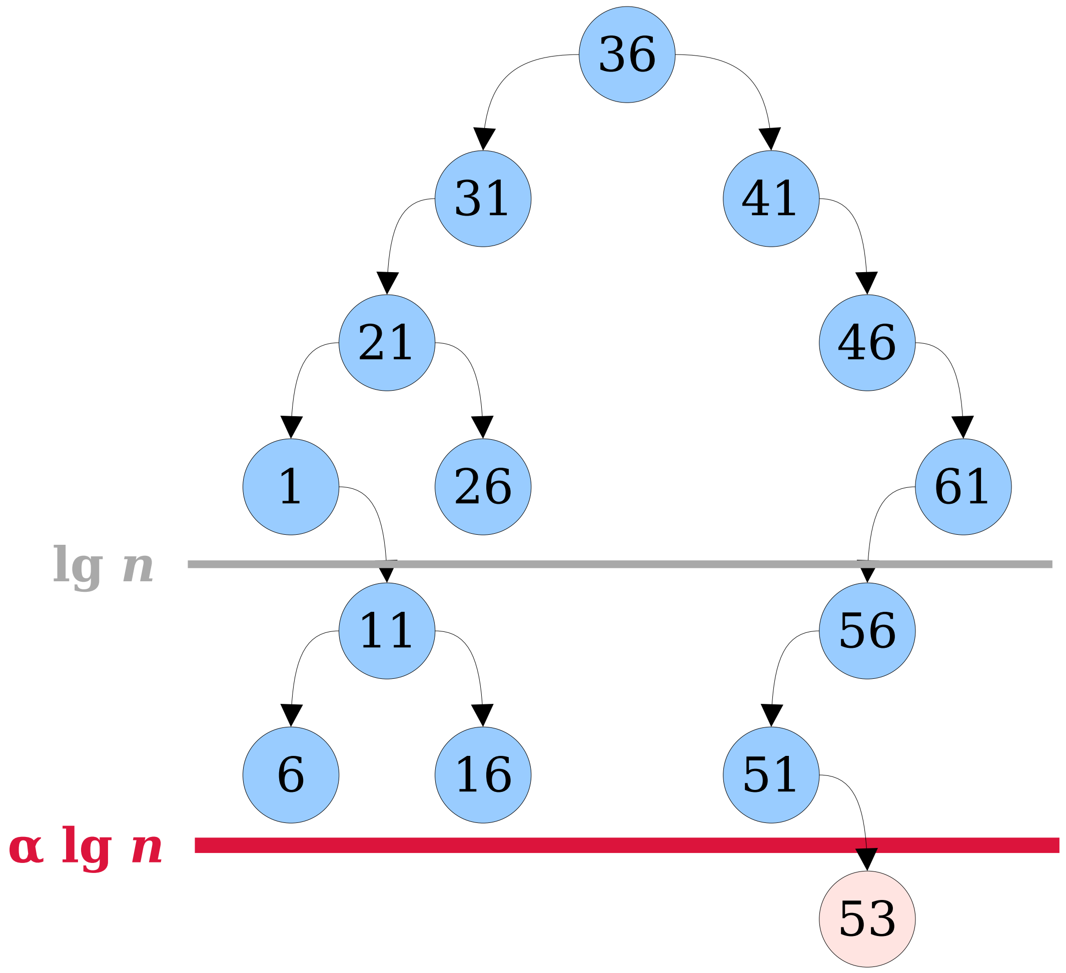

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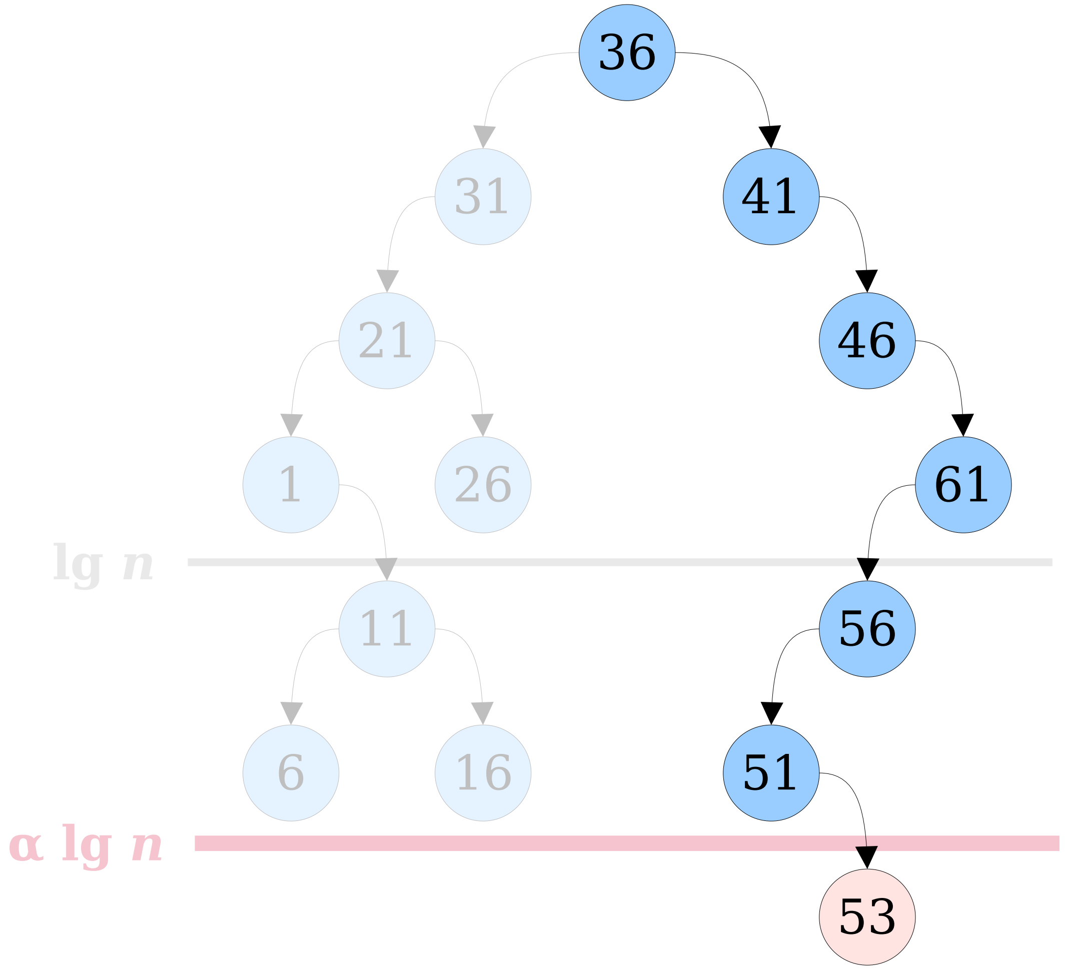

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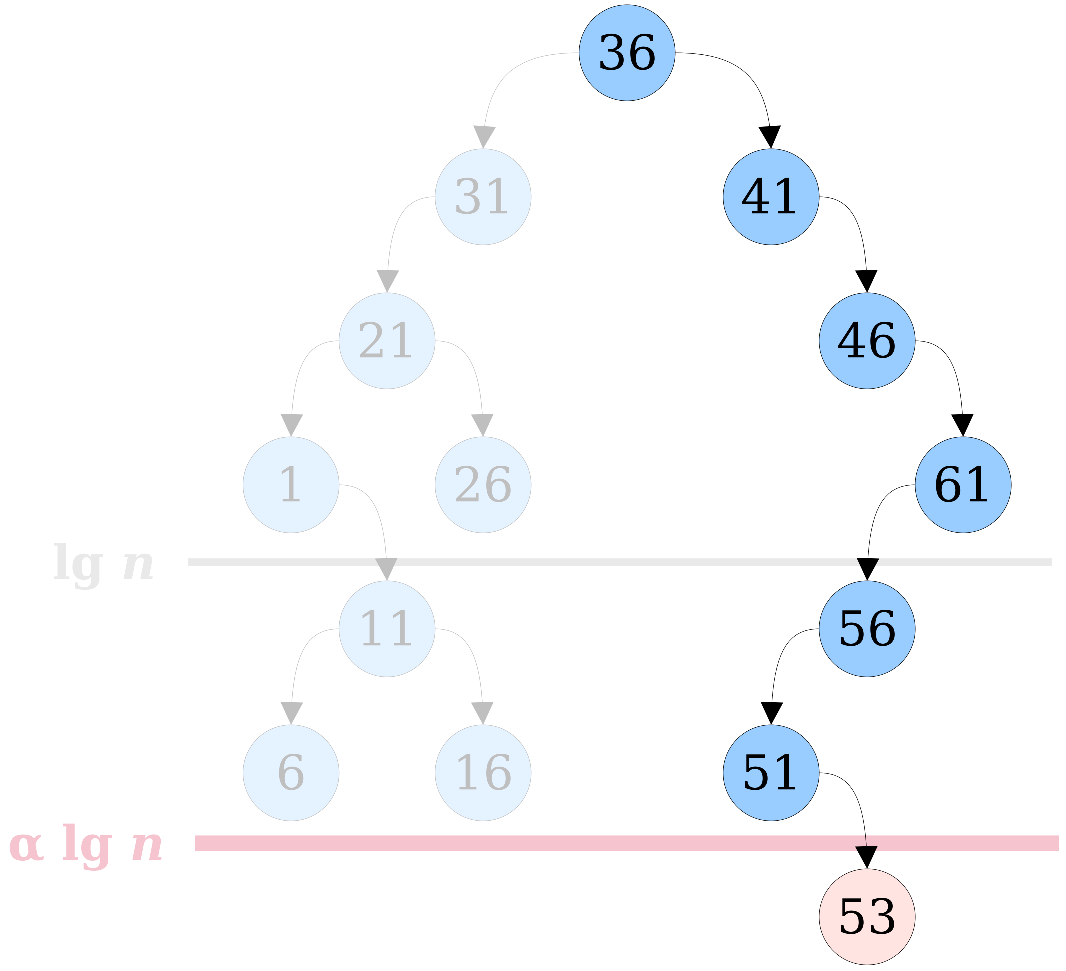
- Look at the access path from the root node to the newly-inserted node.
- We know the root node is not α-balanced, since the overall tree is too tall.



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- Look at the access path from the root node to the newly-inserted node.
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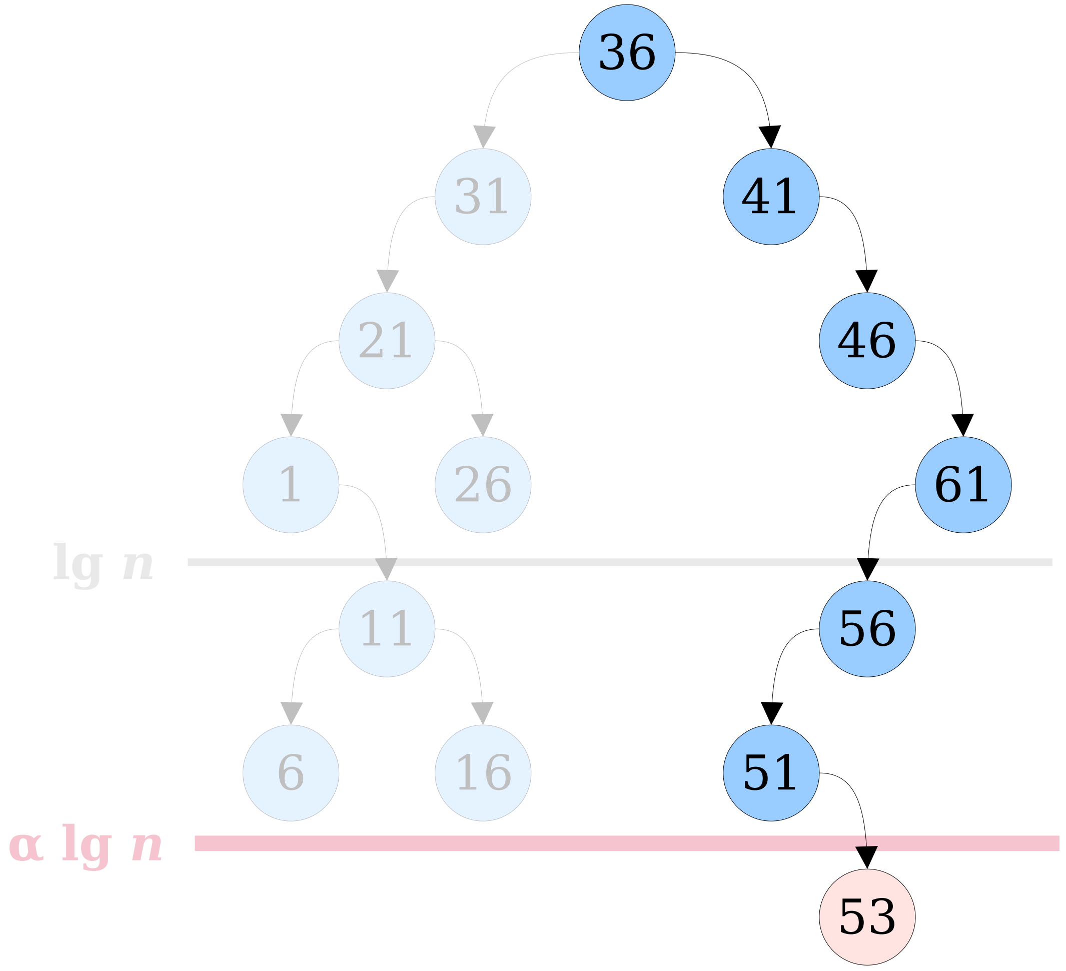
- We also know that the newly-inserted node *is* α-balanced, since it has no children.



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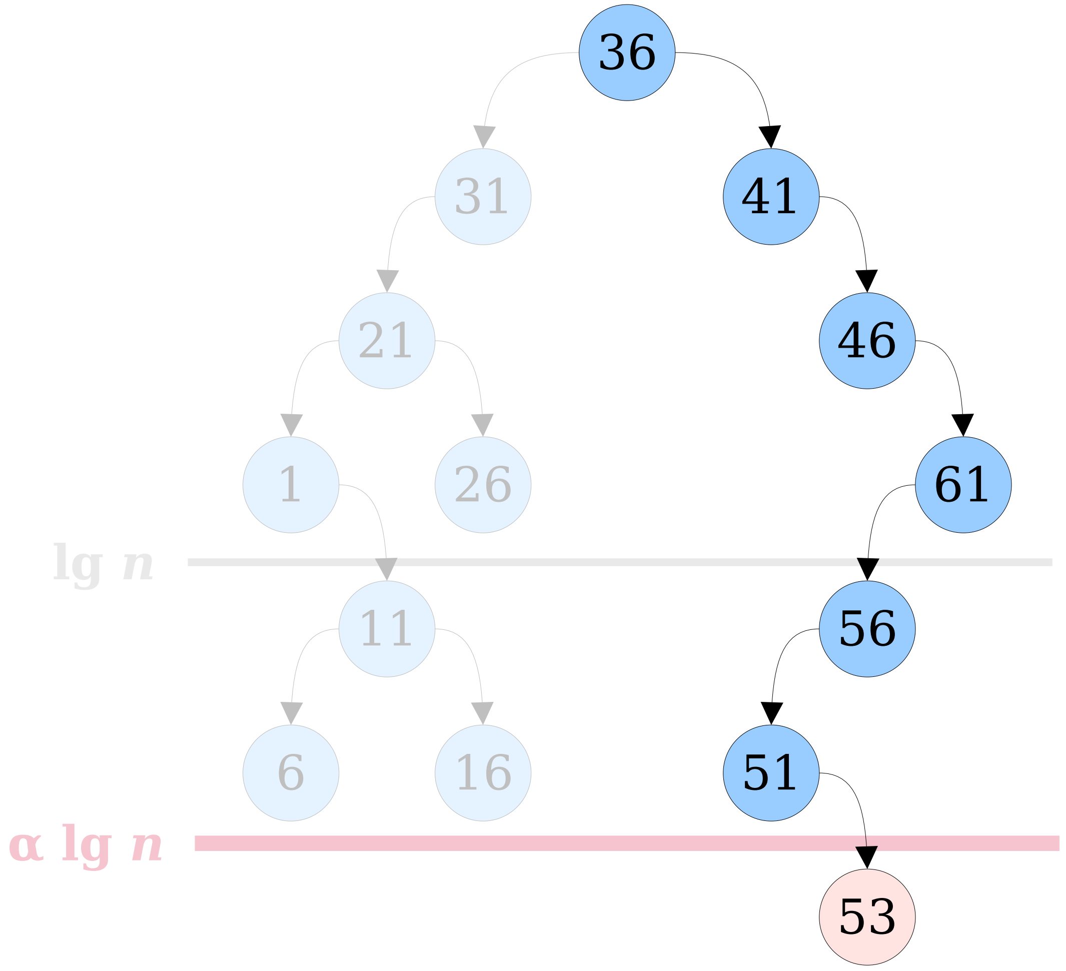

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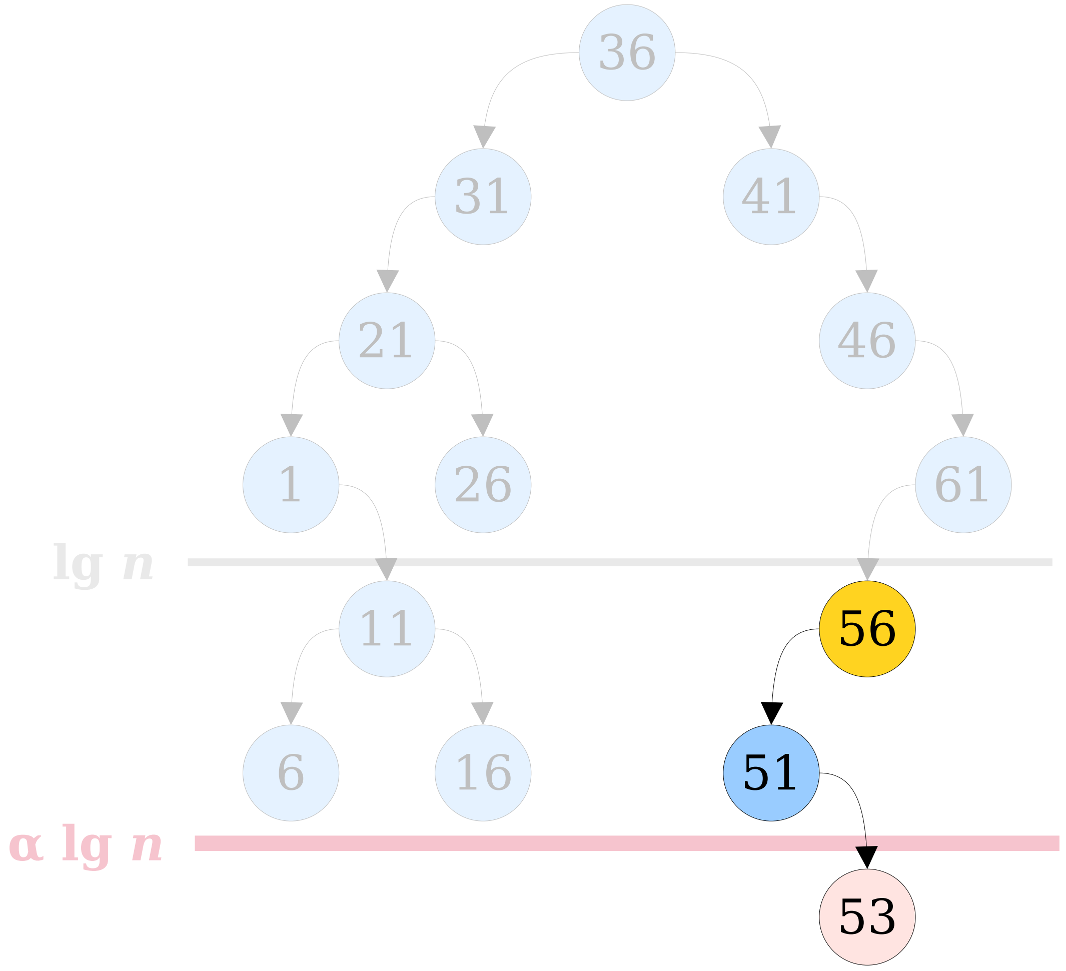

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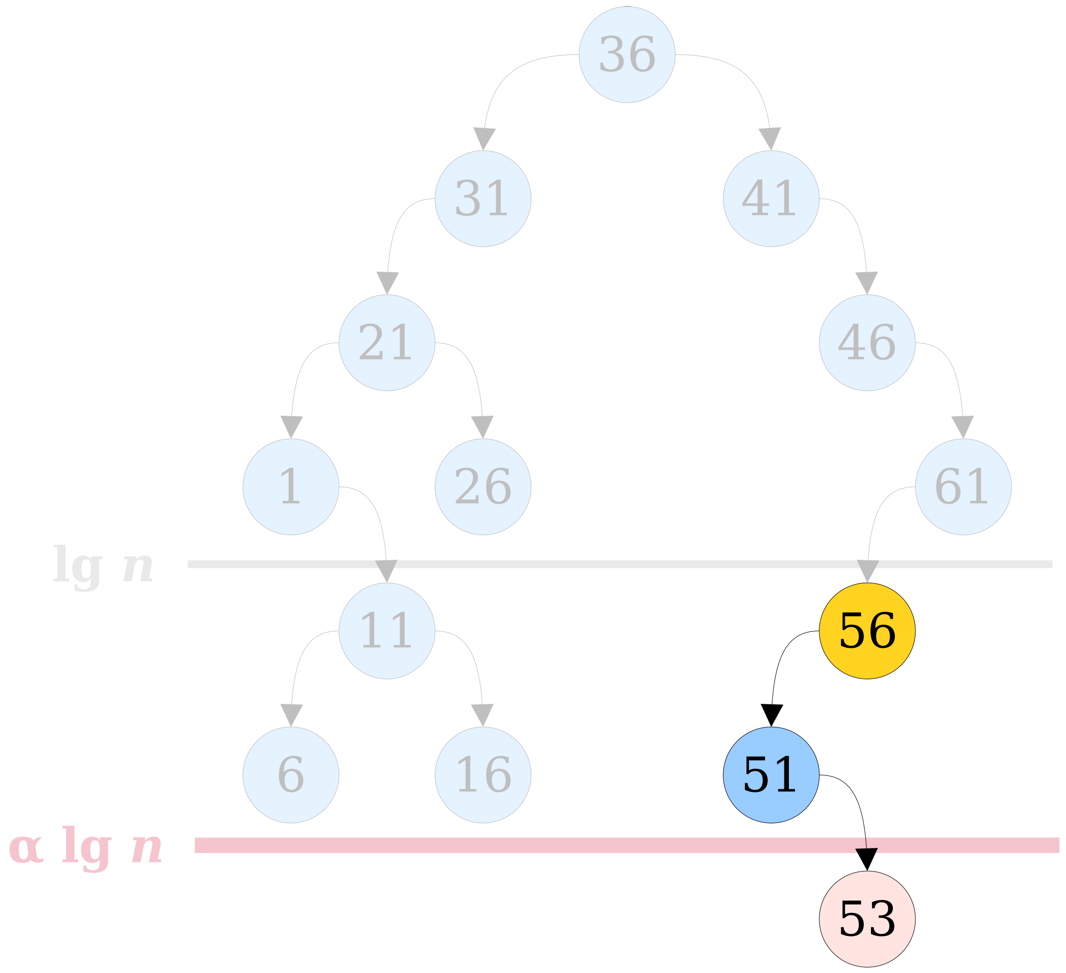
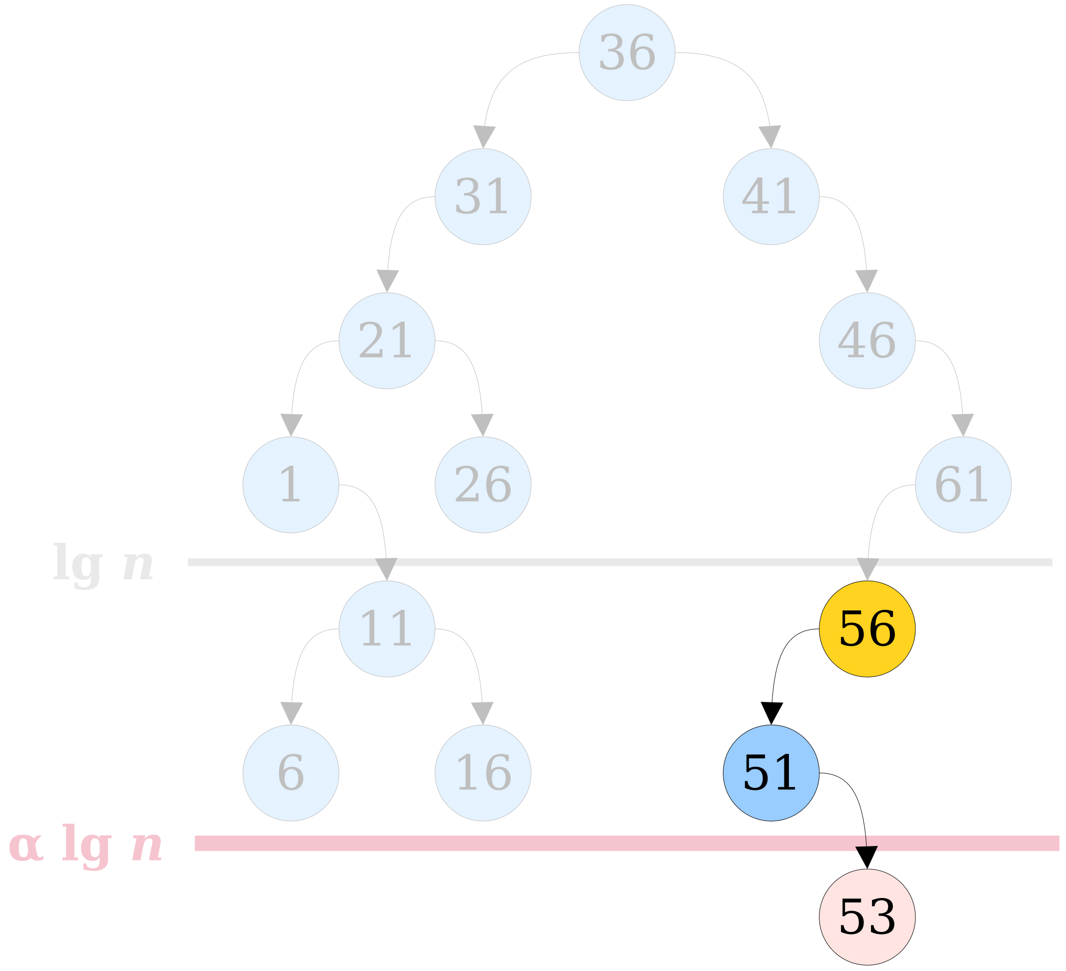
- Therefore, there has to be some deepest node on the access path that isn't α-balanced.
- We can "blame" the imbalance in the overall tree on this subtree. The node chosen this way is called the ***scapegoat***.



Scapegoat Nodes

- We know that the subtree rooted at the scapegoat isn't α-balanced.
- ***Idea:*** Rebuild this tree as a perfectly-balanced BST.
- This will reduce the height of the subtree, which in turn restores the requirement that the height be at most $\alpha \lg n$.



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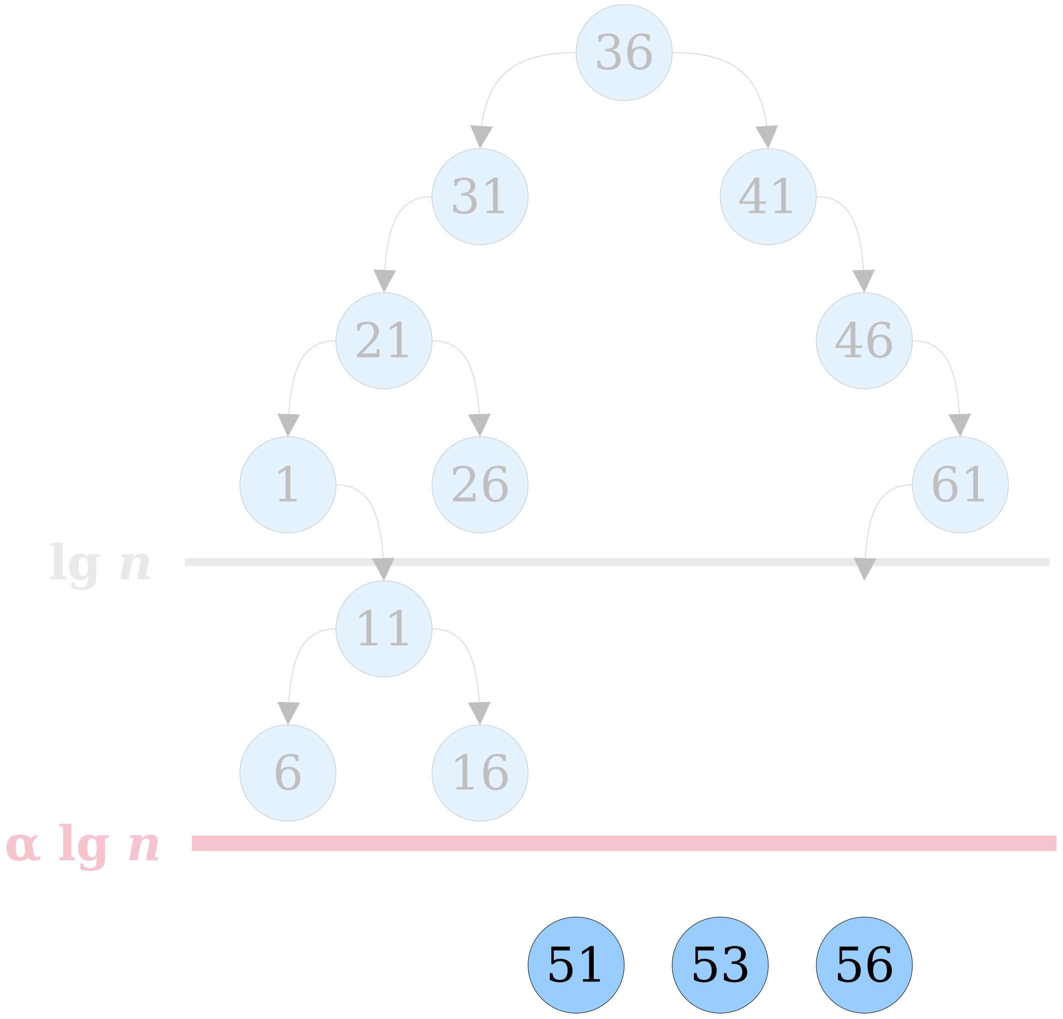

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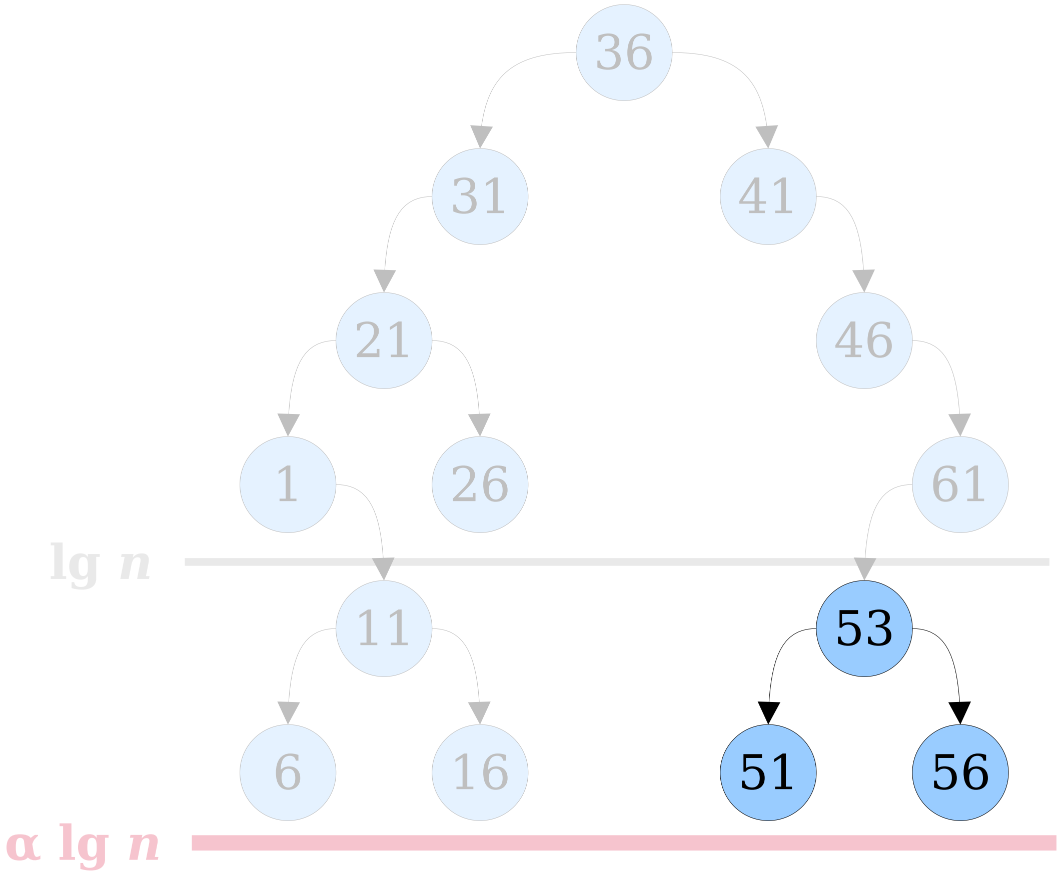

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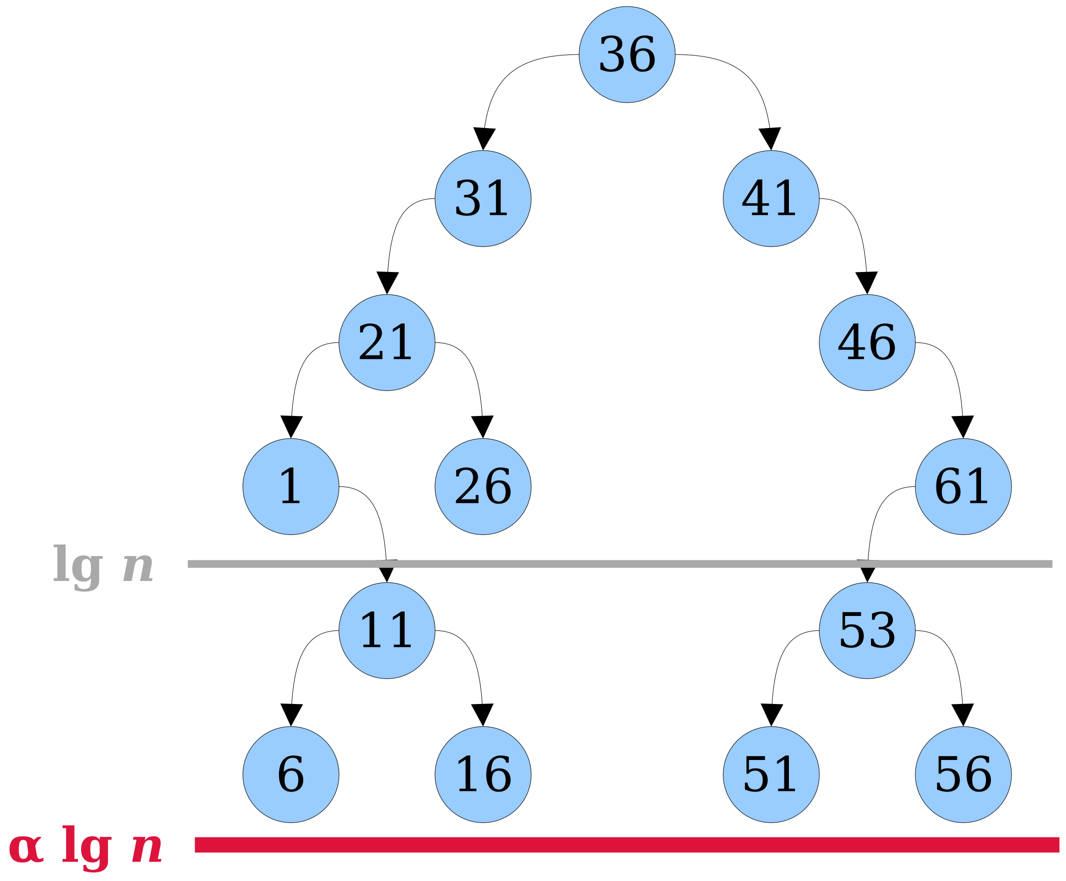

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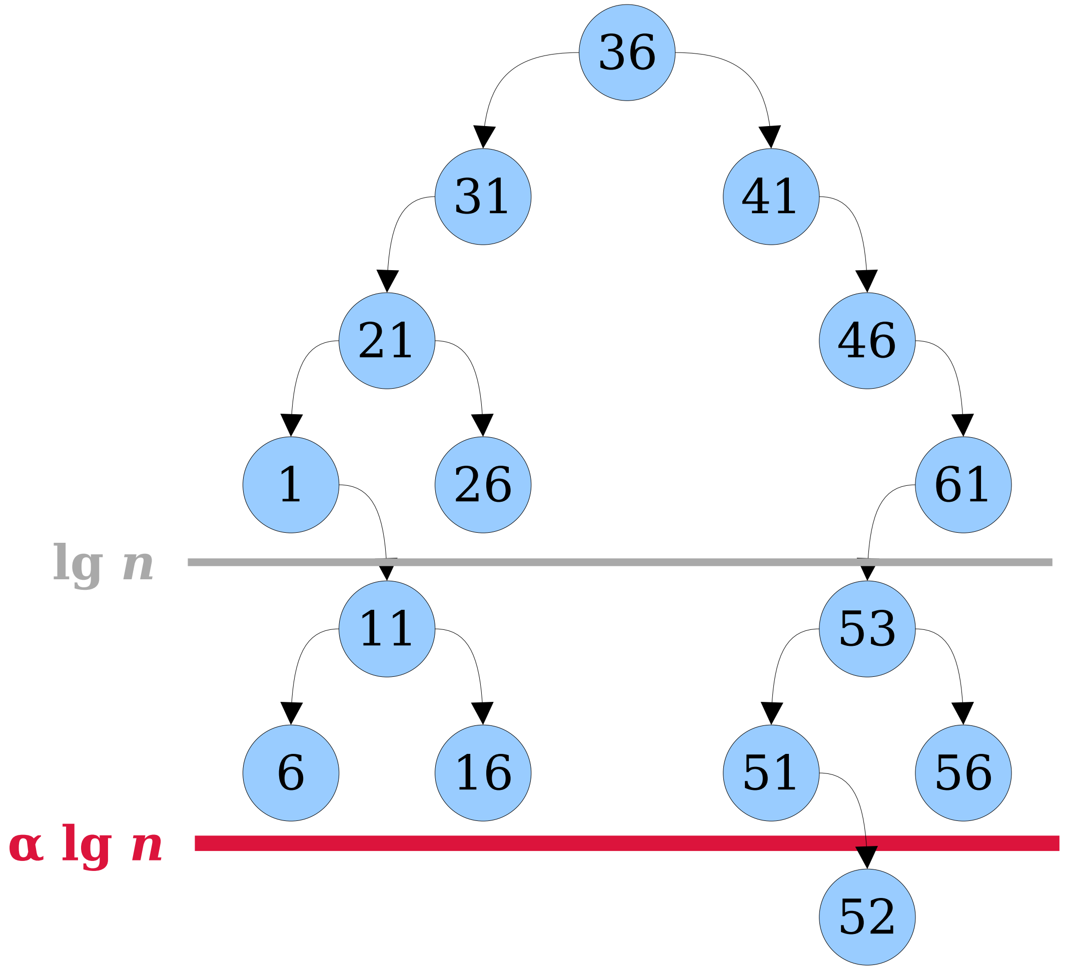

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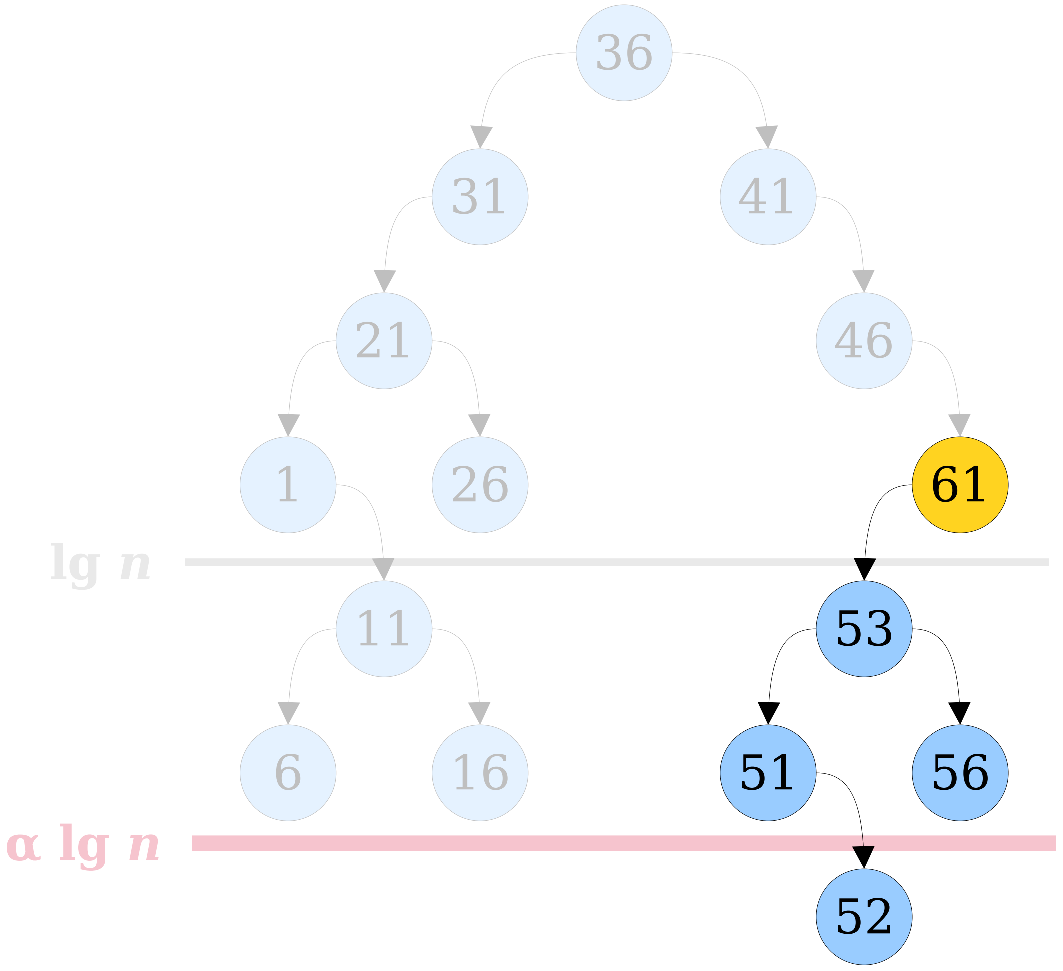

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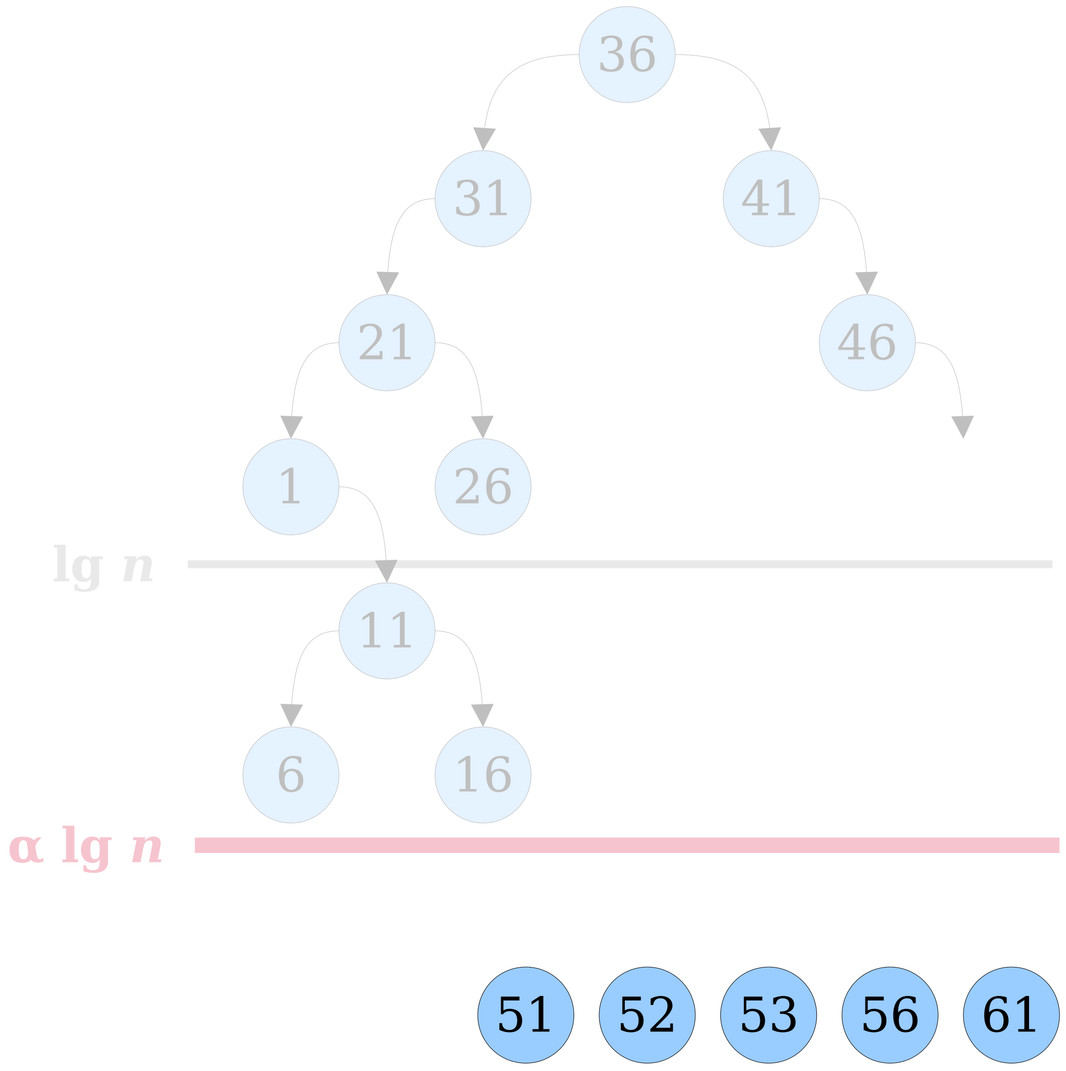

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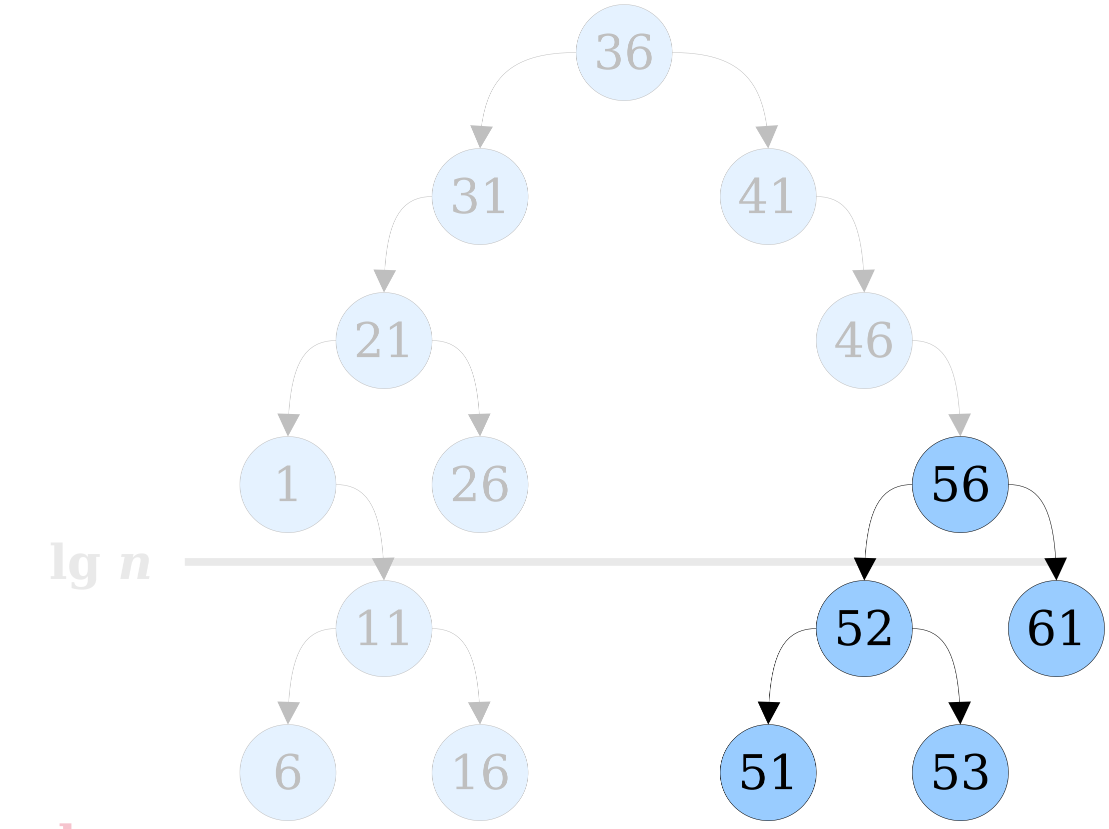

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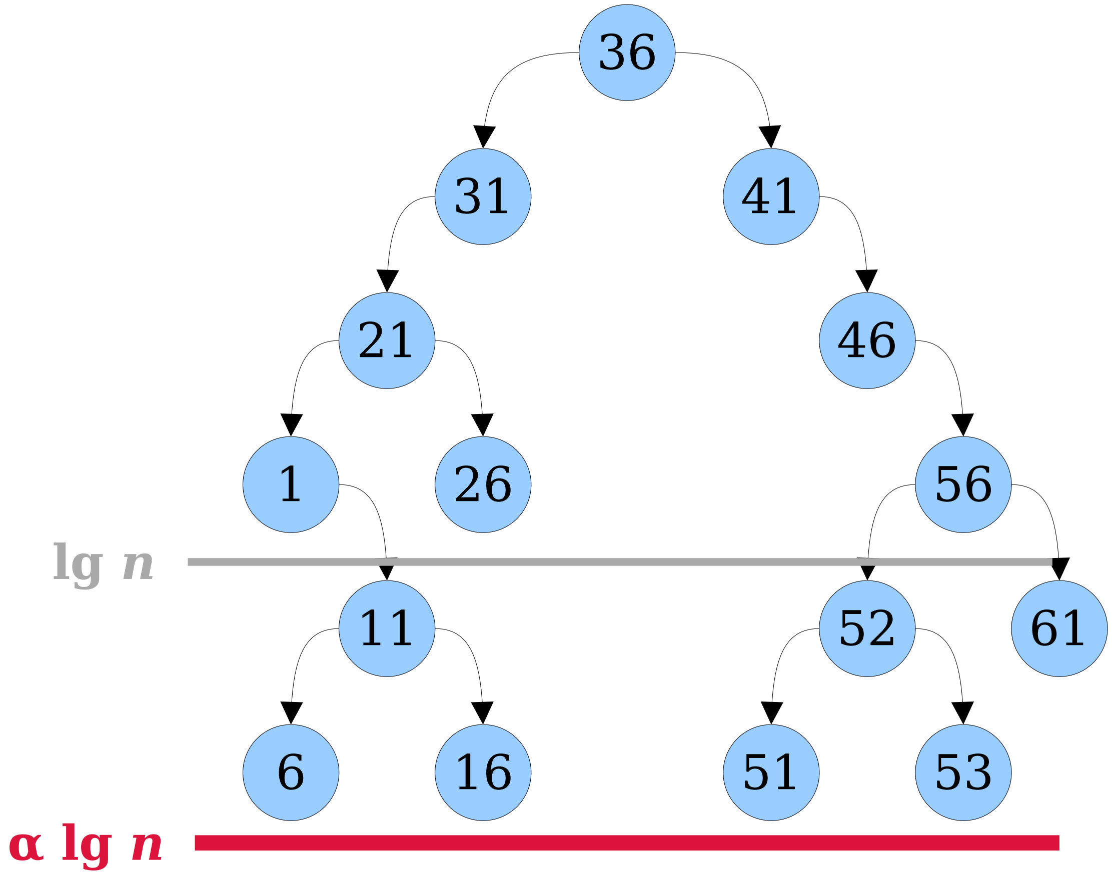

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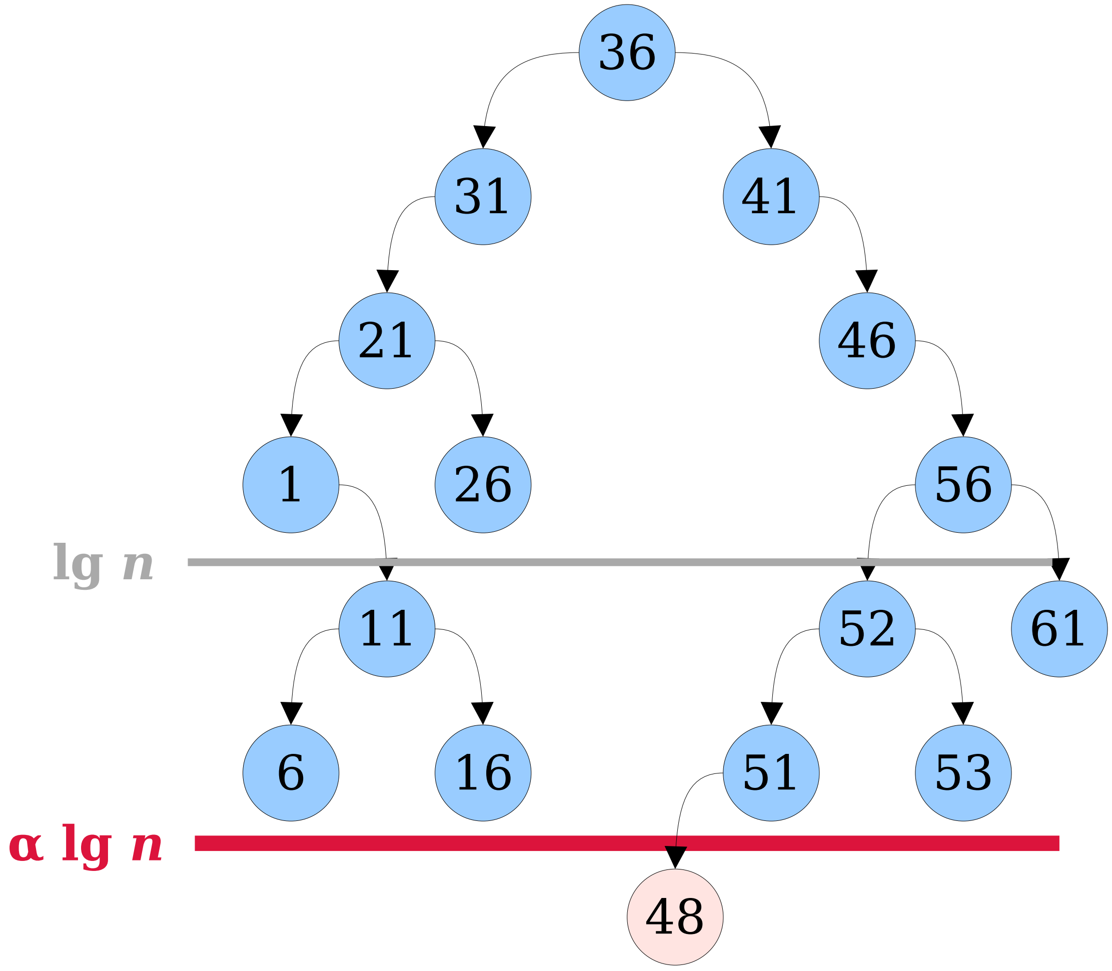

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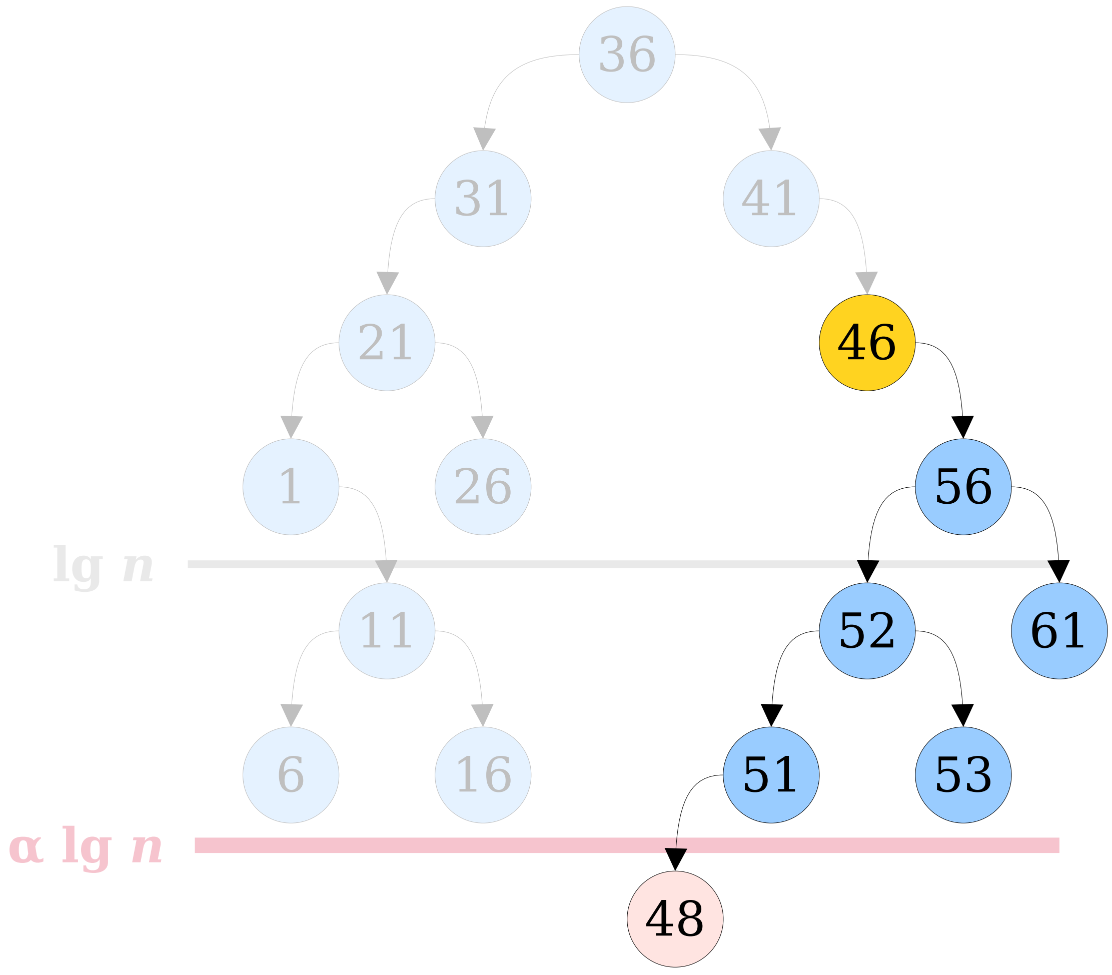

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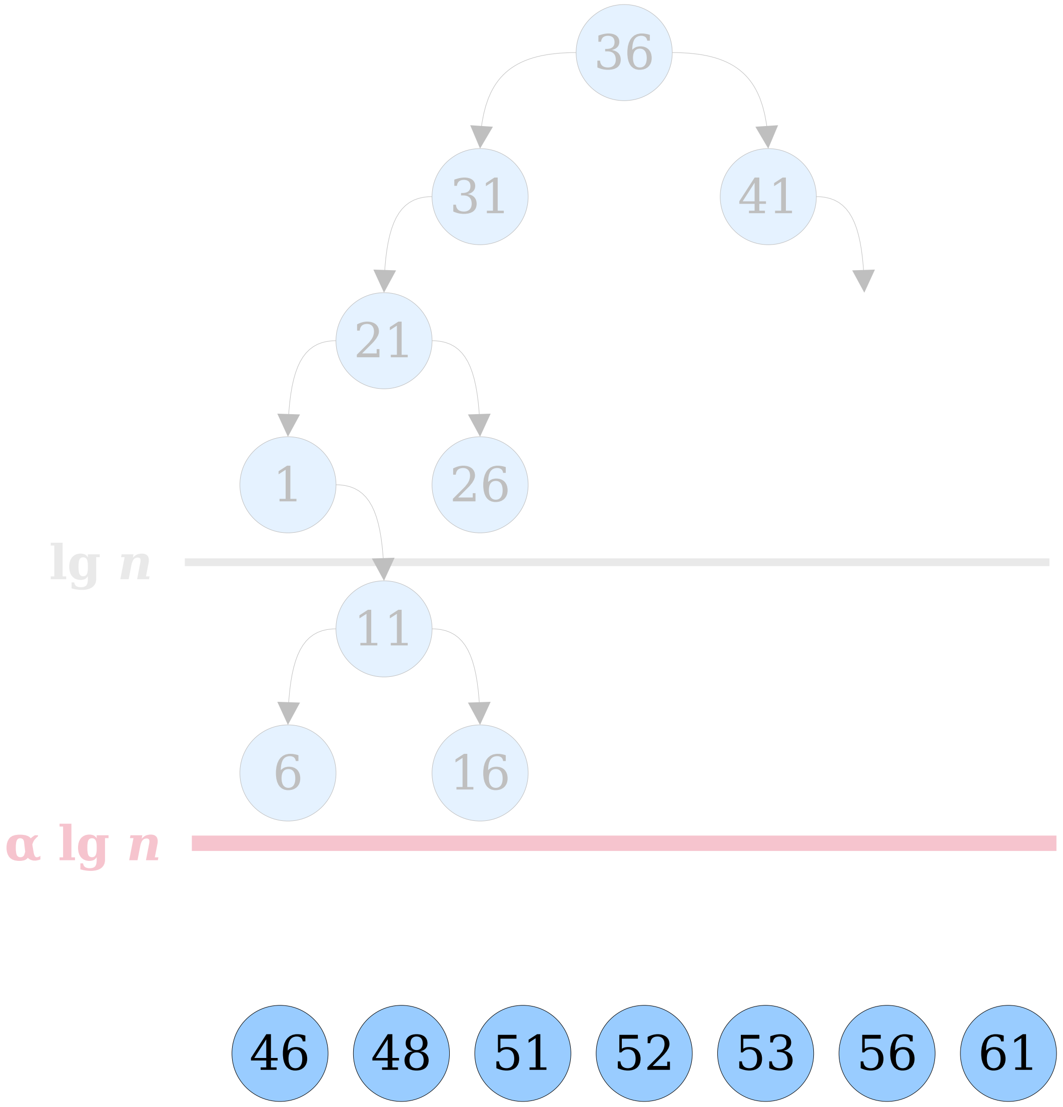

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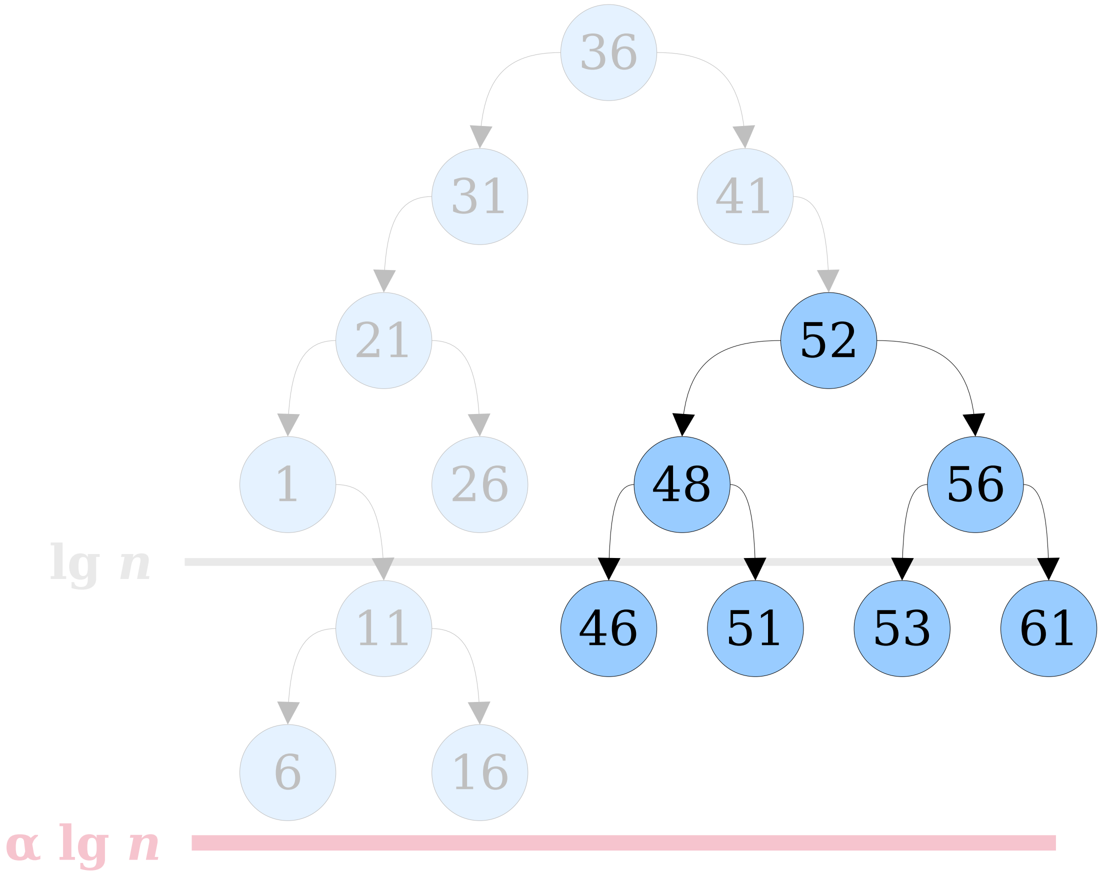

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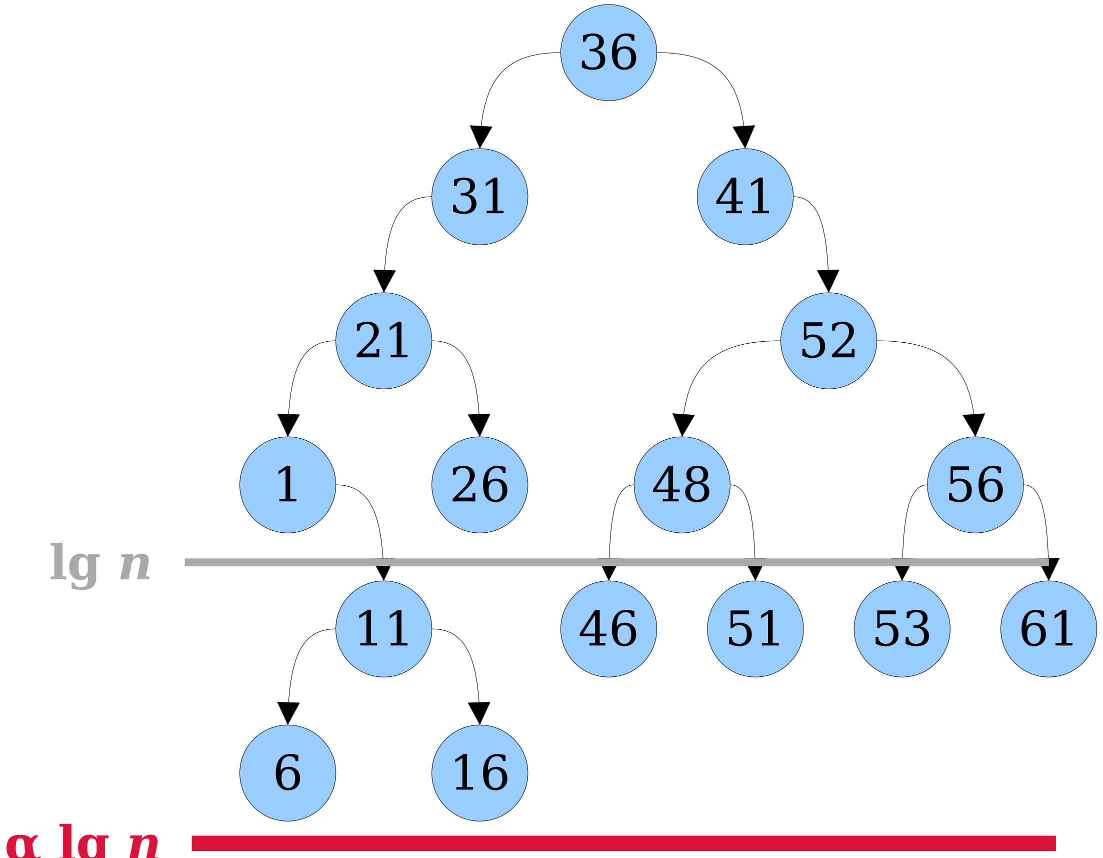
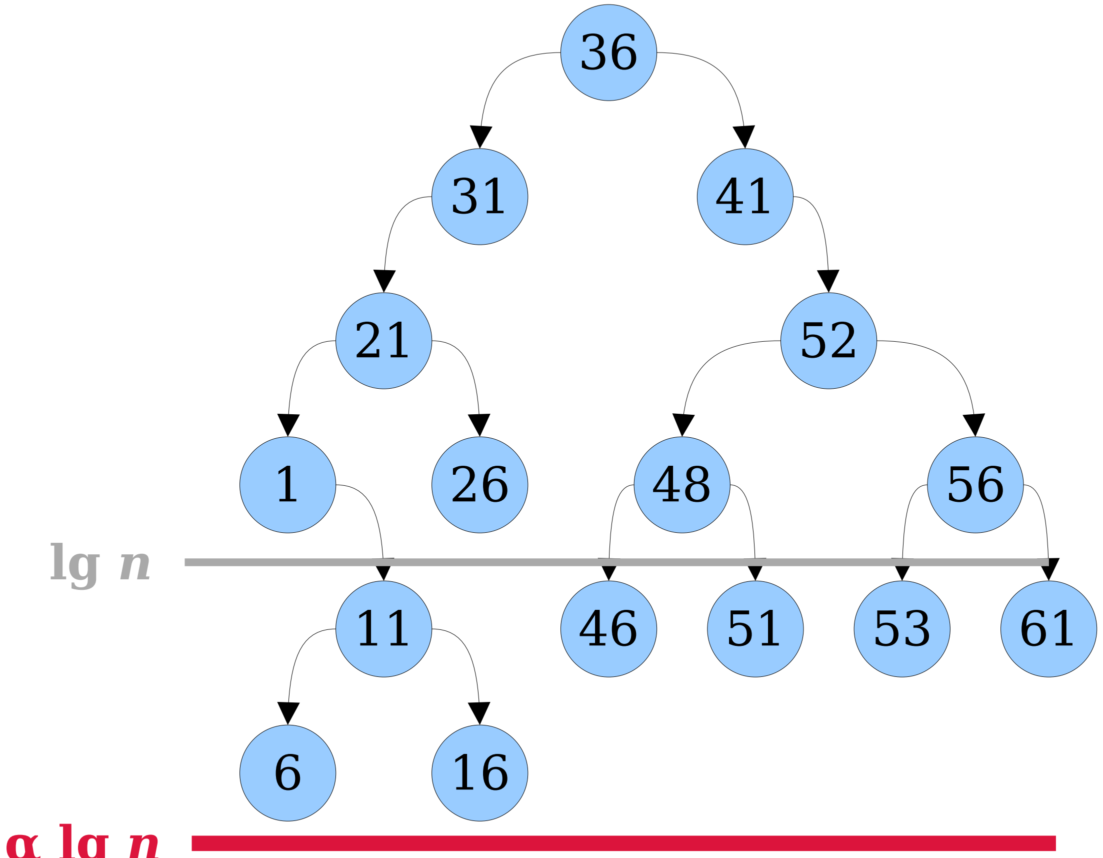


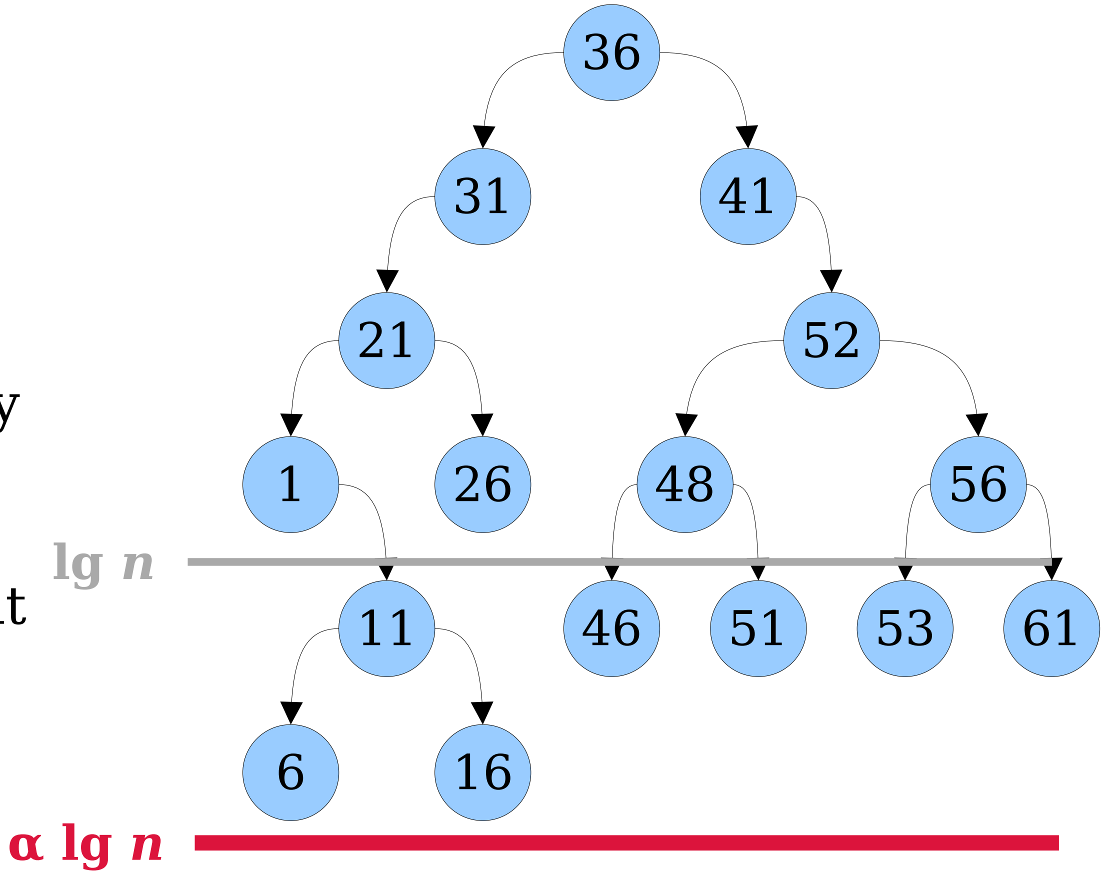
Scapegoat Trees

- A ***scapegoat tree*** is a balanced binary search tree that works as follows:
  - Pick some constant $\alpha > 1$.
  - As long as the tree height is below $\alpha \lg n$, don't do any rebalancing after insertions.
  - Once the tree exceeds that height, find the scapegoat (the deepest α-imbalanced node on the insertion path).
  - Then, optimally rebuild the subtree rooted at that node.
- All that's left now is to work through the details.



Scapegoat Trees

- Questions we need to address:
  - How do we know that optimally rebuilding the scapegoat's subtree will fix the tree height?
  - How quickly can we optimally rebuild the subtree rooted at the scapegoat node?
  - How do we find the scapegoat node?
  - In an amortized sense, how fast is this strategy?
- Let's address each of these in turn.



The Impact of Rebuilding

Scapegoat Rebuilding

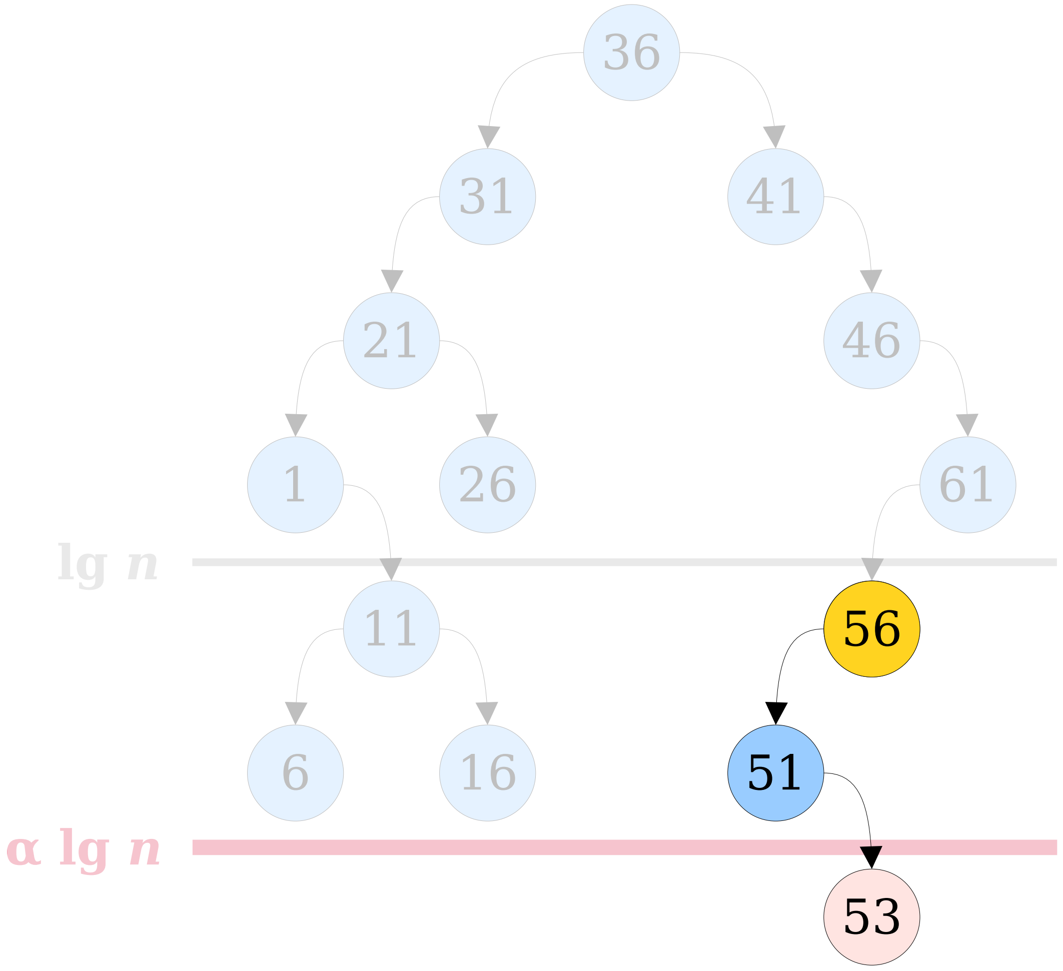
- Our strategy relies on the following claim:

Optimally rebuilding the subtree rooted at the scapegoat node ensures that, as a whole, the tree has height at most α lg *n*.

- This turns out to not be too difficult to prove. Let's break it down into pieces.

Scapegoat Rebuilding

- Suppose we insert a node that causes the $\alpha \lg n$ size limit to be violated.
- Just before we inserted that node, all other nodes in the tree were at height $\alpha \lg n$ or below.
- That means each other node is at depth $\lfloor \alpha \lg n \rfloor$, and our new node is at depth $\lfloor \alpha \lg n \rfloor + 1$.
- Now, look at the scapegoat node and its subtree.
- Because our offending node is only one level too deep, we just need to show that optimally rebuilding the scapegoat subtree reduces its depth by at least one.



Scapegoat Rebuilding

- Let v be our scapegoat node. Since it's not α-balanced, we know that

$$\text{height}_{\text{before}}(v) > \alpha \text{ lg size}(v).$$

- Let r be the root of the subtree we get after rebuilding at v. Because we rebuilt v's tree perfectly, we know that

$$\text{lg size}(v) \geq \text{height}_{\text{after}}(r).$$

- Putting this together gives us that

$$\text{height}_{\text{before}}(v) > \alpha \text{ lg size}(v) > \text{lg size}(v) \geq \text{height}_{\text{after}}(r).$$

- This means that

$$\text{height}_{\text{before}}(v) > \text{height}_{\text{after}}(r).$$

- Therefore, the height of v's subtree after rebuilding has decreased by at least one, so overall balance is restored.

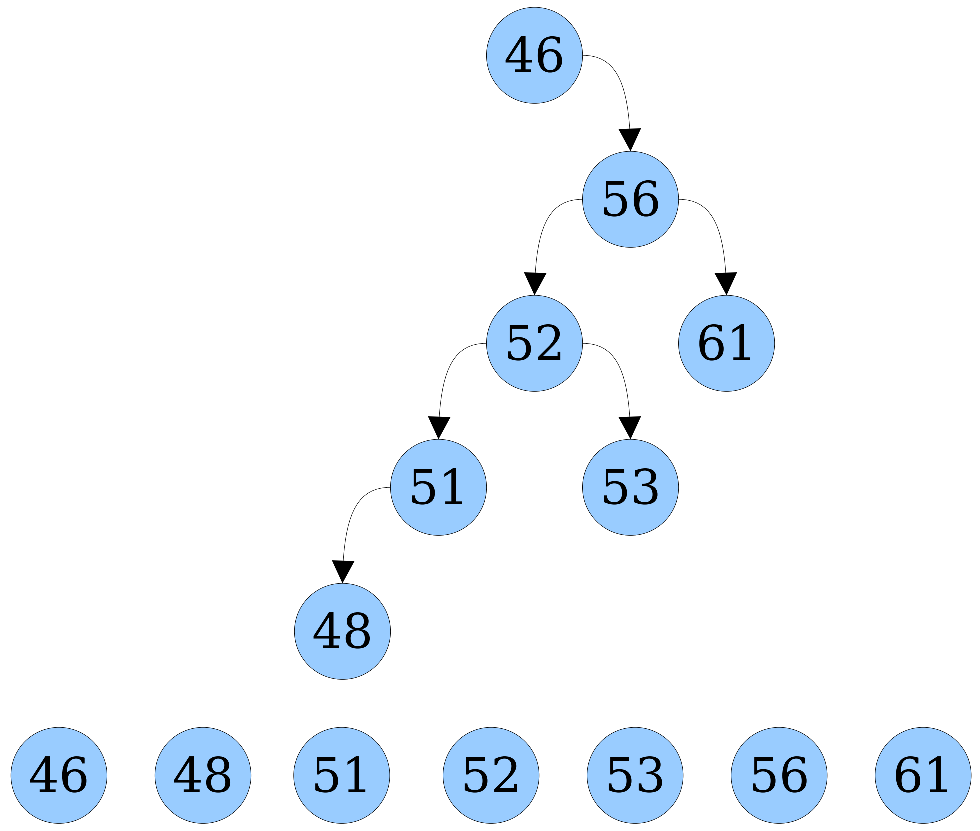
The Cost of Rebuilding

The Cost of Rebuilding

- Once we’ve identified the scapegoat node, we need to rebuild the subtree rooted at that node as a perfectly-balanced BST.
- How quickly can we do this?

The Cost of Rebuilding

- Run an inorder traversal over the subtree and form an array of its nodes in sorted order.



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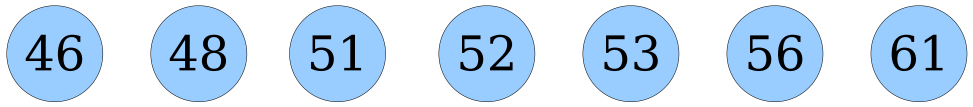

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- Run an inorder traversal over the subtree and form an array of its nodes in sorted order.
- Use the following recursive algorithm to build an optimal tree:
    - If there are no nodes left, return an empty tree.
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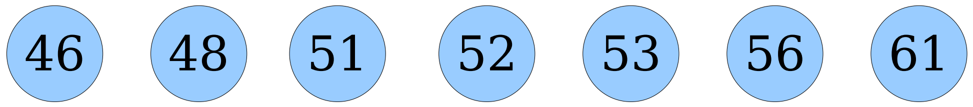

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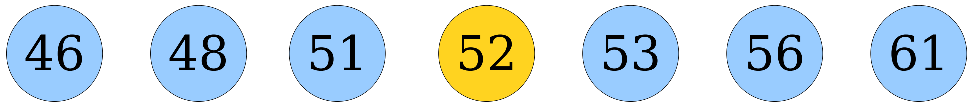

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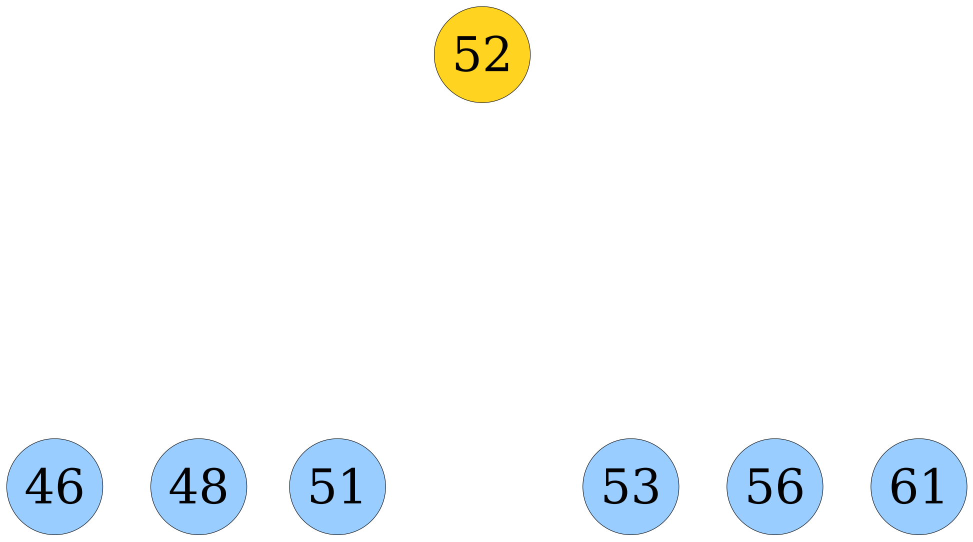

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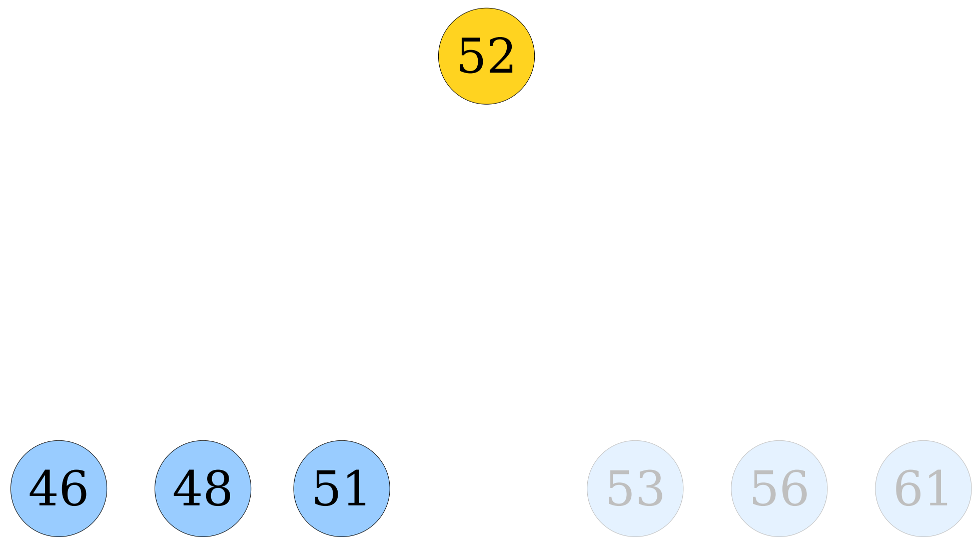

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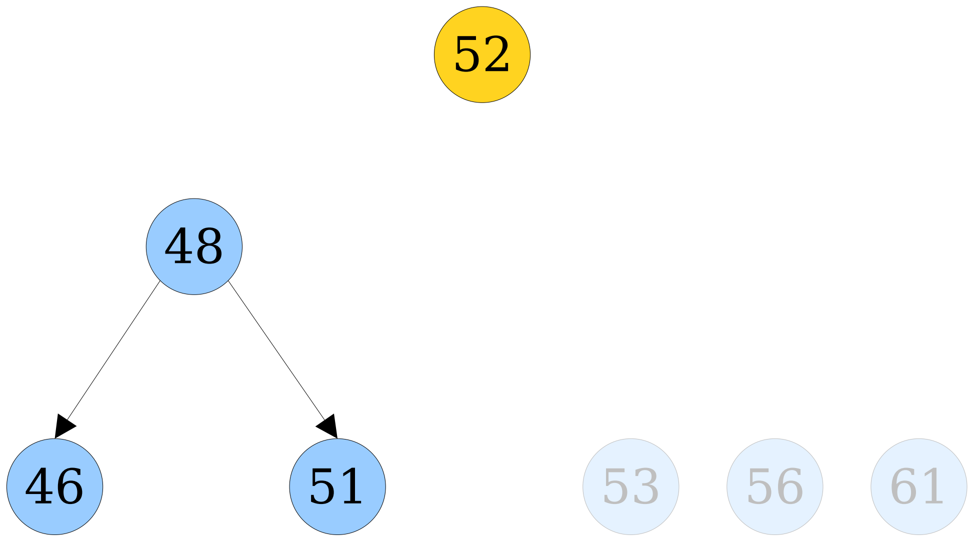

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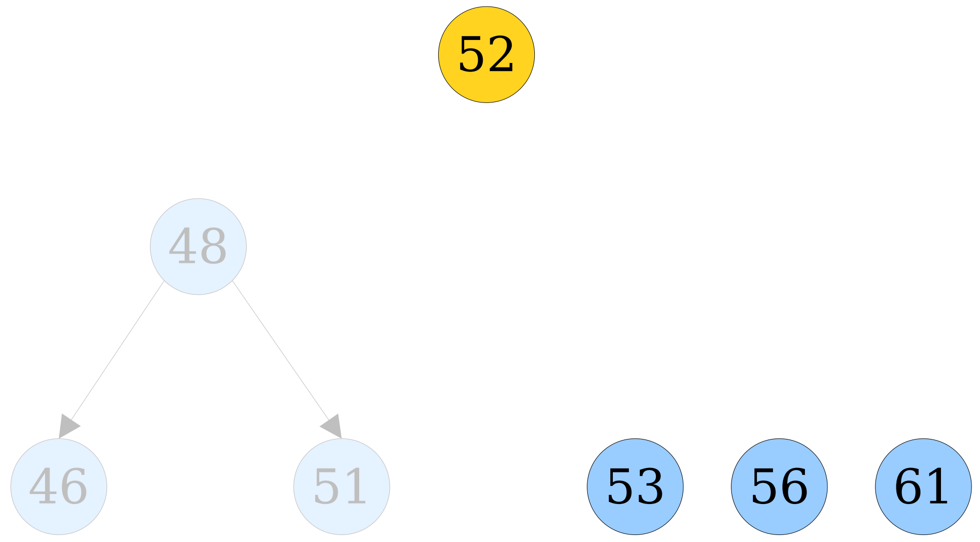

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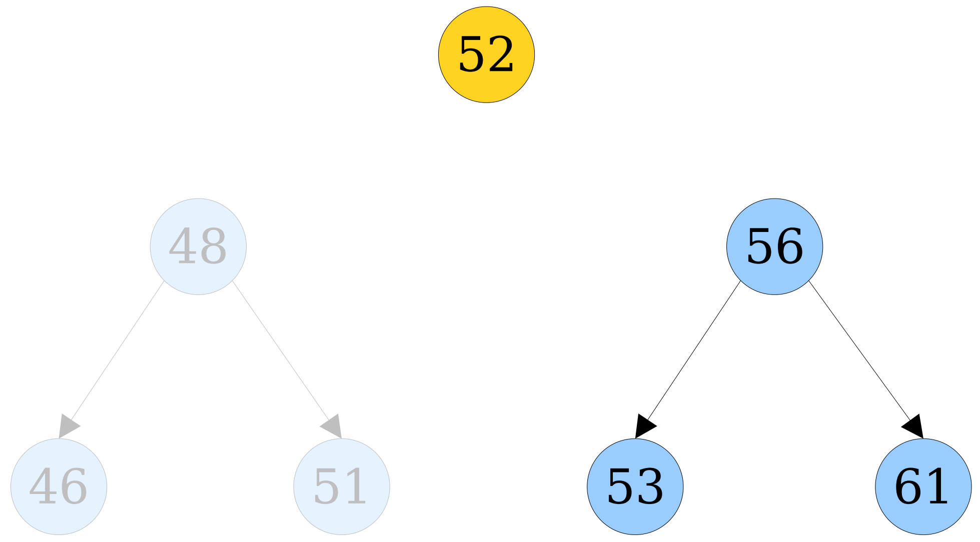

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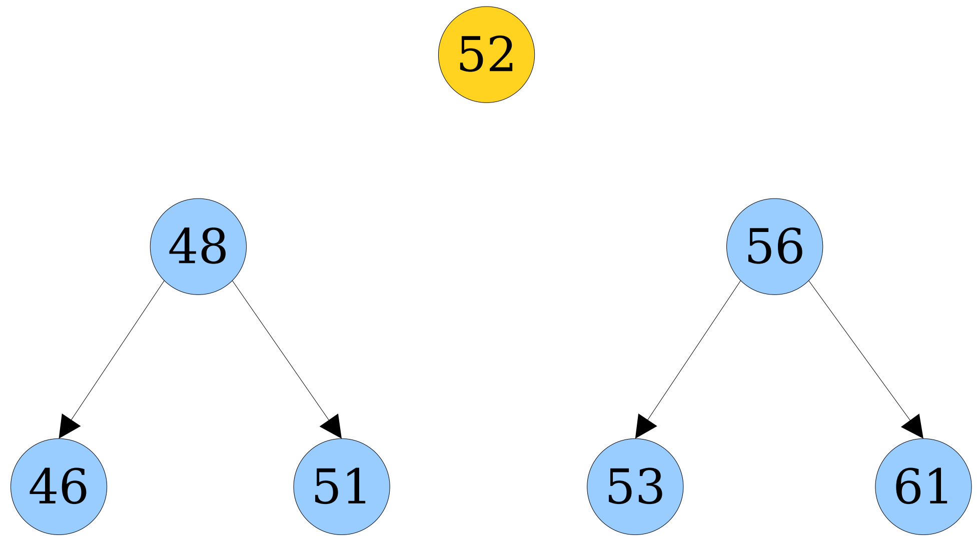

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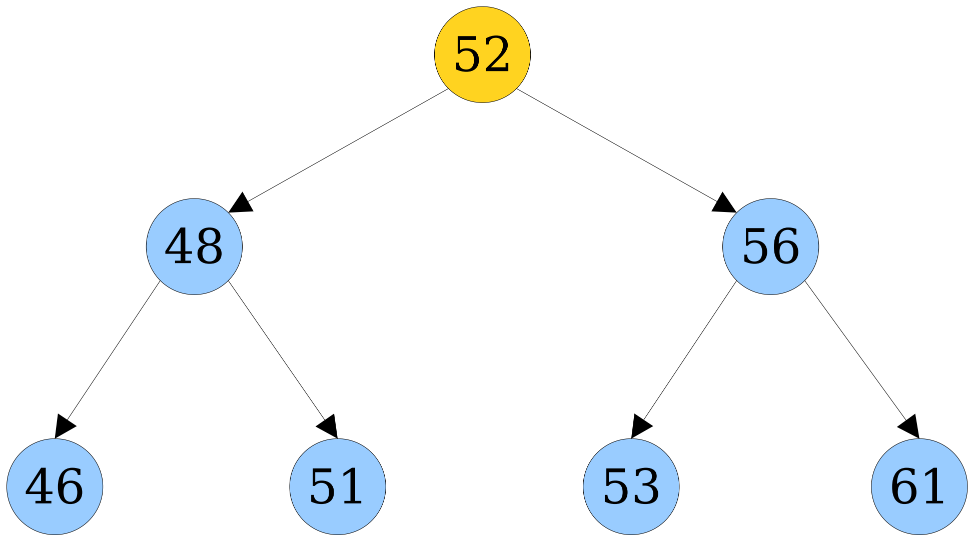

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- Use the following recursive algorithm to build an optimal tree:
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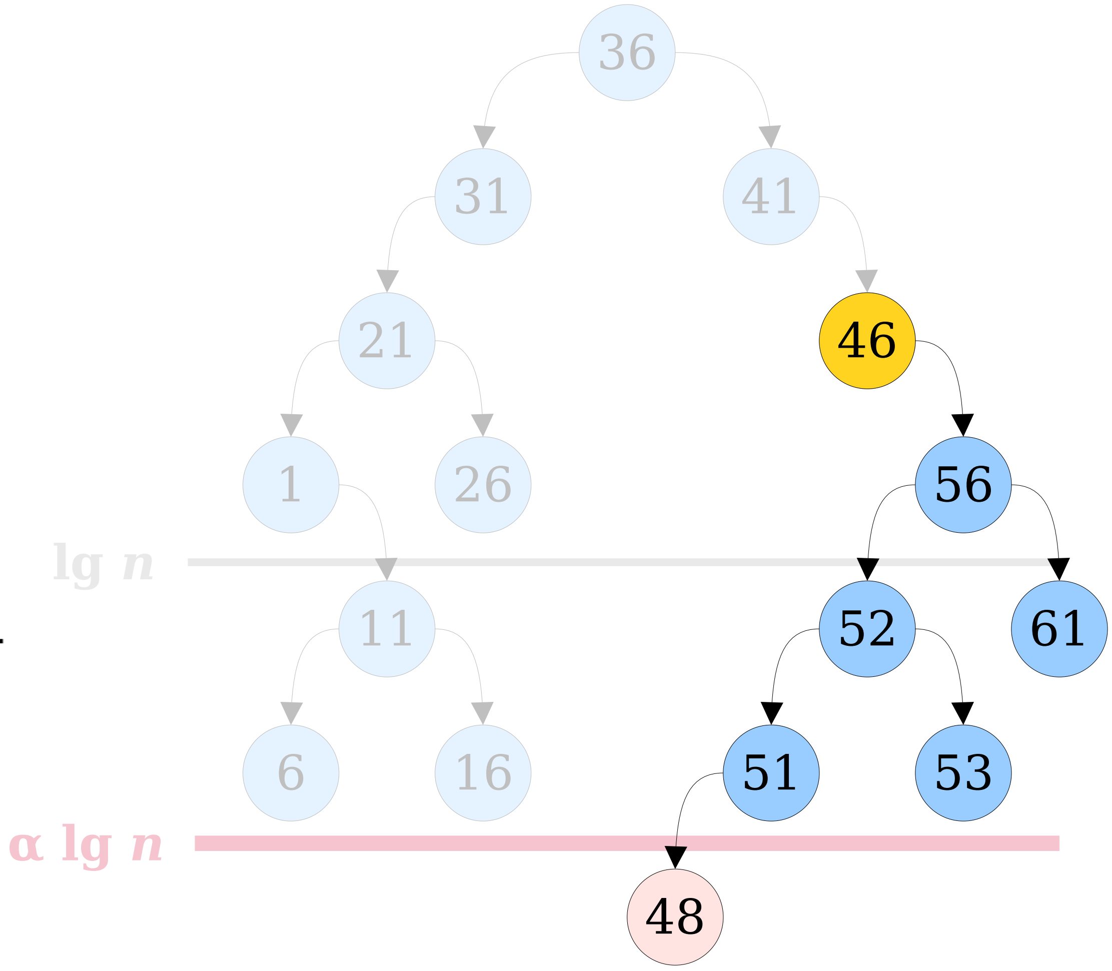
  - Otherwise, put the median element at the root of the tree, and recursively build its left and right subtrees optimally.
- The cost of this strategy is O(size(v)), where v is the node at the root of the subtree.
  - Quick way to see this: the inorder traversal takes time O(size(v)) because there are size(v) nodes visited, and the recursive algorithm has the recurrence $T(m) = 2T(m / 2) + O(1)$.
- This is the simplest algorithm to optimally rebuild the tree, but others exist that are faster in practice or more space-efficient. Look up the ***Galperin-Rivest*** or ***Day-Stout-Warren*** algorithms for other ways to do this in time O(size(v)) in less space.

Finding the Scapegoat Node

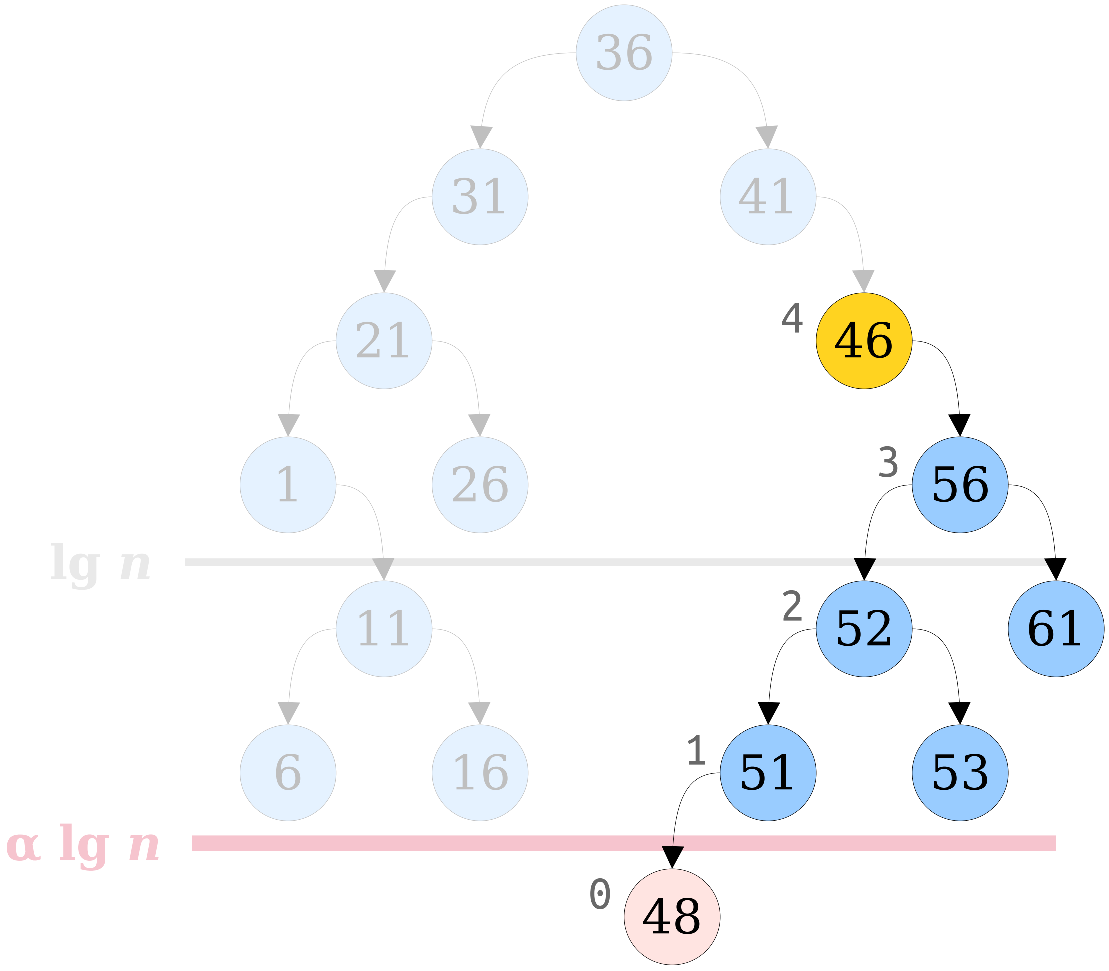
Finding the Scapegoat

- ***Recall:*** The scapegoat node is the deepest node on the access path that isn't α-balanced.
- How efficiently can we identify this node?



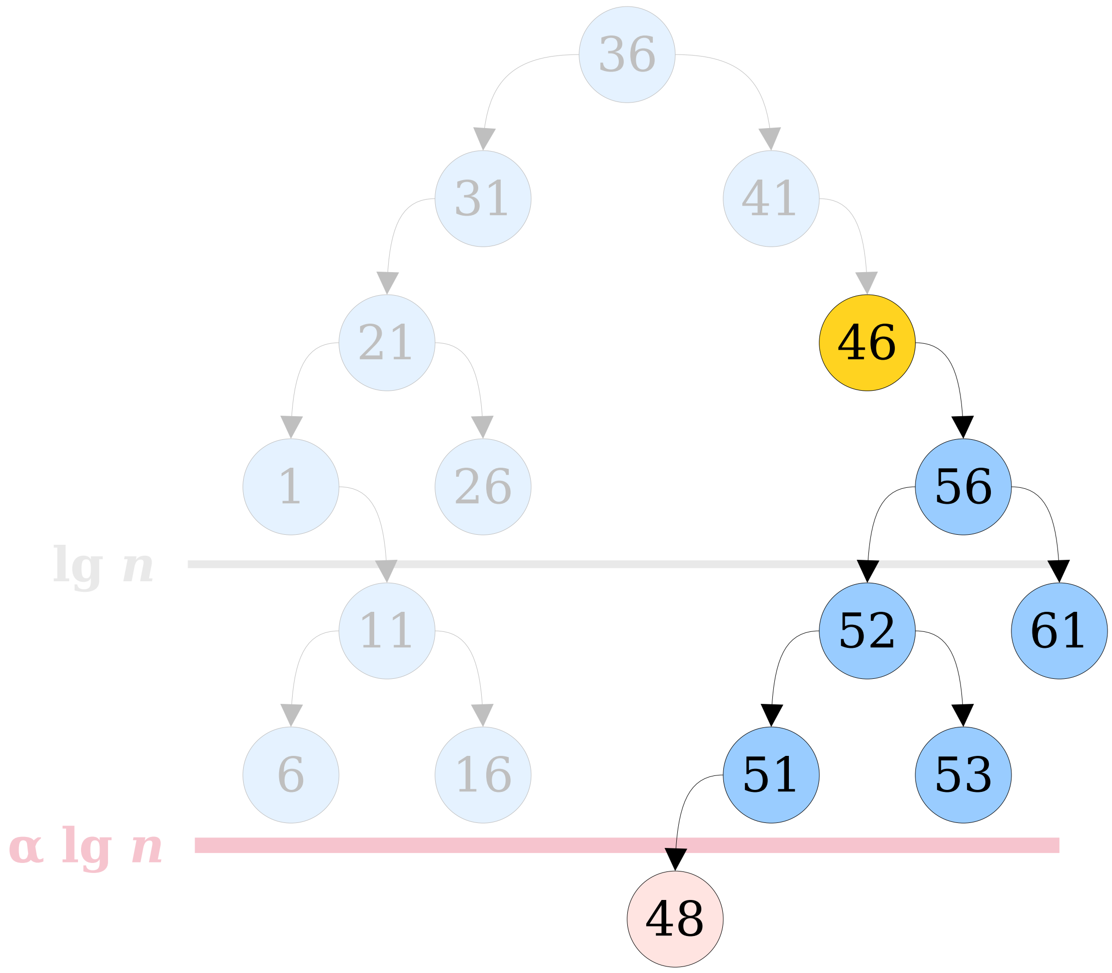
Finding the Scapegoat

- We need to check if height(v) > α lg size(v).
- ***Observation:*** For each node v on the access path, height(v) is the number of steps between v and the newly-added node.
  - This can be computed by counting upward from the new node.
- That just leaves computing size(v).



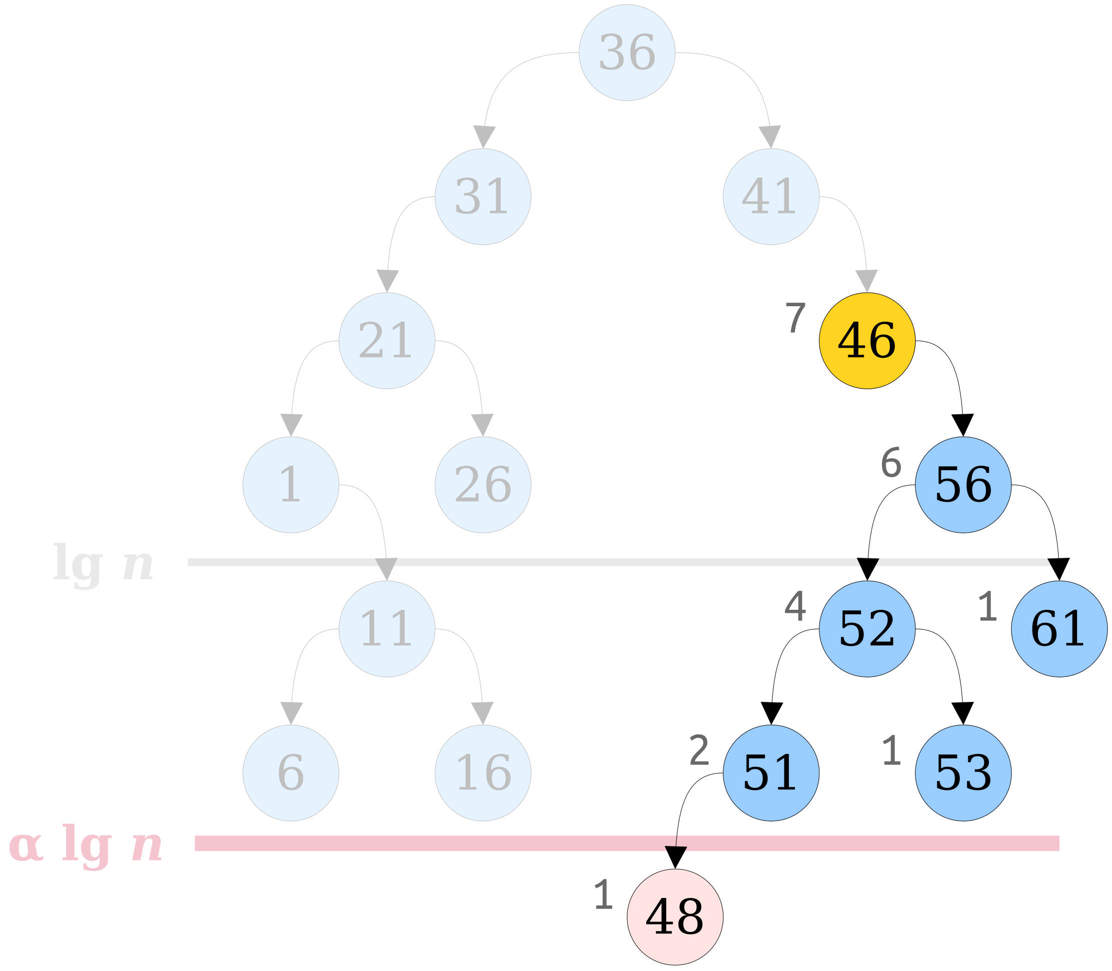
Finding the Scapegoat

- There are two ways we can compute size(*v*) for the nodes on the access path.
- ***Approach 1:*** Augment each node with the number of nodes in its subtree.
  - (This can be done without changing the cost of an insertion or deletion.)
- We can then read size(*v*) by looking at the cached value.
- This has the disadvantage of requiring an extra integer in each node of the tree.



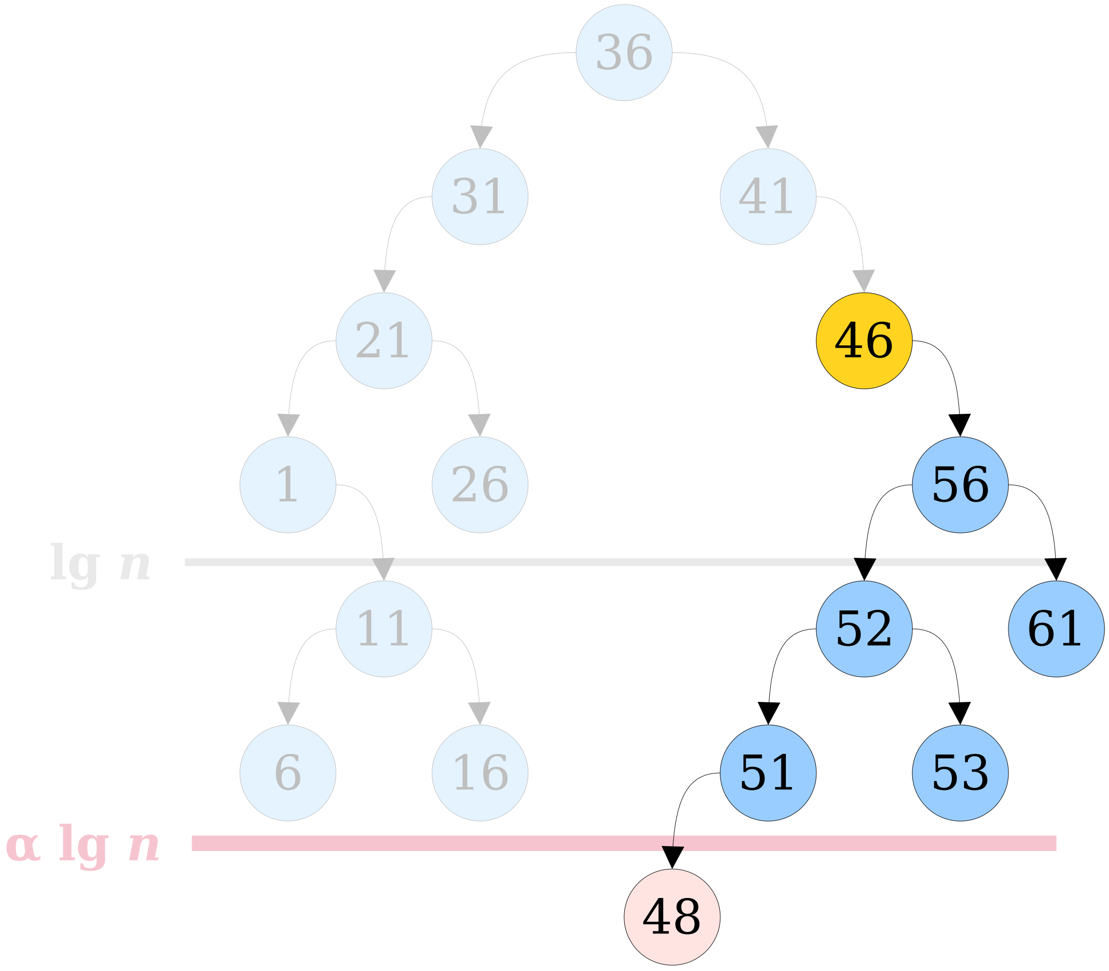
Finding the Scapegoat

- ***Approach 2:*** Compute these values bottom-up.
- Start with a total of 1 for the newly-added node.
- Each time we move upward a step, run a DFS in the opposite subtree to count the number of nodes there.
- Once we hit the scapegoat node *v*, we'll have done O(size(*v*)) total work counting nodes.



Finding the Scapegoat

- ***Approach 1*** does less work, but requires more storage in each node.
- ***Approach 2*** does more work, but means each node just stores data and two child pointers.
- Which of these ends up being more important depends on a mix of engineering constraints and personal preference.



Analyzing Efficiency

Analyzing Efficiency

- Based on what we've seen so far, the cost of an insertion is
  - O(log n) if the insertion keeps us below the α lg n height threshold, and
  - O(log n + size(v)) if we have to rebuild v as a scapegoat.
- The size(v) term can be as large as n, which may happen if the whole tree has to be rebuilt.
- However, it turns out that we can amortize this size(v) term away.

Analyzing Efficiency

- ***Recall:*** To perform an amortized analysis, we do the following:
  - Find a potential function Φ that, intuitively, is small when the data structure is “clean” and large when the data structure is “messy.”
  - Compute the value of $\Delta\Phi = \Phi_{after} - \Phi_{before}$ for each operation.
  - Assign amortized costs as

    $$\textbf{amortized-cost} = \textbf{real-cost} + k \cdot \Delta\Phi$$

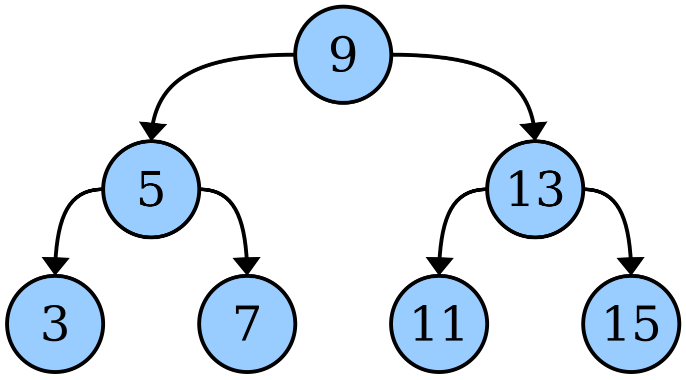
    for some constant *k* we get to pick.
- Our first step is to find a choice of Φ that’s large when our tree is imbalanced and small when it’s balanced.

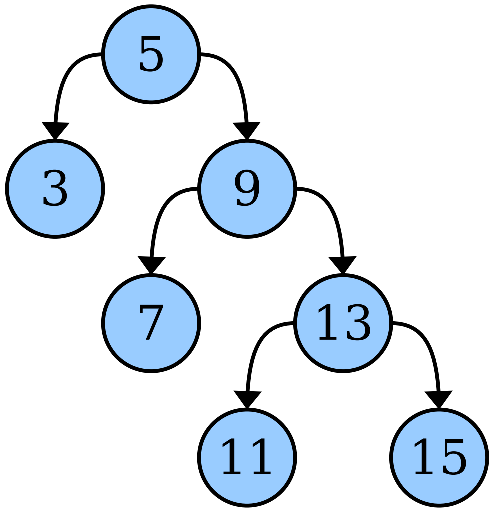
Quantifying Imbalance

- Right before we rebuild a scapegoat subtree, that tree is α-imbalanced.
- Right after we rebuild a scapegoat subtree, that tree is perfectly balanced.
- ***Goal:*** Find a choice of Φ for our tree so that
  - perfectly-balanced trees have low Φ, and
  - α-imbalanced trees have high Φ.
- At this point, we need to do some exploring to see what we find.

Quantifying Imbalance

- When we talk about “perfectly balanced” trees, what exactly is this “balance” in reference to?
- ***Intuition 1:*** A perfectly balanced tree is one where each node has roughly the same number of children in its left subtree as in its right subtree.
- ***Intuition 2:*** An “imbalanced” tree will have nodes whose left and right subtrees have differing numbers of nodes.



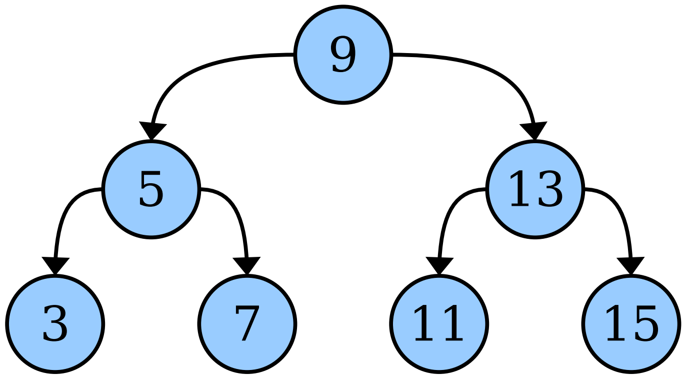


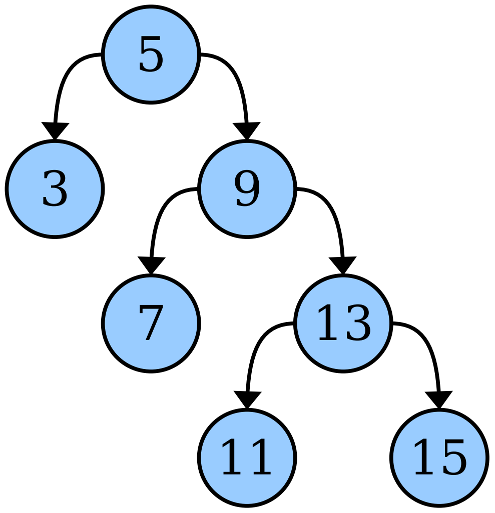
Quantifying Imbalance

- For each node *v*, define the ***imbalance*** of the node as

$$(v) = |\text{size}(v.\text{left}) - \text{size}(v.\text{right})|.$$

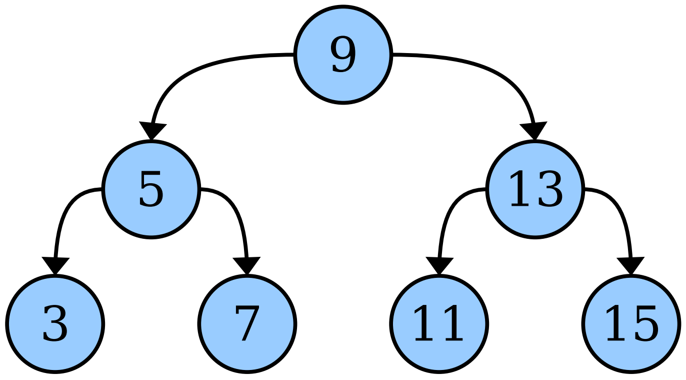
- This gives us a quantitative measure of our more nebulous concept of "imbalance."

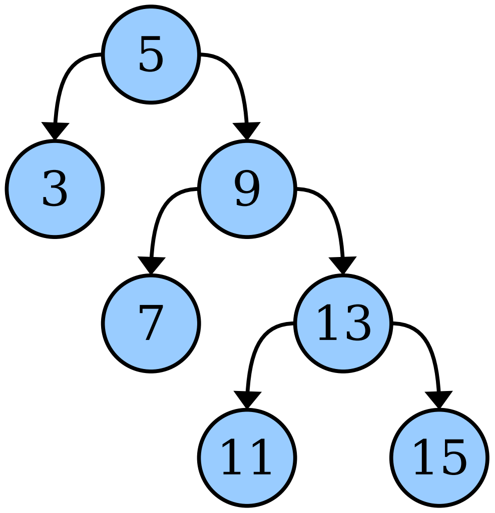




Defining our Potential

- We're looking for a potential function Φ where
  - a perfectly-balanced tree has low Φ, and
  - an imbalanced tree has progressively higher Φ.
- A balanced tree has (v) low for all its nodes.
- An imbalanced tree has (v) high for many nodes.
- ***Initial Idea:*** Define $\Phi = \Sigma_v \quad (v)$.

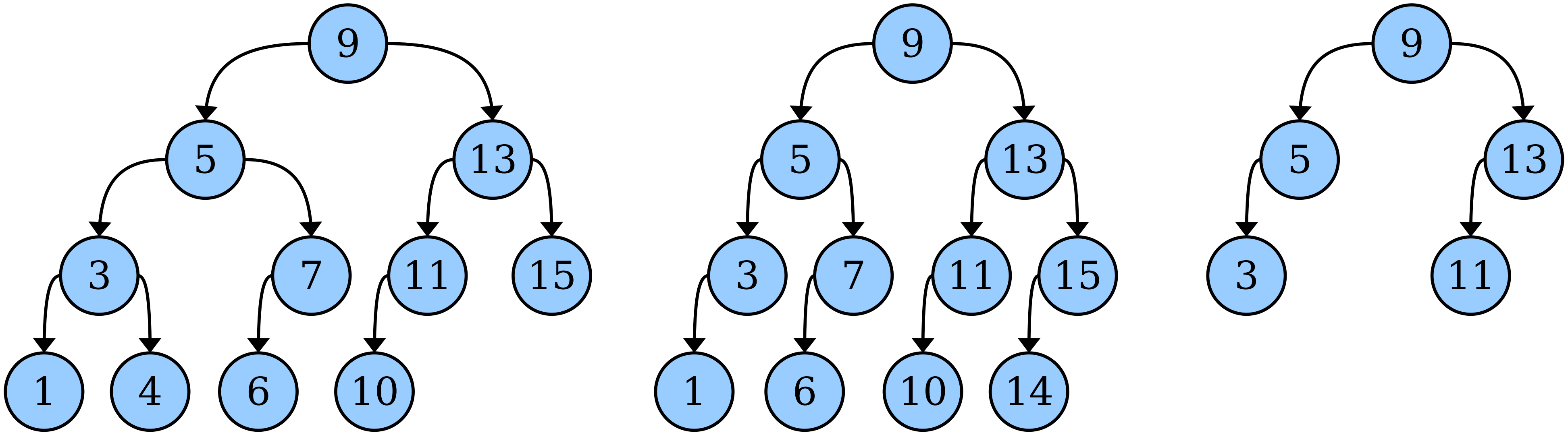




Defining our Potential

- We've set $\Phi = \Sigma_v \quad (v)$.
- What is Φ for the three trees shown below?

Formulate a hypothesis!



Defining our Potential

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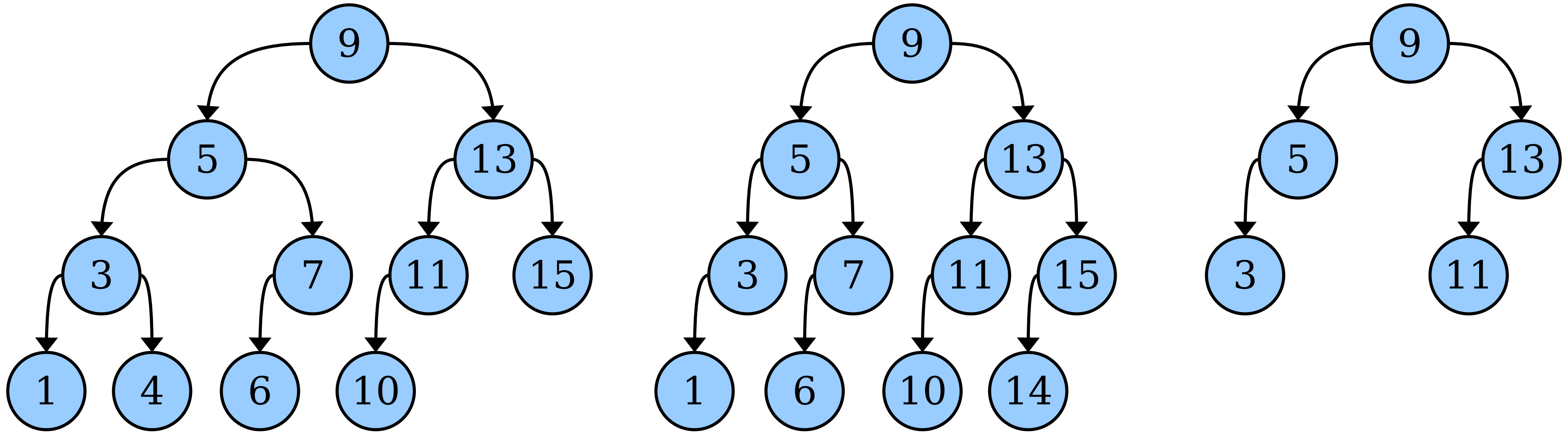
Discuss with your neighbors!



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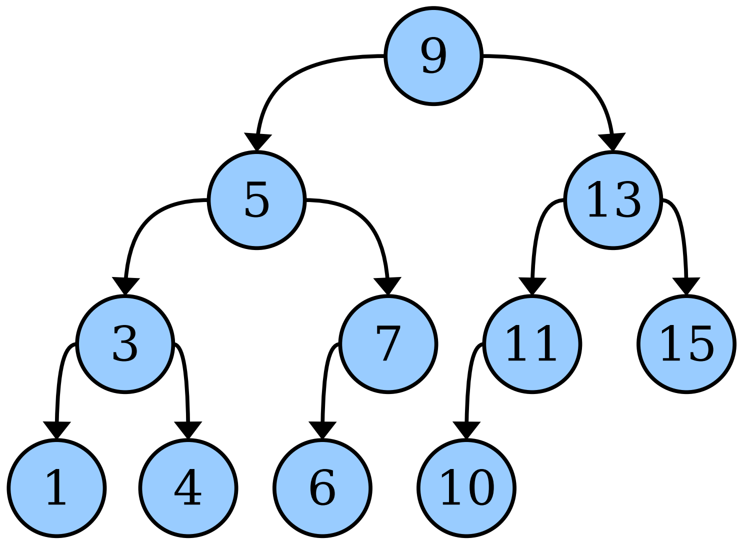
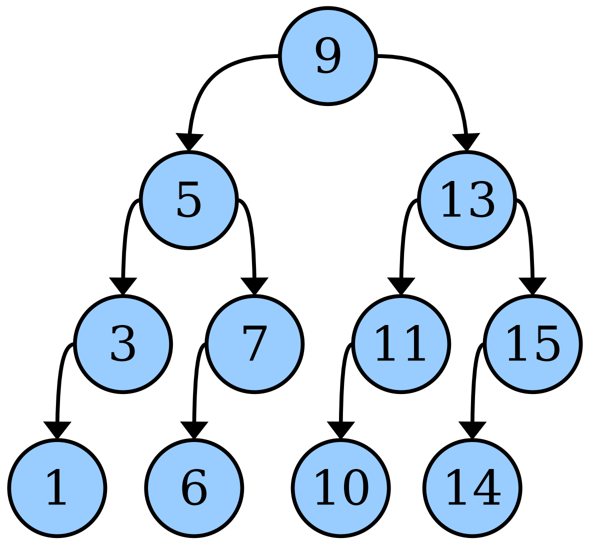
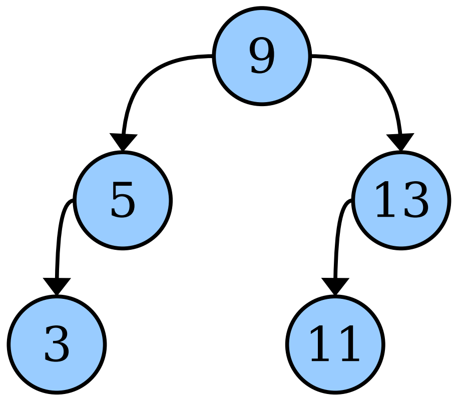


$\Phi = 6$



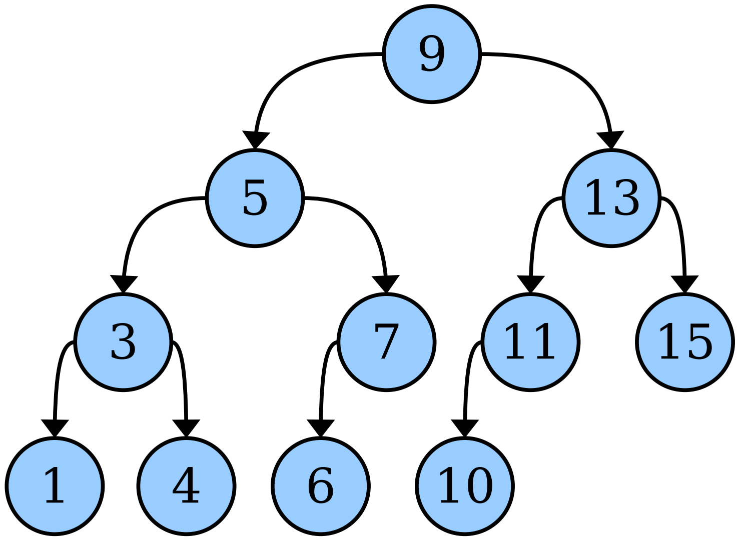
$\Phi = 4$



$\Phi = 2$

Defining our Potential

- ***Observation 1:*** Two trees that fill their rows as efficiently as possible may have different potentials.
- This means that when we rebalance trees, we need to make sure to equalize the number of nodes in the left and right subtrees of each node.



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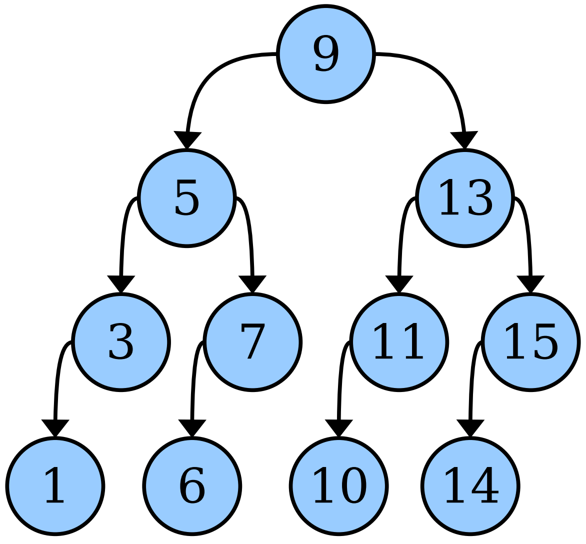

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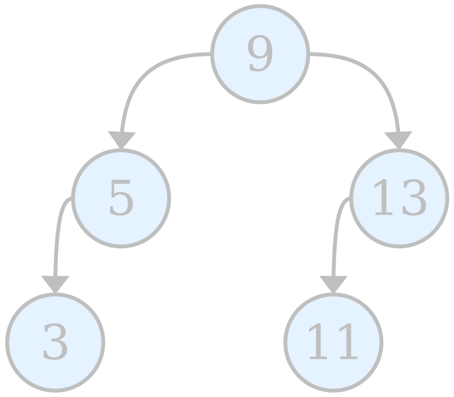
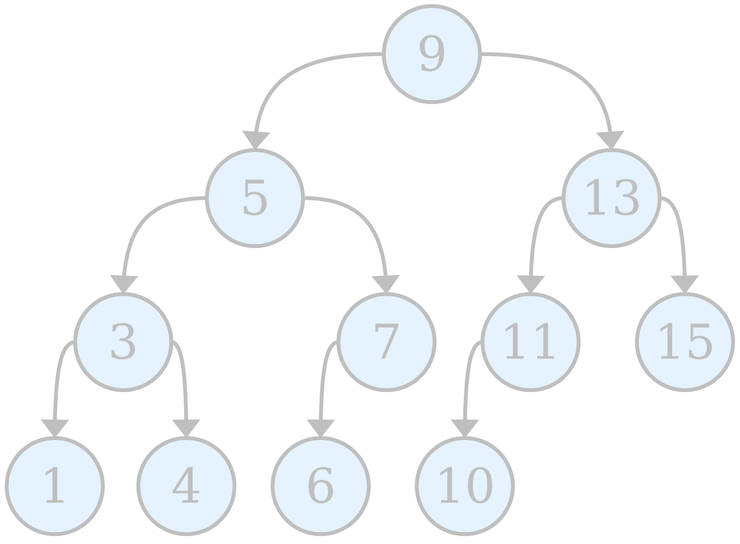


$\Phi = 2$

Defining our Potential

- ***Observation 2:*** The potential of a perfectly-balanced tree can grow as a function of its number of nodes.
- Ideally, both of these trees should have potential 0, indicating "perfectly balanced." The potential shouldn't depend on the number of nodes in the tree.



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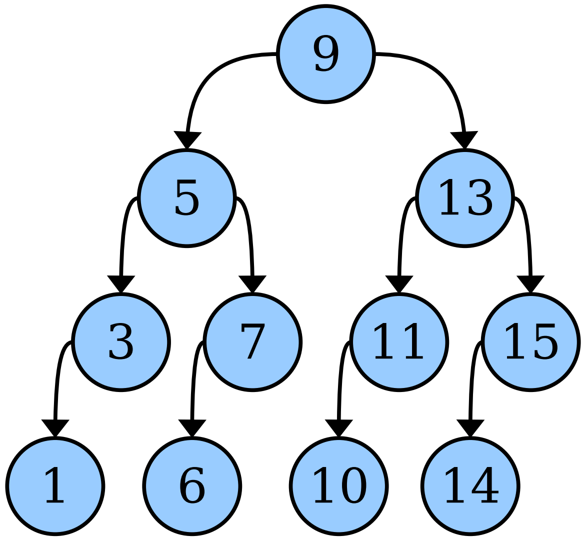

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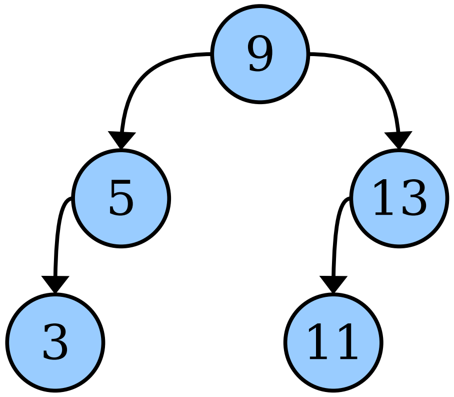


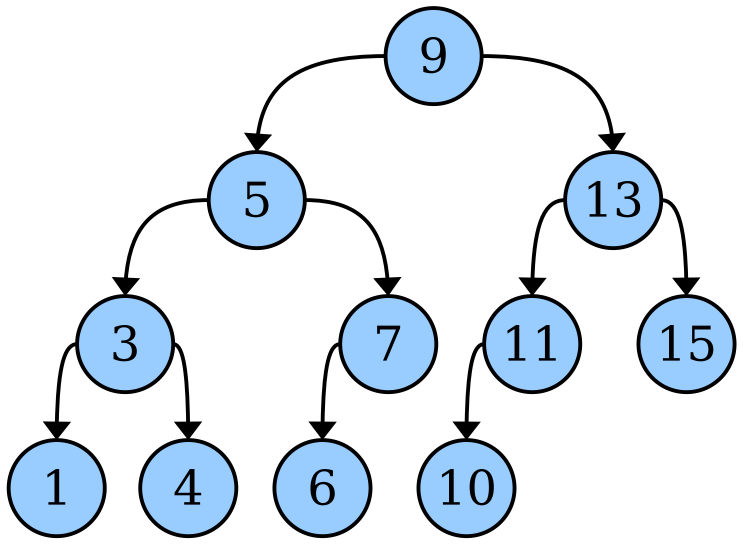
$\Phi = 2$

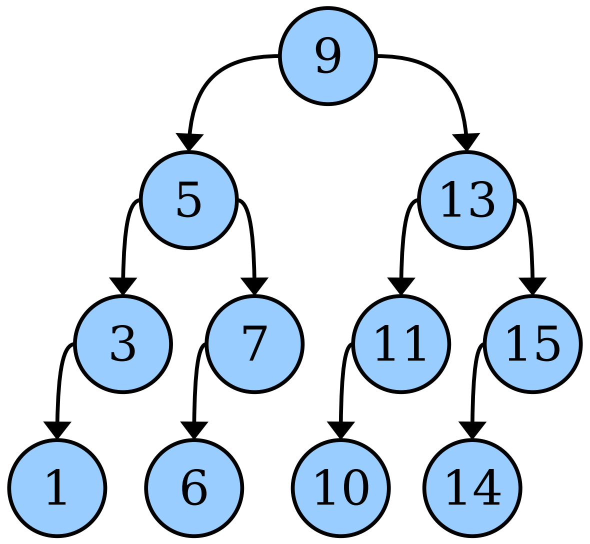
Defining our Potential

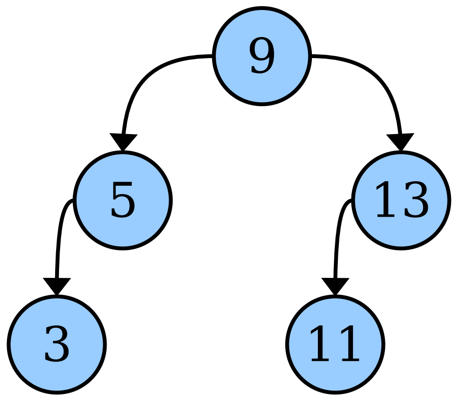
- To account for otherwise balanced trees with extra nodes in their bottom layers, let's define ’(v) as
  - ’(v) = 0 if (v) ≤ 1.
  - ’(v) = (v) otherwise.
- ***Revised Idea:*** Set $\Phi = \Sigma_v$ ’(v).

Defining our Potential

- We’re now using $\Phi = \Sigma_v$ ’(v).
- What is Φ for the three trees shown below?
- ***Intuition:*** If a subtree rooted at v is perfectly balanced, then ’$(v) = 0$.


$\Phi = 2$


$\Phi = 0$


$\Phi = 0$

Analyzing Scapegoat Trees

- Now that we have a definition of Φ, we can look at the amortized cost of an insertion.
- We need to consider two cases:
    - ***Case 1:*** The insertion doesn't trigger a rebuild.
    - ***Case 2:*** The insertion triggers a rebuild.
- Intuitively, we're hoping that Case 1 has a small positive ΔΦ (messes accumulate slowly) and that Case 2 has a large negative ΔΦ (messes get cleaned up quickly).
- Let's run the numbers!

Analyzing Scapegoat Trees

- ***Case 1:*** Our insertion does not trigger a rebuild.
- Recall that

$$\textit{amortized-cost} = \textit{real-cost} + k \cdot \Delta\Phi$$

for a constant k that we get to pick.

What are *real-cost* and $\Delta\Phi$,
as a function of n?

Formulate a hypothesis!

Analyzing Scapegoat Trees

- ***Case 1:*** Our insertion does not trigger a rebuild.
- Recall that

$$\textit{amortized-cost} = \textit{real-cost} + k \cdot \Delta\Phi$$

for a constant k that we get to pick.

What are *real-cost* and ΔΦ,
as a function of n?

Discuss with your neighbors!

Analyzing Scapegoat Trees

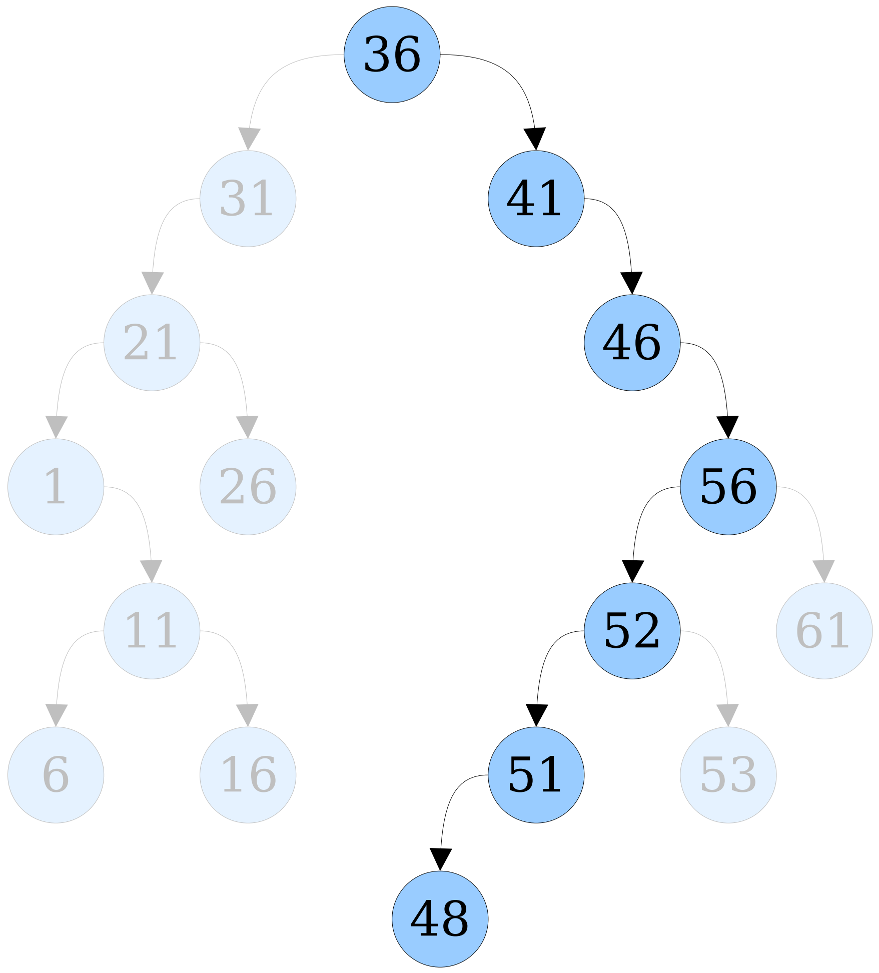
- ***Case 1:*** Our insertion does not trigger a rebuild.
- Recall that

$$\textit{amortized-cost} = \textit{real-cost} + k \cdot \Delta\Phi$$

  for a constant k that we get to pick.
- We're inserting into a tree of height at most α lg n, so our *real-cost* is O(log n).
- When we insert the node, it changes   (v) by ±1 for each node v on its access path.
- There are O(log n) nodes on this access path, and   (v) increases by at most one for each of those nodes. This means   '(v) increases by at most two for each of those nodes.
- Therefore, $\Delta\Phi$ = O(log n).
- Amortized cost: O(log n) + k · O(log n) = **O(log *n*)**.

Analyzing Scapegoat Trees

- ***Case 1:*** Our insertion does not trigger a rebuild.
- In this case, $\Delta\Phi = O(\log n)$.
- Focus on any one of the new node's ancestors.
- If we rebuild the subtree rooted at that node in the future, we have to do some work to move the new node.
- ***Intuition:*** The $O(\log n)$ added potential corresponds to paying $O(1)$ work in advance to each of $O(\log n)$ future rebuilds.



Analyzing Scapegoat Trees

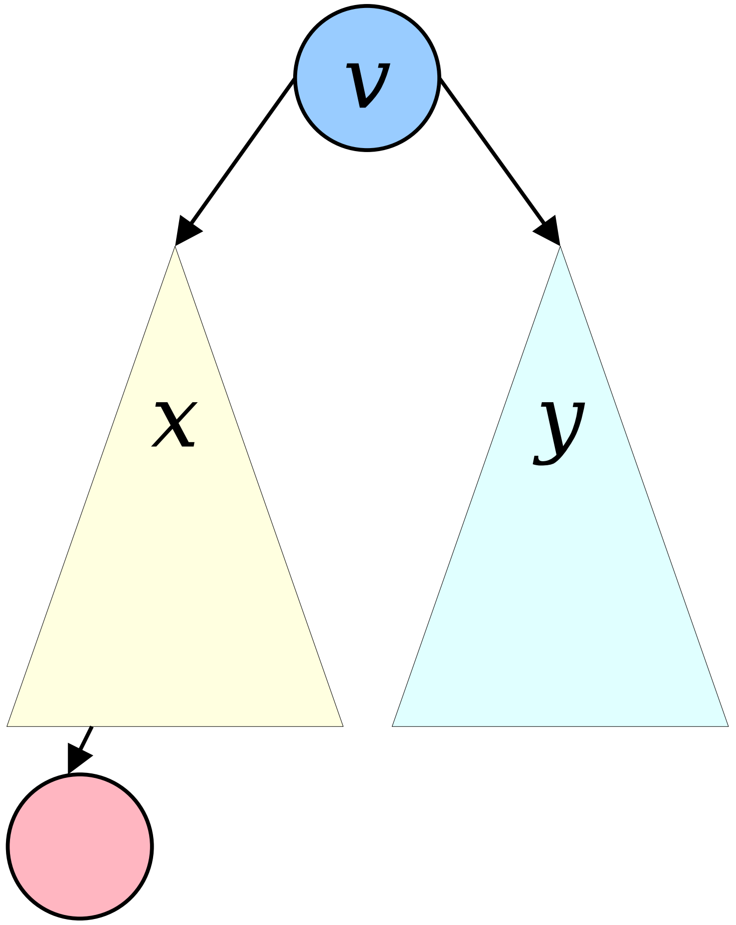
- ***Case 2:*** Our insertion triggers a rebuild.
- Recall that

  $$\textbf{\textit{amortized-cost}} = \textbf{\textit{real-cost}} + \boldsymbol{k} \cdot \Delta\Phi$$

  for a constant k that we pick.
- Here, *real-cost* is O(log n + size(v)), where v is the scapegoat node.
  - The O(log n) comes from the cost of the actual insertion.
  - The O(size(v)) is for the cost of rebuilding.
- For this to amortize away, we need ΔΦ to be -Ω(size(v)).
- Our previous intuition tells us this should be the case.
- Let's run the numbers to check.

Analyzing Scapegoat Trees

- Let *v* be the scapegoat node. We're interested in (v).
- One of *v*'s children is a tree containing our newly-inserted node. Call that subtree *x*.
- Call *v*'s other child *y*.
- ***Goal:*** Determine (v) = |size(x) – size(y)|.



Analyzing Scapegoat Trees

- Since v is α-imbalanced, we know

$$\text{height}(v) > \alpha \lg \text{size}(v).$$

- v is the ***deepest*** α-imbalanced node on the access path. This means x is α-balanced, so

$$\text{height}(x) \le \alpha \lg \text{size}(x).$$

- Since the newly-inserted node is the deepest node in v's subtree, we know that

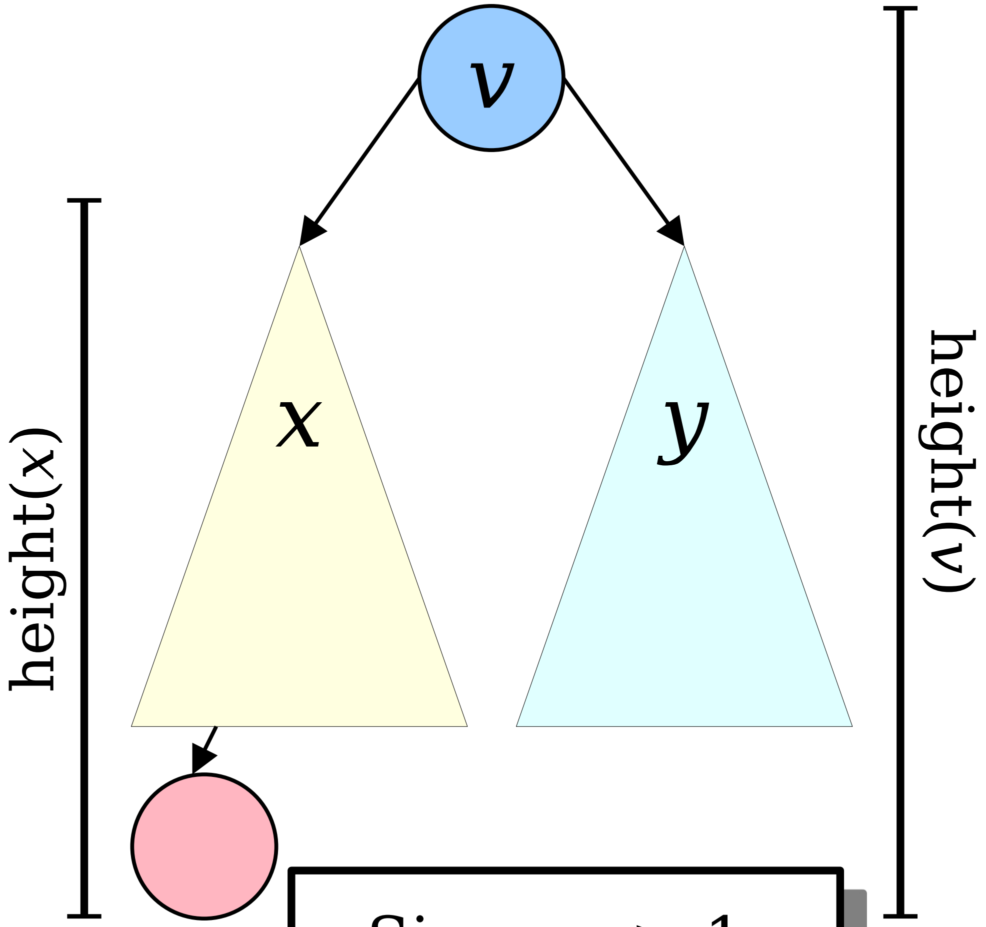
$$\text{height}(v) = \text{height}(x) + 1.$$

- Putting all this together gives

$$\alpha \lg \text{size}(v) < \alpha \lg \text{size}(x) + 1.$$

- That in turn means that

$$\text{size}(v) < \text{size}(x) \cdot 2^{1/\alpha}.$$



Since $\alpha > 1$, we know that $2^{1/\alpha} \in (1, 2)$

Analyzing Scapegoat Trees

- We just proved that

$$\text{size}(v) < \text{size}(x) \cdot 2^{1/\alpha}.$$

- We also know that

$$\text{size}(v) = 1 + \text{size}(x) + \text{size}(y).$$

- That means

$$\text{size}(x) + \text{size}(y) < \text{size}(x) \cdot 2^{1/\alpha}.$$

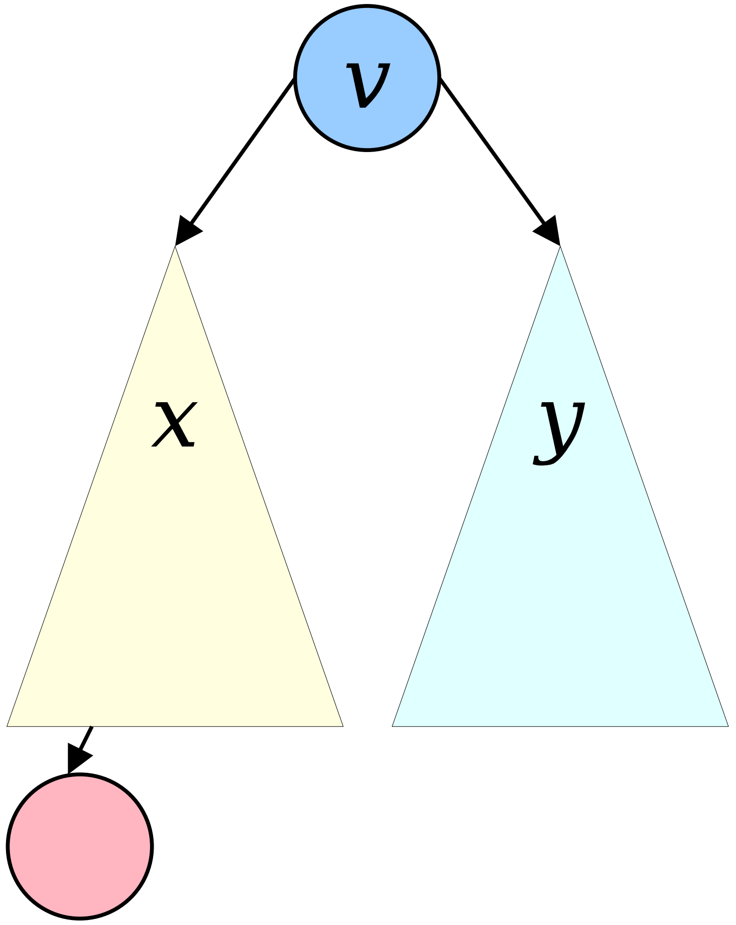
- Therefore,

$$\text{size}(y) < \text{size}(x) \cdot (2^{1/\alpha} - 1).$$

Since $2^{1/\alpha} \in (1, 2)$, we know $2^{1/\alpha} - 1 \in (0, 1)$.

So y must have fewer nodes than x.

(Surprising, but true! Explore and see why!)



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- That means

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- Therefore,

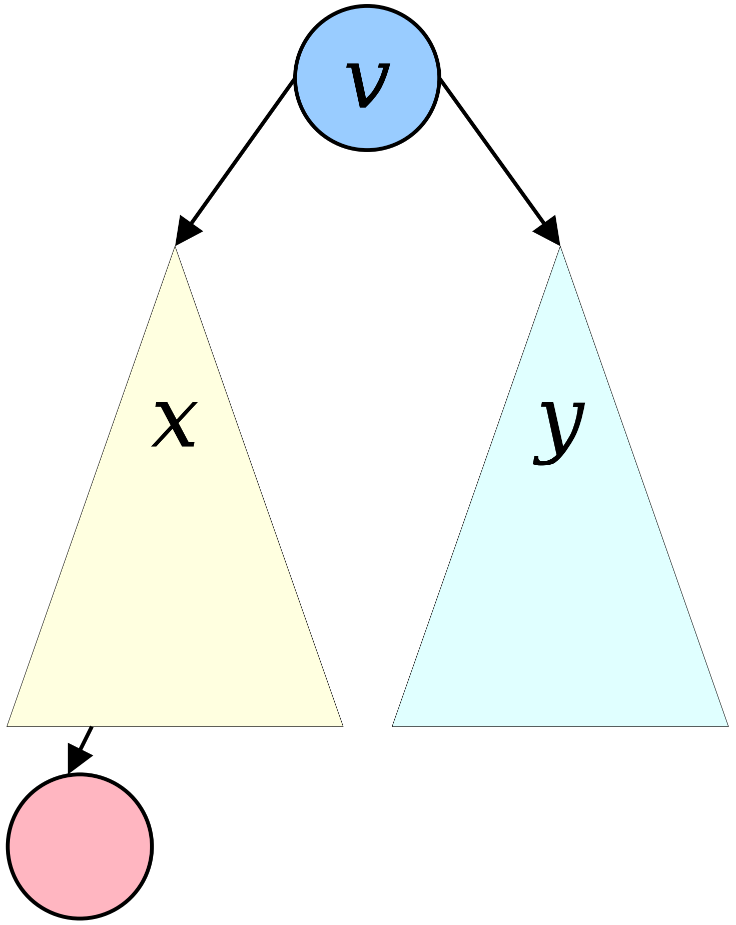
$$\text{size}(y) < \text{size}(x) \cdot (2^{1/\alpha} - 1).$$

- This means that

$$\begin{aligned}(v) &= |\text{size}(x) - \text{size}(y)| \\ &> \text{size}(x) - \text{size}(x) \cdot (2^{1/\alpha} - 1) \\ &= \text{size}(x) \cdot (2 - 2^{1/\alpha}).\end{aligned}$$

- Combined with the initial inequality, this gives us that

$$(v) > \text{size}(v) \cdot (2^{1 - 1/\alpha} - 1).$$



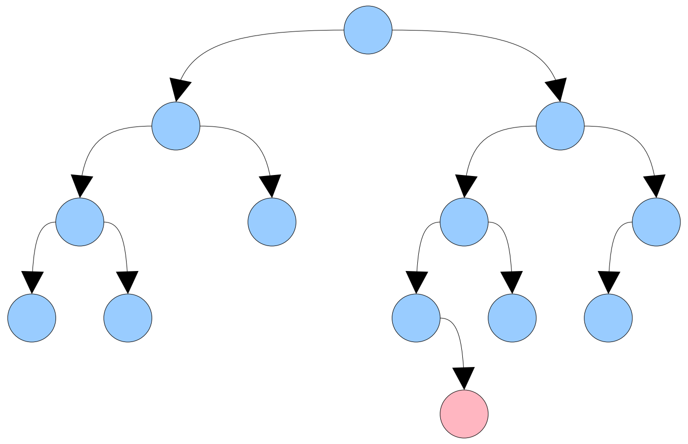
$2^{1 - 1/\alpha} \in (1, 2)$,
So $2^{1 - 1/\alpha} - 1 \in (0, 1)$.

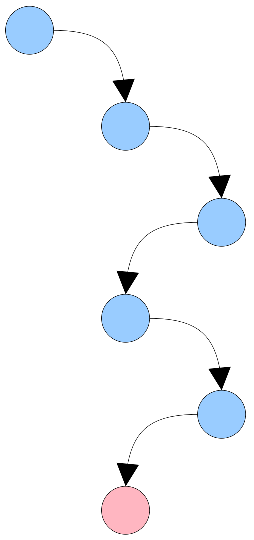
Analyzing Scapegoat Trees

- We've just concluded that

  $(v) > \text{size}(v) \cdot (2^{1 - 1/\alpha} - 1)$

- Let's take a minute to check our math.
- If α is close to 1, we're requiring the trees to be very tightly balanced. Therefore, when an imbalance occurs, we'd expect (v) to be small relative to size(v).
- If α is large, we're allowing for huge imbalances in the trees. Therefore, when a node is too deep, we expect the tree it's a part of to be highly imbalanced, so we'd expect (v) to be large relative to size(v).





Analyzing Scapegoat Trees

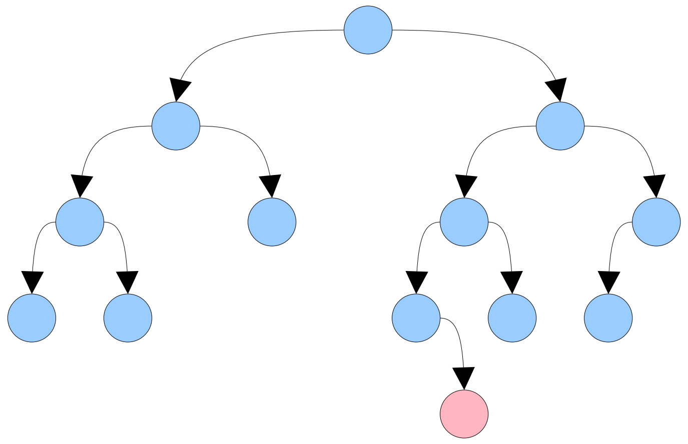
- We've just concluded that

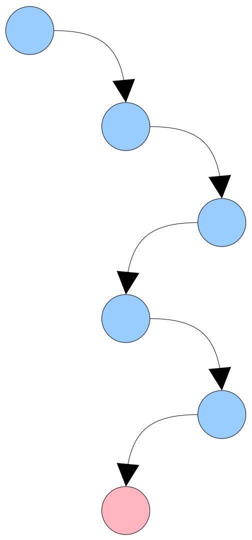
$$(v) > \text{size}(v) \cdot (2^{1 - 1/\alpha} - 1)$$

- Notice that for any fixed value of α that we have

$$(v) = \Omega(\text{size}(v)).$$

- In other words, the scapegoat node always has an imbalance that is (at least) linear in the size of its subtree.
- We can then backcharge the linear work required to optimally rebuild it to the operations that caused the imbalance in the first place.





Analyzing Scapegoat Trees

- We can now work out the amortized cost of an insertion that triggers a rebuild.
  - Actual cost of inserting a new node: O(log n).
  - Actual cost of rebuilding at the scapegoat node: O(size(v)).
  - Change in potential: $\Delta\Phi < -\Omega(\text{size}(v))$.
- Amortized cost:

$$O(\log n) + O(\text{size}(v)) - k \cdot \Omega(\text{size}(v)).$$

- By tuning k based on the hidden constant factors in the O and Ω terms, we can get them to cancel, leaving an amortized cost of **O(log *n*)**.

Where We Stand

- Here's the current scorecard for scapegoat trees.
- Intuitively:
  - If you pick α to be smaller, you get a more balanced tree (faster lookups), but the overhead to optimally rebuild subtrees gets bigger (slower insertions).
  - If you pick α to be larger, you get a less balanced tree (slower lookups), but the overhead to optimally rebuild trees is smaller (faster insertions).
- Tuning α appropriately now becomes a matter of engineering.
- ***Question:*** What about deletions?

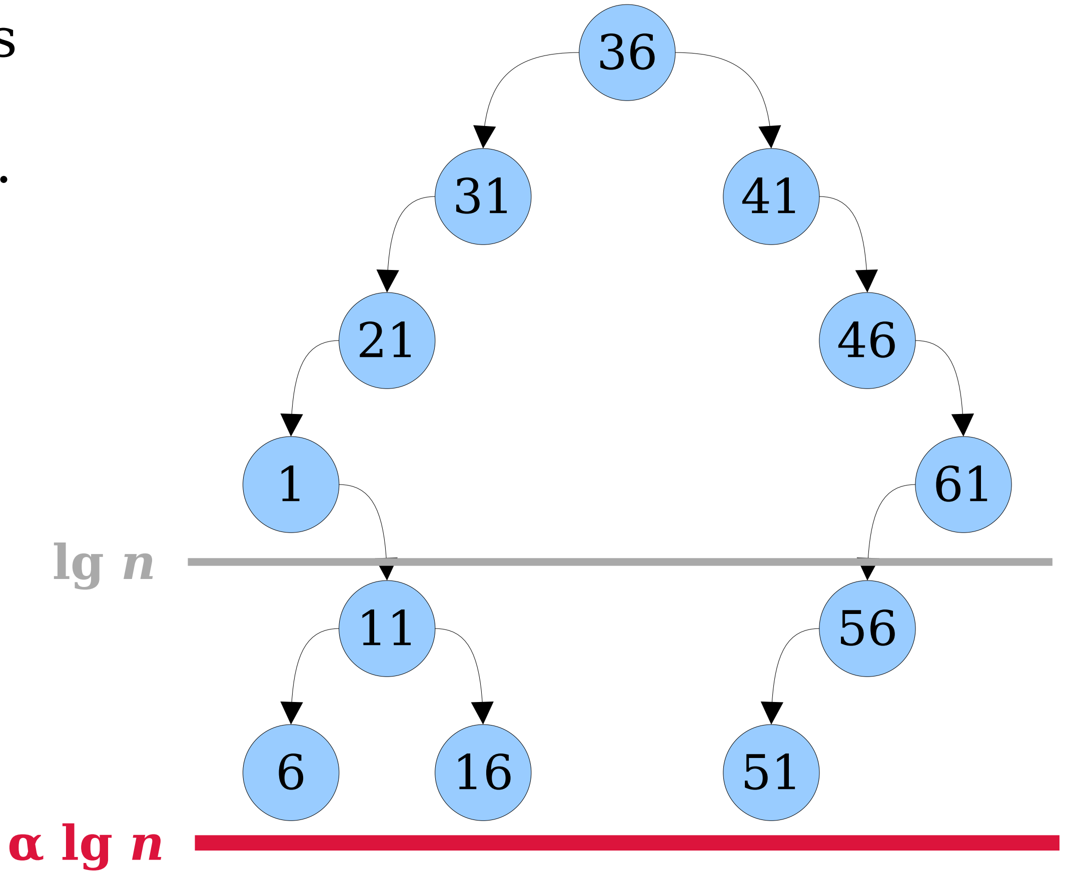
Scapegoat Tree

Lookup:	$O(\log n)$
Insert:	$O(\log n)^*$

* *amortized*

Why Deletions are Different

- In the insert-only case, we can easily detect when the height is violated, and we know which node exceeded the height limit.
- Neither of these are true with deletions.
  - Deleting one node may make an unrelated node height above the threshold.
  - Deleting one node may make multiple unrelated nodes exceed the threshold.



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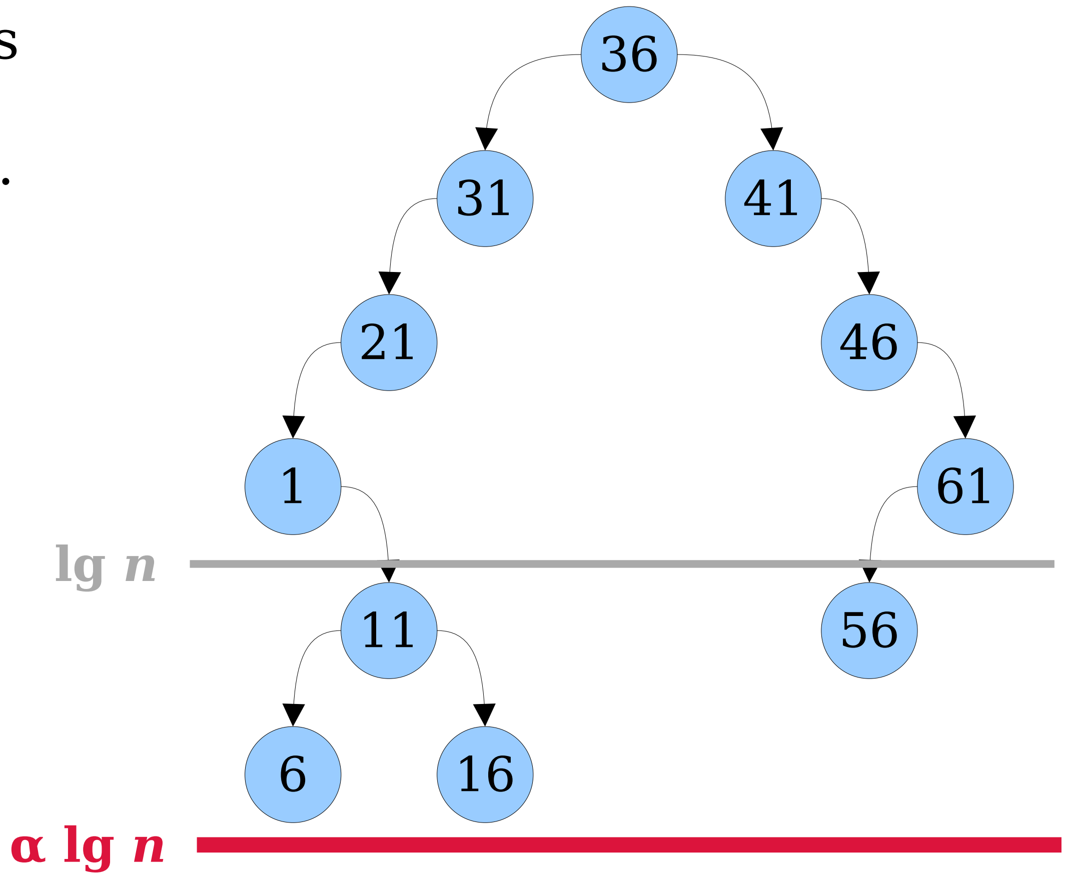

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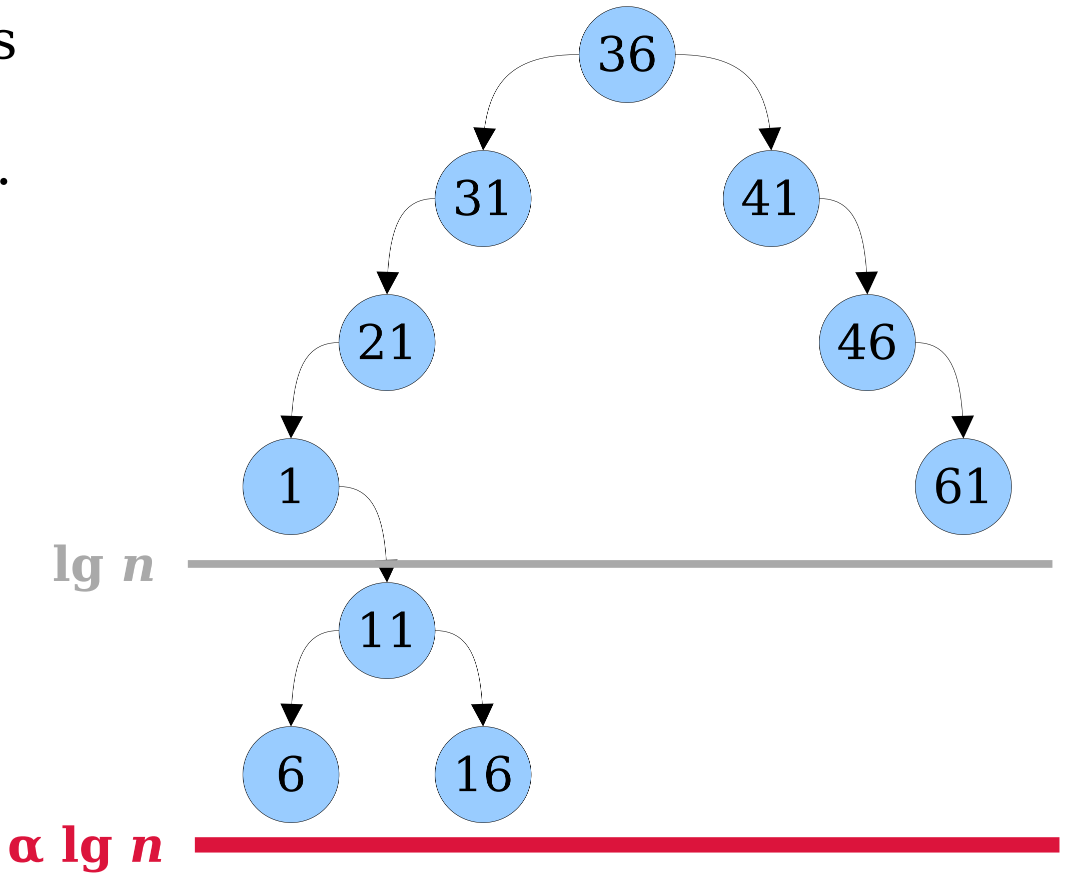

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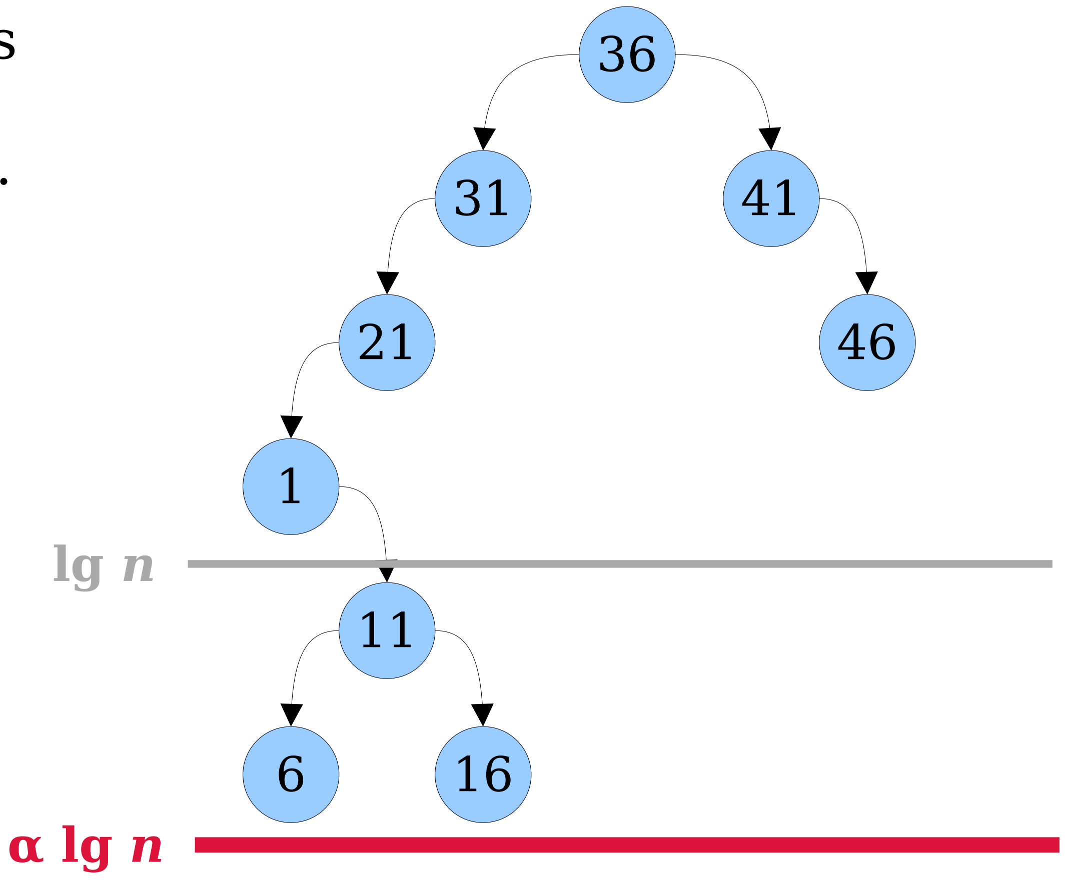

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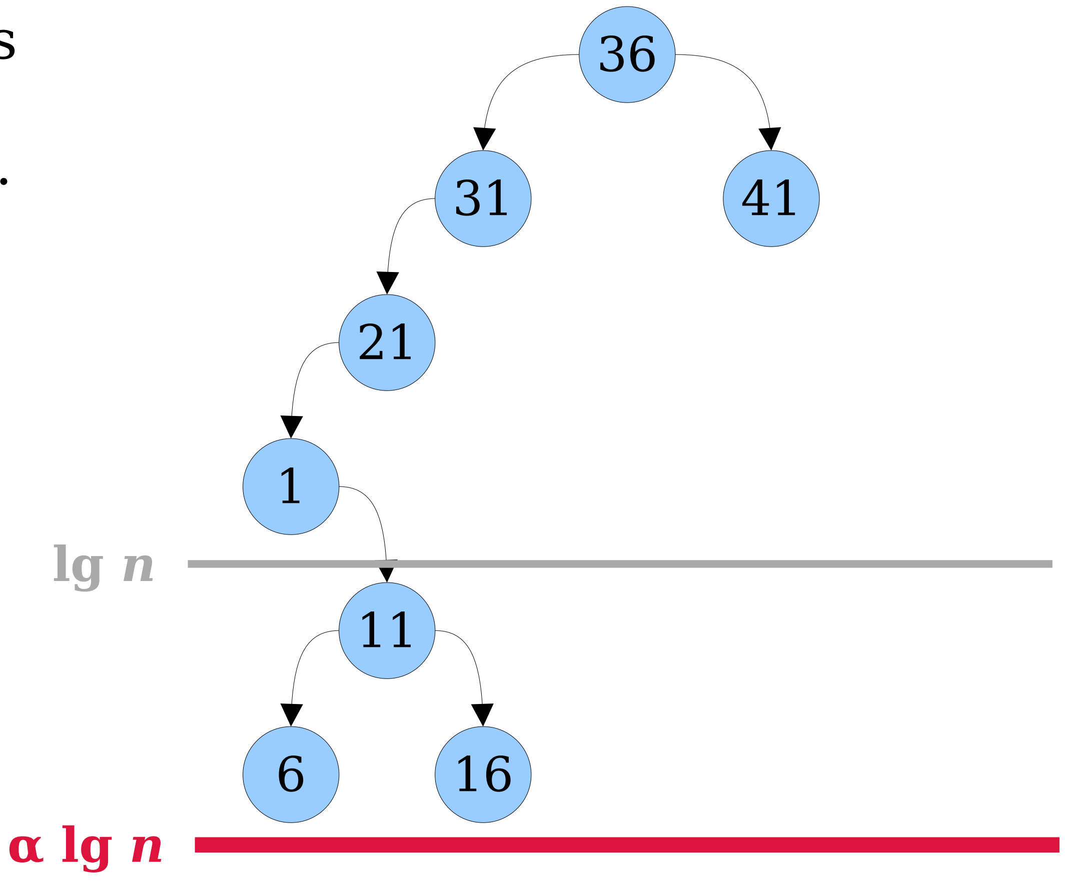

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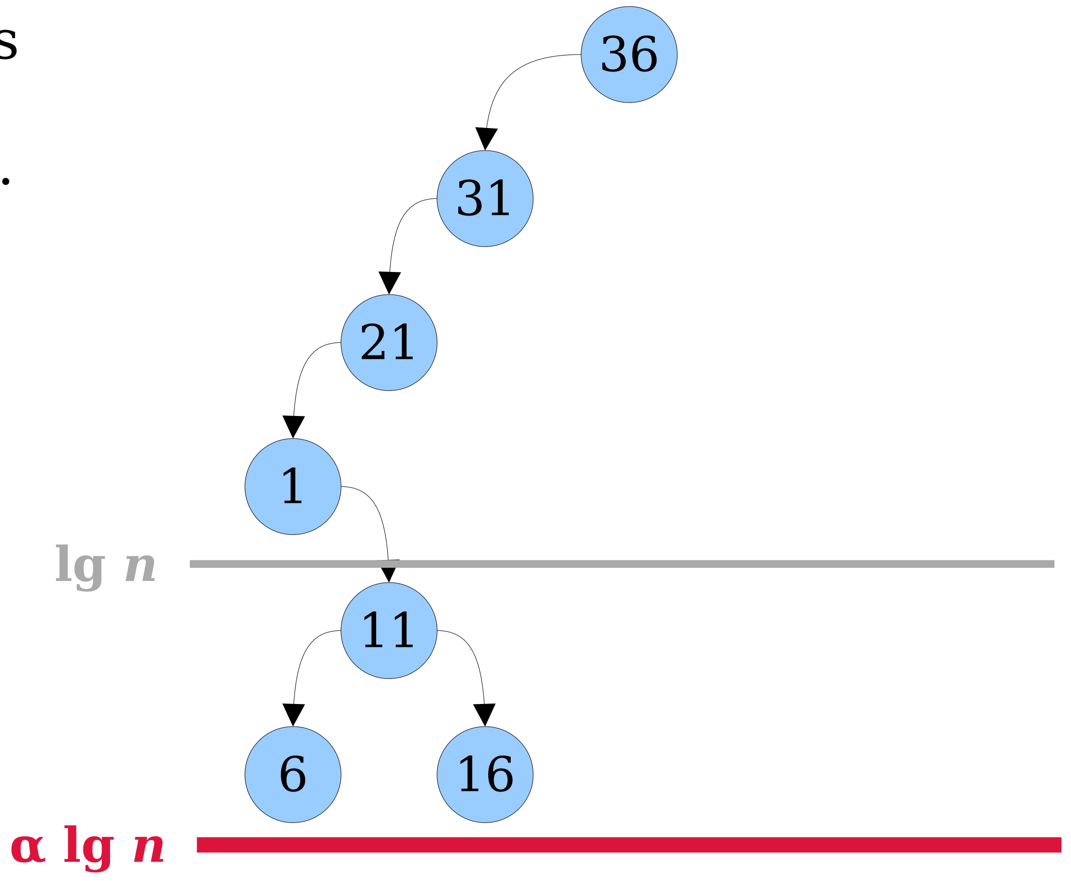

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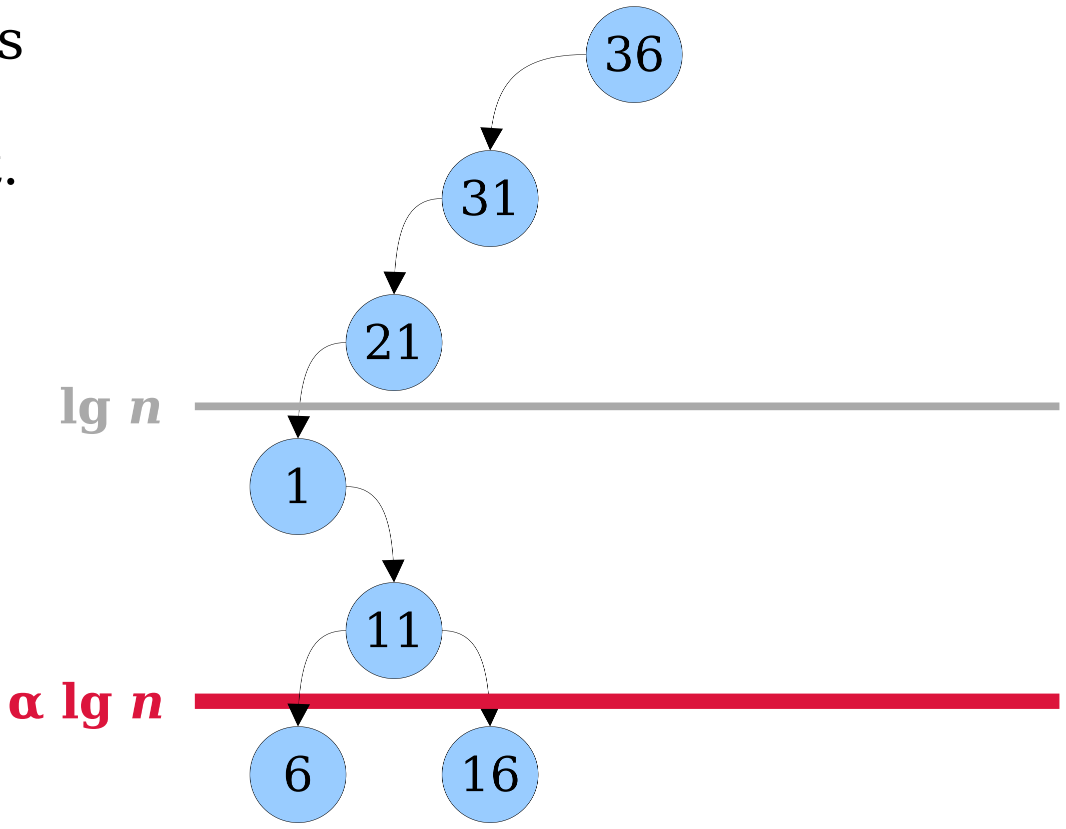

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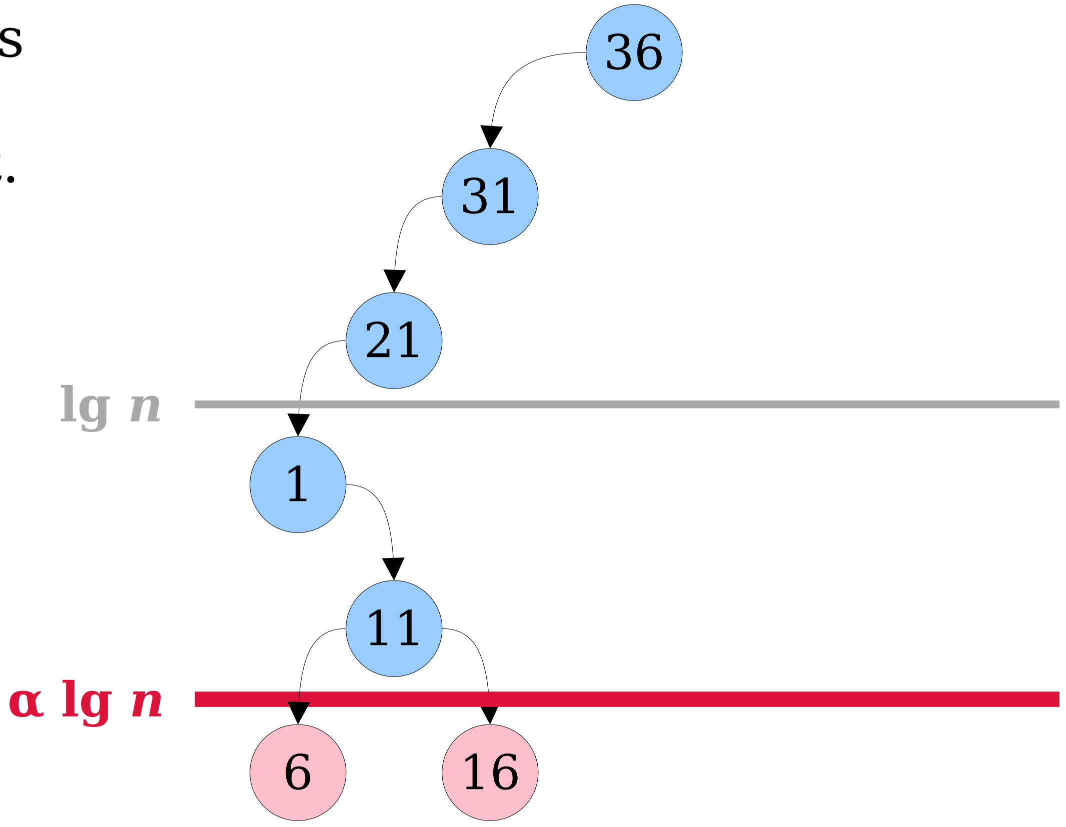
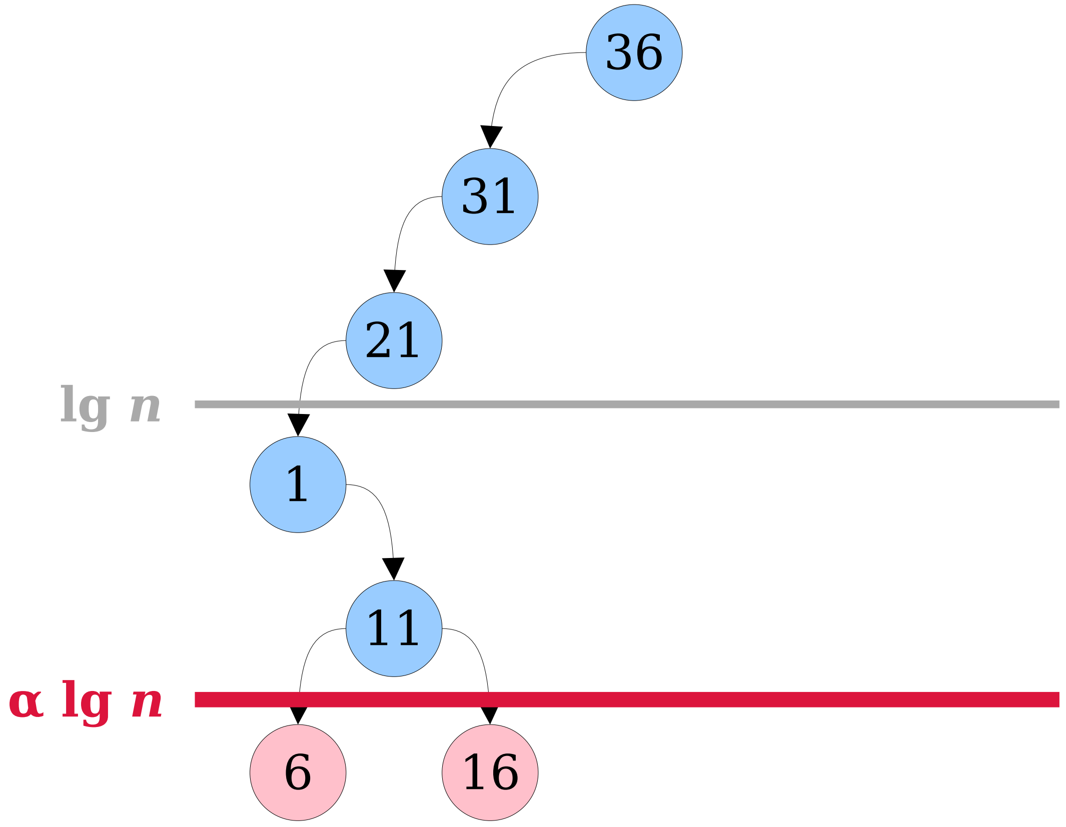
- Neither of these are true with deletions.
  - Deleting one node may make an unrelated node height above the threshold.
  - Deleting one node may make multiple unrelated nodes exceed the threshold.
- ***Intuition:*** Deletions will require some sort of *global* rebuilding of the tree, rather than the *local* rebuilding we saw earlier.



Why Deletions are Different

- As we delete nodes from our BST, the value of $\alpha \lg n$ will decrease, but it does so slowly.
- Leaf nodes will be the first to exceed the $\alpha \lg n$ threshold.
- However, a very large number of nodes need to be deleted before non-leaves cross the threshold.
- Let's quantify this.



Why Deletions are Different

- Suppose our tree currently has n nodes in it. We'll perform some number of deletions and arrive at a tree with n_{new} nodes.
- At what value of n_{new} is it possible for non-leaf nodes to have a depth greater than $\alpha \lg n_{new}$?

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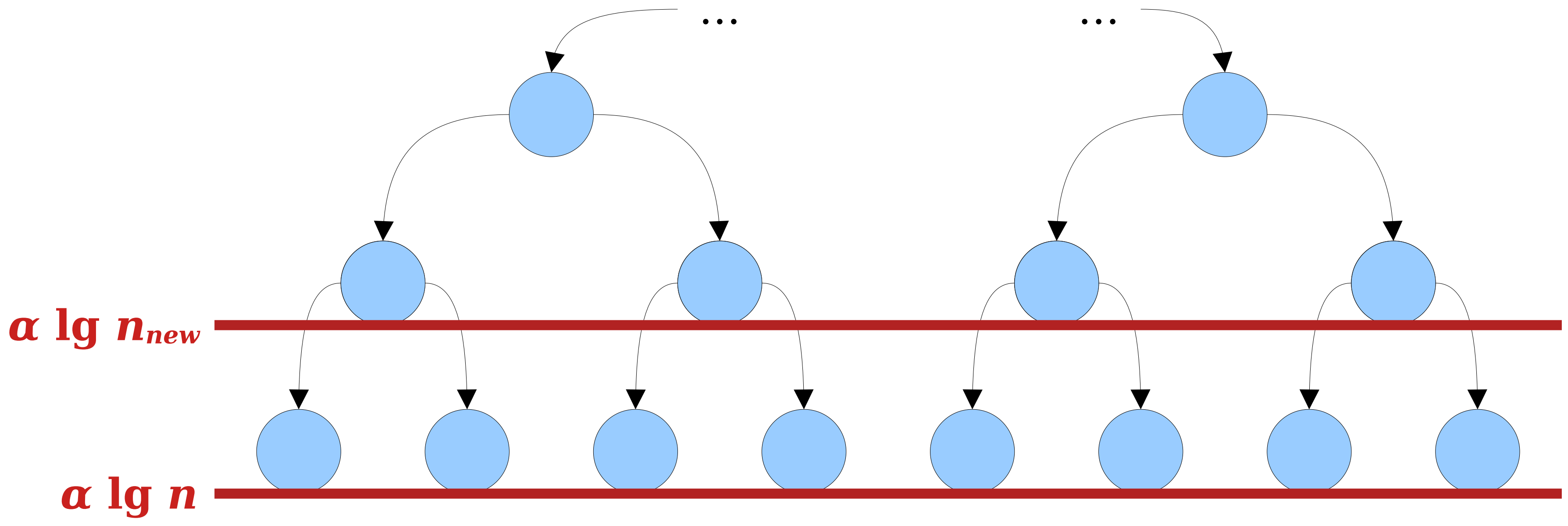

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- Suppose our tree currently has n nodes in it. We'll perform some number of deletions and arrive at a tree with n_{new} nodes.
- At what value of n_{new} is it possible for non-leaf nodes to have a depth greater than $\alpha \lg n_{new}$?
- We need to solve

$$\alpha \lg n_{new} < \alpha \lg n - 1.$$

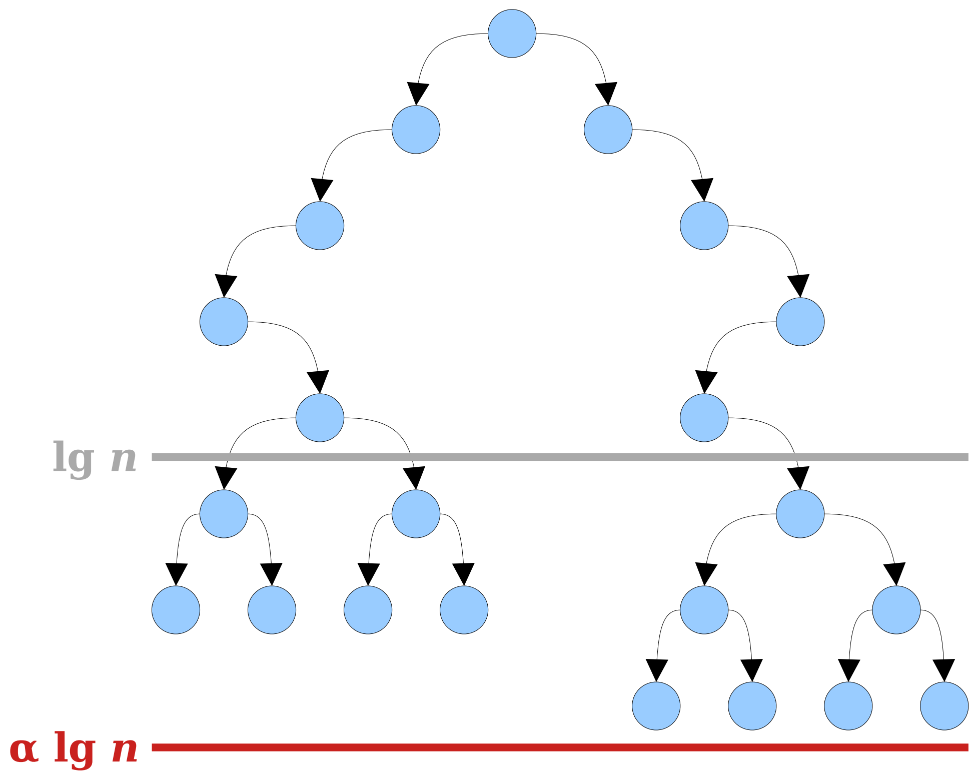
- Rearranging gives us that

$$n_{new} < n \cdot 2^{-1/\alpha}.$$

- Note that $2^{-1/\alpha} \in (½, 1)$ for any $\alpha > 1$.
- We need to delete at least a constant fraction (specifically, a $1 - 2^{-1/\alpha}$ fraction) of the nodes before nodes one layer above the bottom could exceed the $\alpha \lg n$ limit.

Why Deletions are Different

- ***Idea:*** Don't worry about rebalancing until we lose a $(1 - 2^{-1/\alpha})$ fraction of the nodes.



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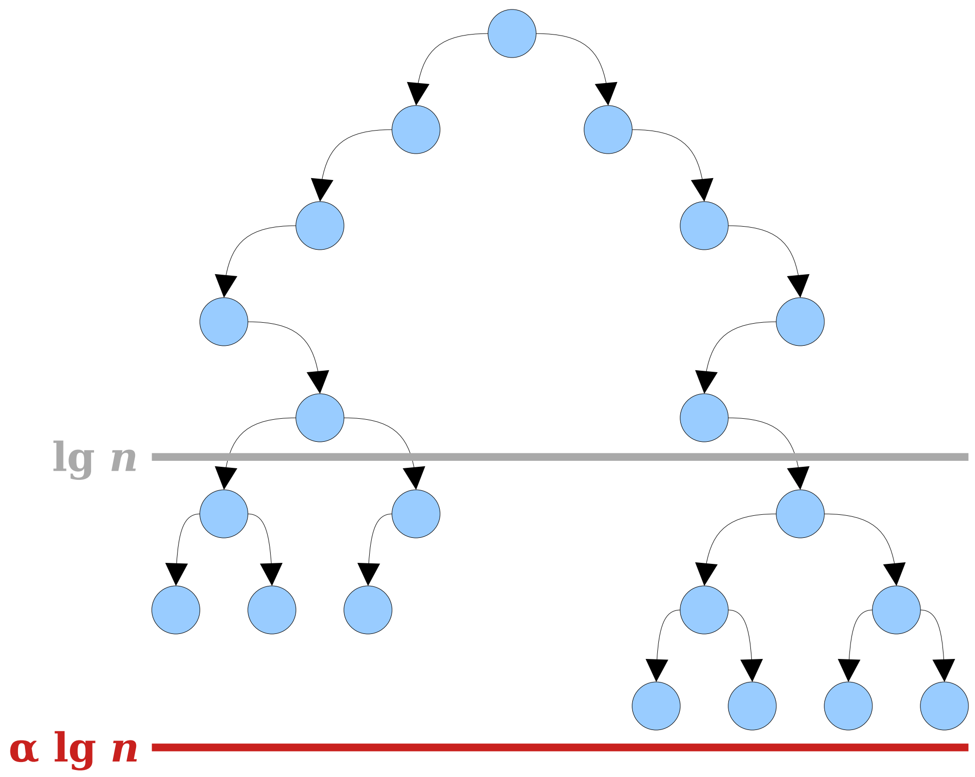

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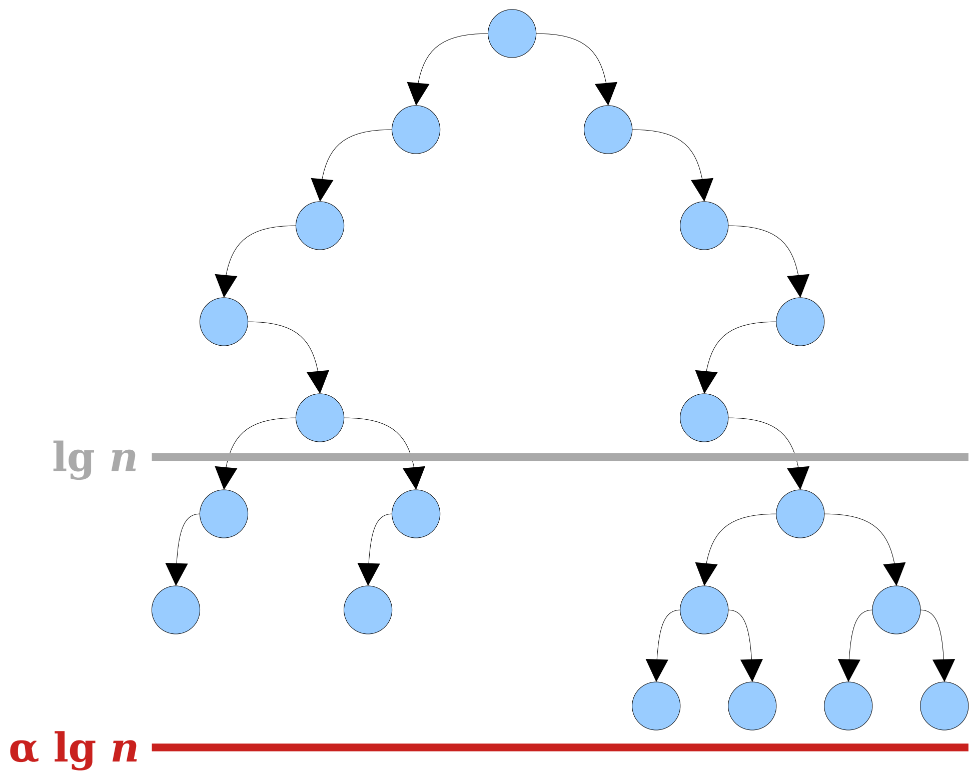

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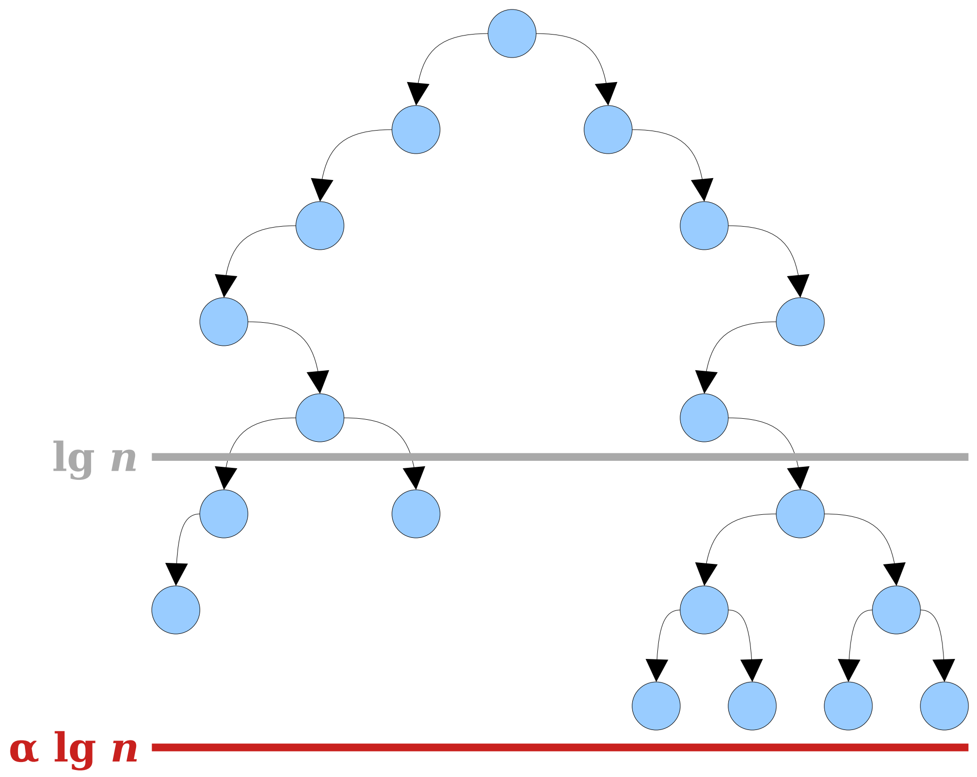

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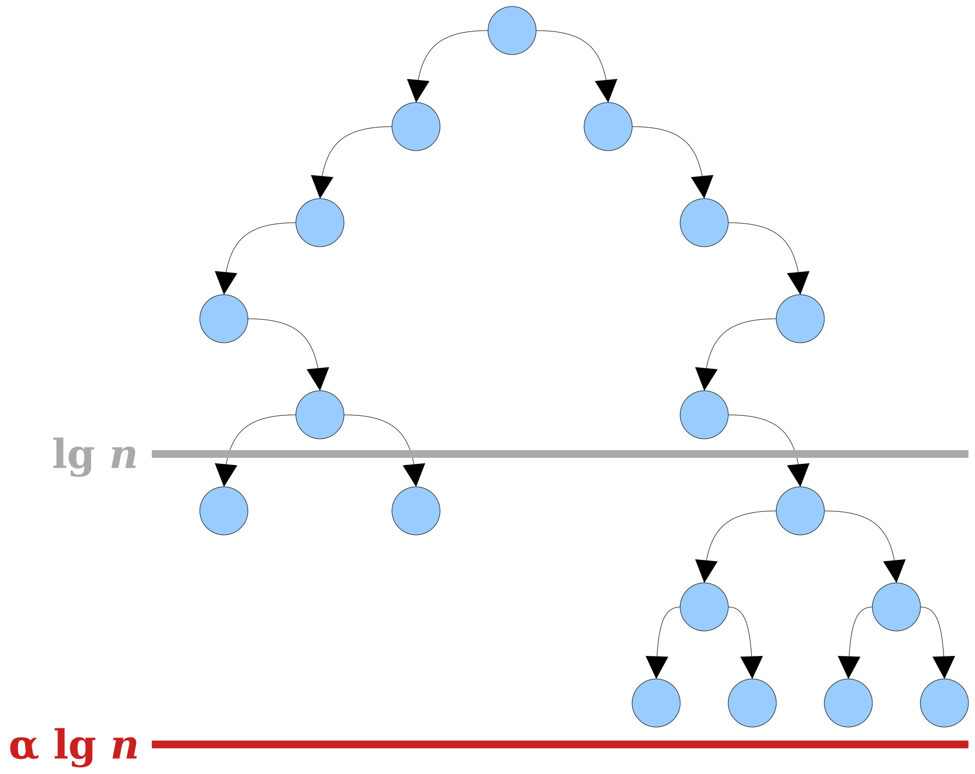
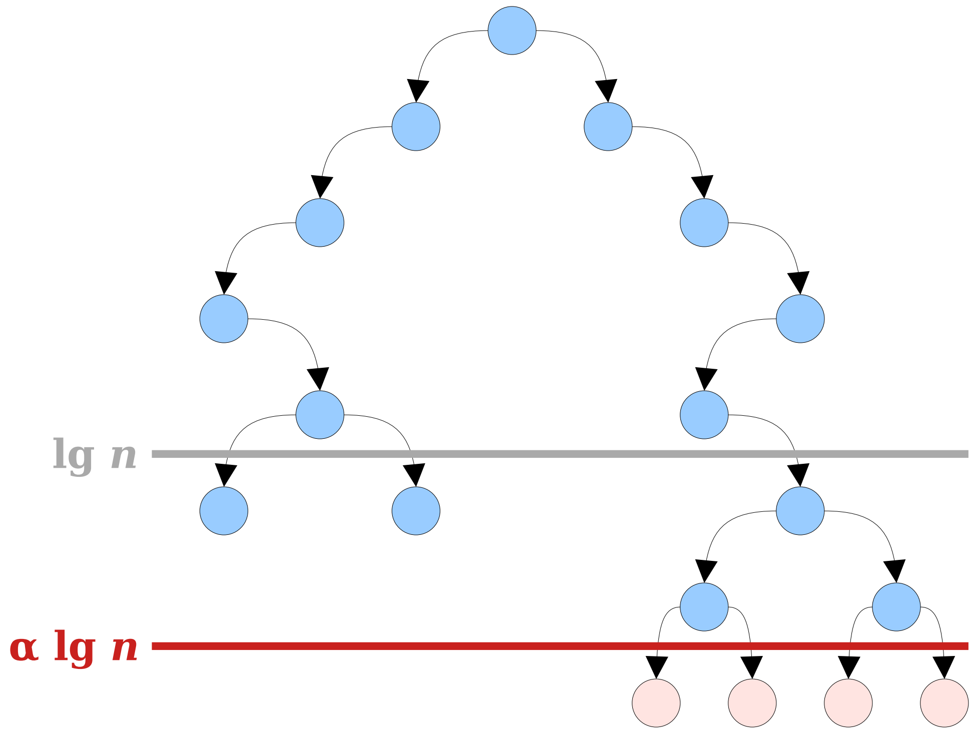


Why Deletions are Different

- ***Idea:*** Don't worry about rebalancing until we lose a $(1 - 2^{-1/\alpha})$ fraction of the nodes.
- Assuming we lose fewer than this many nodes, all nodes in the tree will be at depth at most $\alpha \lg n + 1$.
  - Focus on any node. Assume there were n_0 nodes at the point when the node was inserted. The node depth is then at most $\alpha \lg n_0$.
  - As long as we haven't lost at least a $(1 - 2^{-1/\alpha})$ fraction of the nodes, the current value of n is such that $\alpha \lg n \geq \alpha \lg n_0 - 1$.
- This still gives us lookups that run in time O(log n), and insertions still work properly.



Why Deletions are Different

- ***Idea:*** Don't worry about rebalancing until we lose a $(1 - 2^{-1/\alpha})$ fraction of the nodes.
- Assuming we lose fewer than this many nodes, all nodes in the tree will be at depth at most $\alpha \lg n + 1$.
    - Focus on any node. Assume there were n_0 nodes at the point when the node was inserted. The node depth is then at most $\alpha \lg n_0$.
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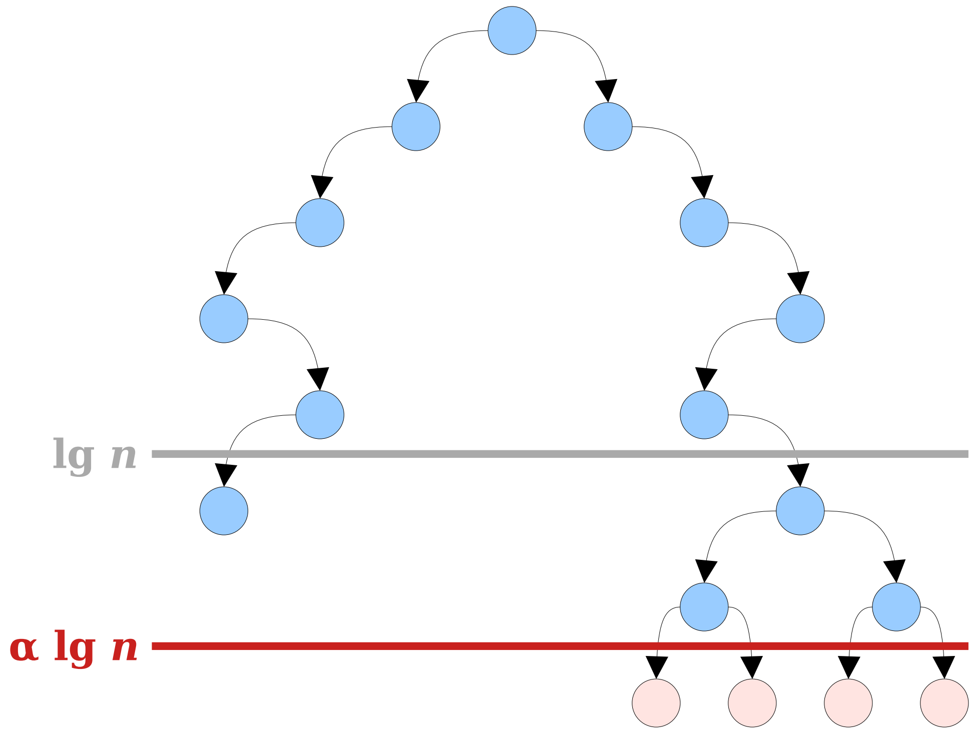

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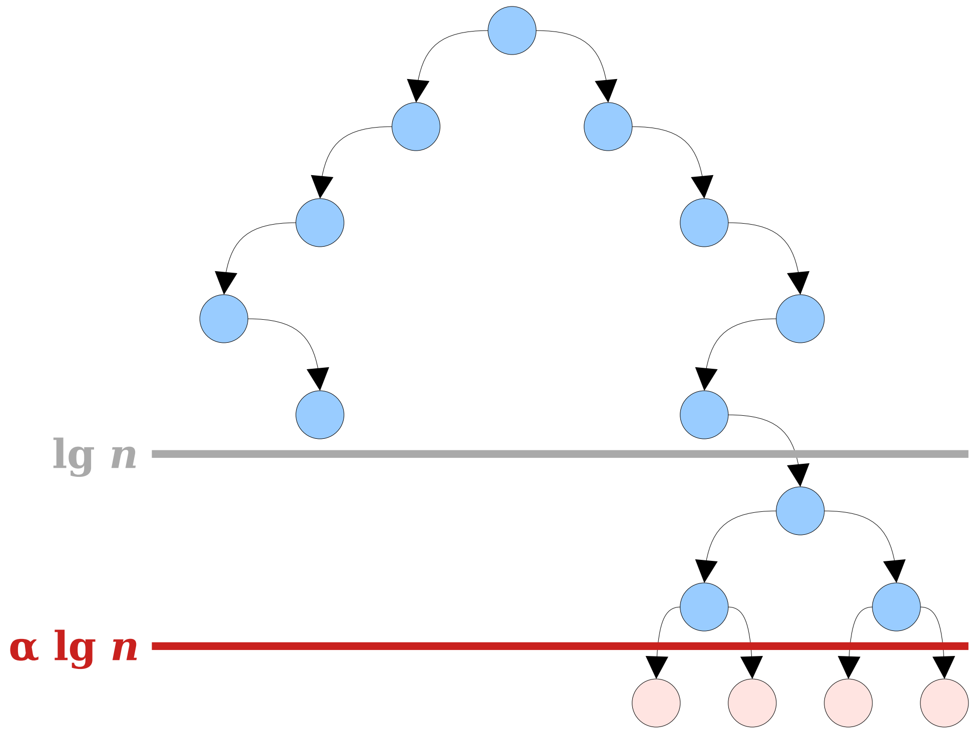

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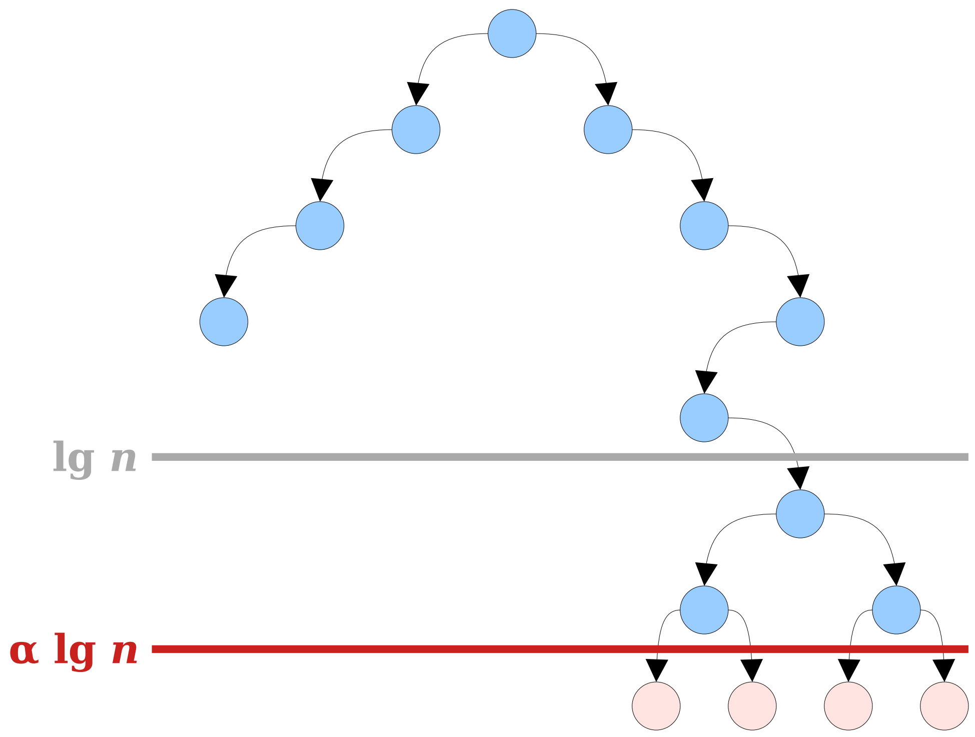

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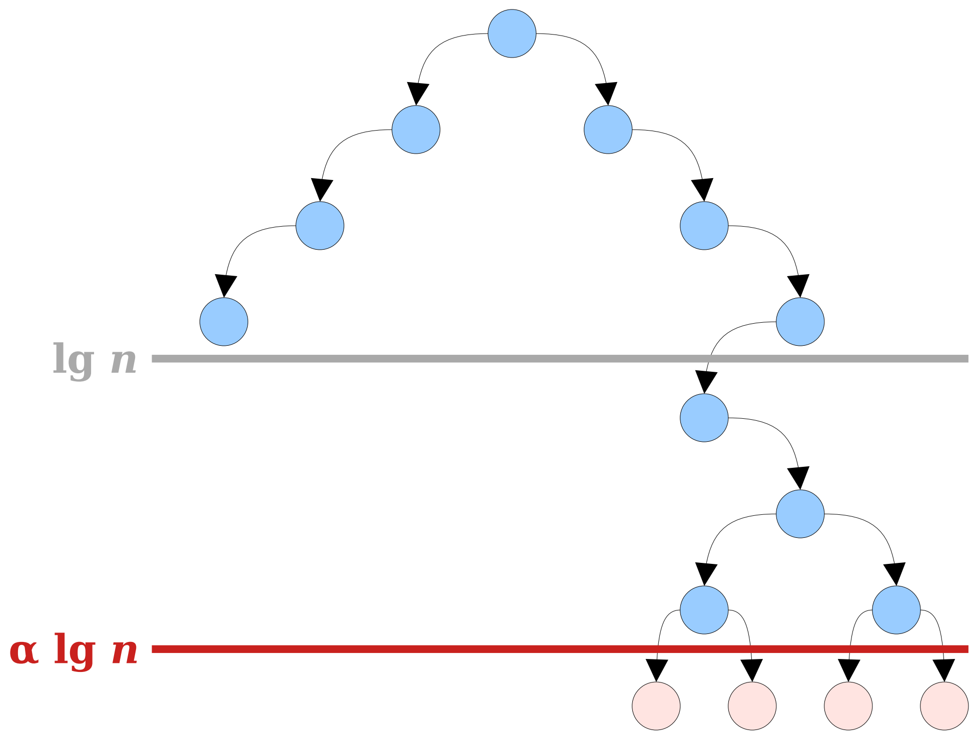

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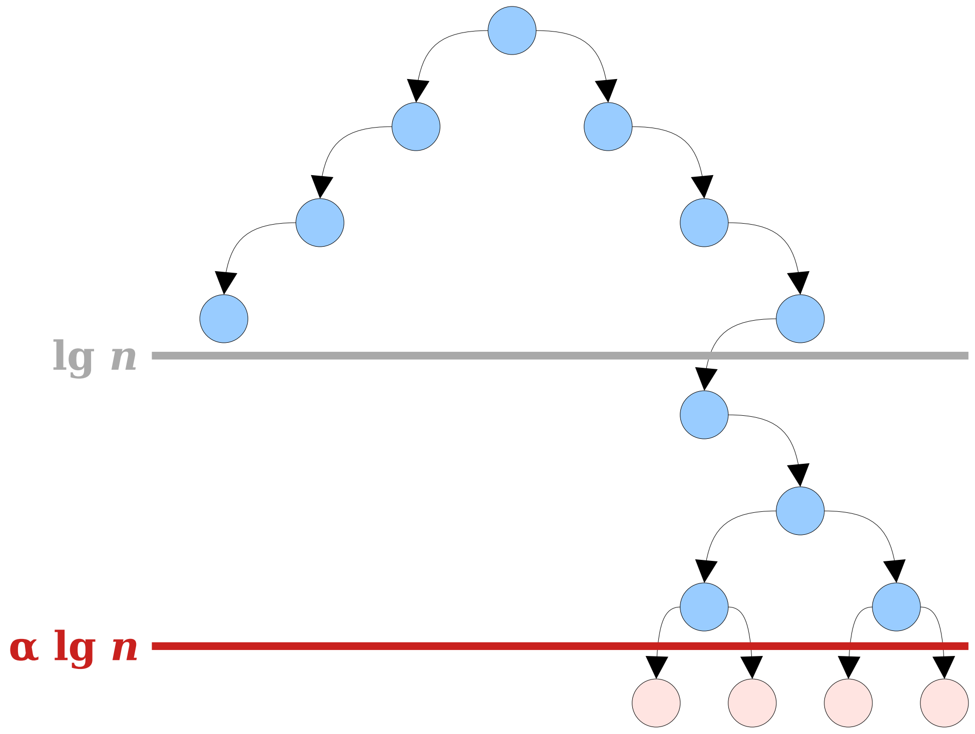

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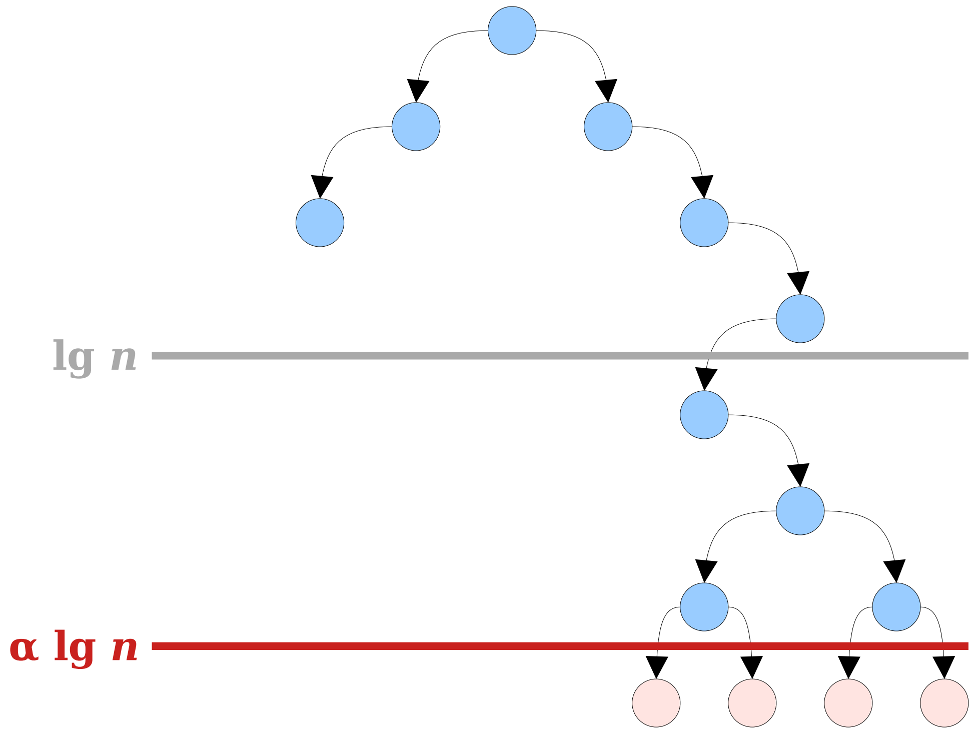

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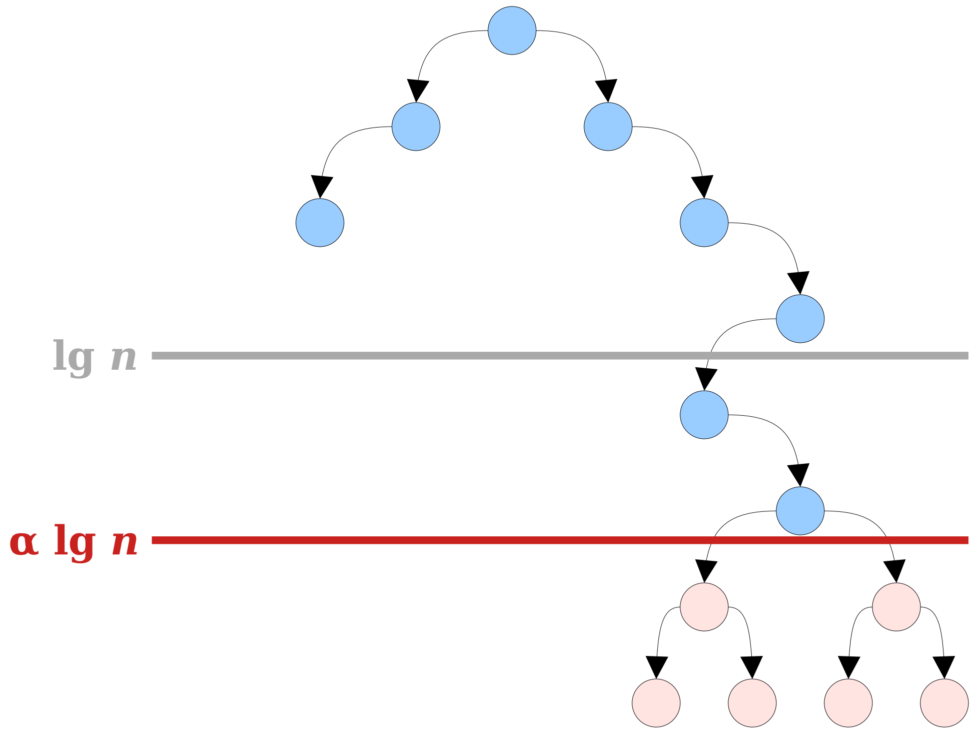

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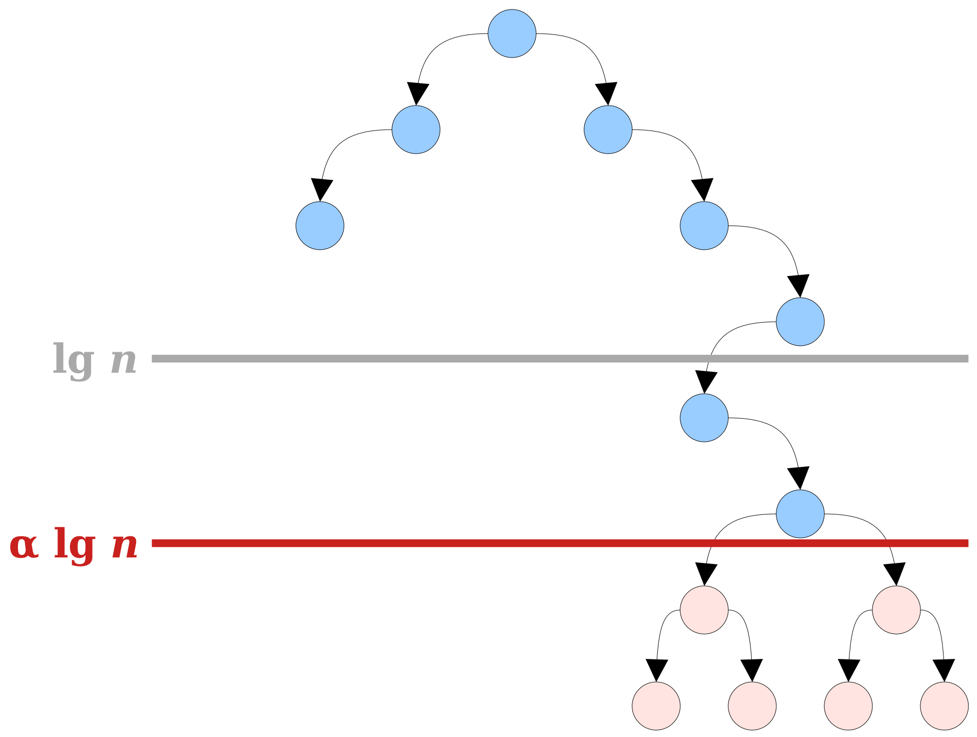
- Once we've lost a $(1 - 2^{-1/\alpha})$ fraction of the nodes, we need to worry about rebalancing the tree.
- We won't know much about the tree shape.
    - It could have a large number of deep nodes.
    - It could be perfectly balanced.
- ***Idea:*** Don't try to analyze the tree. Just rebuild the entire tree from scratch.



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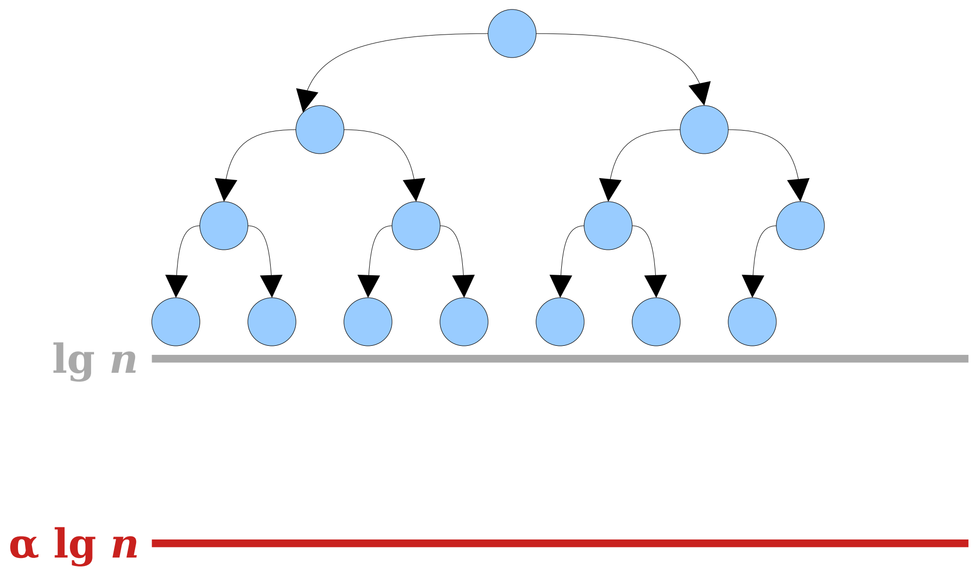

Scapegoat Tree Deletions

- Here's how this approach will work.
  - Keep track of the maximum number of nodes the tree has had since it was last globally rebuilt. (Call this n_{max}).
  - If the number of nodes drops to a $n_{max} \cdot 2^{-1/\alpha}$, globally rebuild the tree as a perfectly balanced tree, then reset n_{max} to the current tree size.
- Although rebuilding the tree is an expensive operation, intuitively we expect to be able to "backcharge" the work to the lazy delete operations that triggered it.

Scapegoat Tree Deletions

- Our goal now is to work out the amortized cost of doing global rebuilds on deletions.
- ***Recall:*** Our current potential function is

$$\Phi = \Sigma_v \quad '(v),$$

  which we chose to make the cost of local rebuilds on insertions amortize away.
- We need to adjust this potential function to account for the fact that deleted nodes slowly lead us to do a global rebuild of the whole tree.
- ***Idea:*** Change our potential to

$$\Phi = D + \Sigma_v \quad '(v),$$

  where D is the number of deletions that have been performed since we last did a global rebuild.

Scapegoat Tree Deletions

- What is the amortized cost of a deletion when we don’t trigger a global rebuild?
- Actual cost: O(log n), since the tree height is at most α lg n + 1.
- Change in potential (recall that $\Phi = D + \Sigma_v$ ’(v)):
  - D increases by one, since we’ve performed a deletion.
  - ’(v) changes by at most two for each node on the access path of the removed node, and there are O(log n) such nodes.
  - Net change: O(log n).
- Amortized cost:

$$\mathrm{O}(\log n) + k \cdot \mathrm{O}(\log n) = \mathbf{O(\log \boldsymbol{n})}.$$

Scapegoat Tree Deletions

- What is the amortized cost of a deletion when we *do* trigger a global rebuild?
- We picked

$$\Phi = D + \Sigma_v \quad '(v).$$

- After the rebuild, we have $\Sigma_v \quad '(v) = 0$. Therefore, there is an unknown but nonpositive change in potential for this term.
- How much does D change?
  - At the point where we start the rebuild, we have $n = n_{max} \cdot 2^{-1/\alpha}$ nodes left in the tree.
  - This means that $D \geq n_{max} \cdot (1 - 2^{-1/\alpha})$.
  - Rewriting in terms of n, this means $D \geq n \cdot (2^{1/\alpha} - 1) = \Omega(n)$.
  - Since after this step we drop D to zero, we have $\Delta D \leq -\Omega(n)$.
- Overall, we have $\Delta\Phi \leq -\Omega(n)$.

Scapegoat Tree Deletions

- Actual cost of the deletion:
  - O(log n) for the actual deletion logic.
  - O(n) to rebuild the tree.
- Amortized cost:

$$O(\log n) + O(n) - k \cdot \Omega(n).$$

- As before, we can tune k based on the hidden constant factors in the O and Ω terms to make them cancel out and leave behind an amortized cost of **O(log *n*)**.

The Final Scorecard

- Here's the final scorecard for our scapegoat tree.
- It matches the time bounds we'd expect of a red/black tree, in an amortized sense, with a *dramatically* simpler implementation.
- This gives a sense of just how useful a technique amortization can be!

Scapegoat Tree

Lookup:	O(log n)
Insert:	O(log n)*
Delete:	O(log n)*

* *amortized*

Further Exploration

- I haven't seen much work done into building an optimized scapegoat tree implementation. How fast can you make this idea work? Is it competitive with a red/black tree?
- We've treated α as a constant. What if you allow it to vary based on the workflow (say, decreasing it as more lookups happen and increasing it as more deletions/insertions happen)? A past CS166 project team looked into this in 2014, and I'm curious to see it on modern hardware.
- Are there other, less aggressive strategies besides rebuilding the scapegoat subtree that can be used to restore balance?
- Are there other ways of picking a scapegoat node that work better in practice? For example, could you pick a scapegoat higher up in the tree that would do a better job rebalancing things?
- What is the practical time/space tradeoff between the two approaches for calculating size(v) when finding a scapegoat?
- The version of scapegoat trees described here is a hybrid between two approaches: the original developed by Galperin and Rivest and a simplification by Jeff Erickson. The Galperin/Rivest version has tighter structural constraints, while Erickson's version uses a different deletion strategy. Can you remix this ideas in other ways?
- Because there are no rotations, it should be way easier to augment a scapegoat tree than it is to augment a red/black tree. Can you find a weaker set of requirements for augmenting a BST if you assume the tree you're augmenting is a scapegoat tree?

Next Time

- ***Tournament Heaps***
  - A simple and fast priority queue.
- ***Lazy Tournament Heaps***
  - An asymptotically faster priority queue.