# Lecture 3: Policy Evaluation Without Knowing How the World Works / Model Free Policy Evaluation

CS234: RL

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Material builds on structure from David SIlver's Lecture 4: Model-Free Prediction: http://www0.cs.ucl.ac.uk/staff/d.silver/web/Teaching.html . Other resources: Sutton and Barto Jan 1 2018 draft (http://incompleteideas.net/book/the-book-2nd.html) Chapter/Sections: 5.1; 5.5; 6.1-6.3

# Class Structure

- Last Time:
  - Markov reward / decision processes
  - Policy evaluation & control when have true model (of how the world works)
- **Today:**
  - **Policy evaluation when don’t have a model of how the world works**
- Next time:
  - Control when don’t have a model of how the world works

# This Lecture: Policy Evaluation

- **Estimating the expected return of a particular policy if don't have access to true MDP models**

- Dynamic programming
- Monte Carlo policy evaluation
  - Policy evaluation when don't have a model of how the world work
    - Given on-policy samples
    - Given off-policy samples
- Temporal Difference (TD)
- Metrics to evaluate and compare algorithms

# Recall

- **Definition of return $G_t$ for a MDP under policy π**:
  - Discounted sum of rewards from time step t to horizon when following policy π(a|s)
  - $G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \ldots$
- **Definition of state value function $V^\pi(s)$ for policy π**:
  - Expected return from starting in state s under policy **π**
  - $V^\pi(s) = \mathbb{E}_\pi [G_t | s_t = s] = \mathbb{E}_\pi [r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \ldots | s_t = s]$
- **Definition of state-action value function $Q^\pi(s,a)$ for policy π**:
  - Expected return from starting in state s, taking action a, and then following policy **π**
  - $Q^\pi(s,a) = \mathbb{E}_\pi [G_t | s_t = s, a_t = a] = \mathbb{E}_\pi [r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \ldots | s_t = s, a_t = a]$

# Dynamic Programming for Policy Evaluation

- Initialize $V_0(s) = 0$ for all s
- For k=1 until convergence
    - For all s in S:

$$V_k^\pi(s) = R^\pi(s) + \gamma \sum_{s' \in S} P^\pi(s'|s) V_{k-1}^\pi(s')$$

$$R^\pi(s) = \sum_{a \in A} \pi(a|s) R(s,a)$$

$$P^\pi(s'|s) = \sum_{a \in A} \pi(a|s) P(s'|s,a)$$

# Dynamic Programming for Policy Evaluation

- Initialize $V_0(s) = 0$ for all s
- For k=1 until convergence
  - For all s in S:

**Bellman backup for a particular policy**

$$V_k^\pi(s) = B^\pi V_{k-1}^\pi(s) = R^\pi(s) + \gamma \sum_{s' \in S} P^\pi(s'|s) V_{k-1}^\pi(s')$$

$$R^\pi(s) = \sum_{a \in A} \pi(a|s) R(s,a)$$

$$P^\pi(s'|s) = \sum_{a \in A} \pi(a|s) P(s'|s,a)$$

# Dynamic Programming for Policy π Value Evaluation

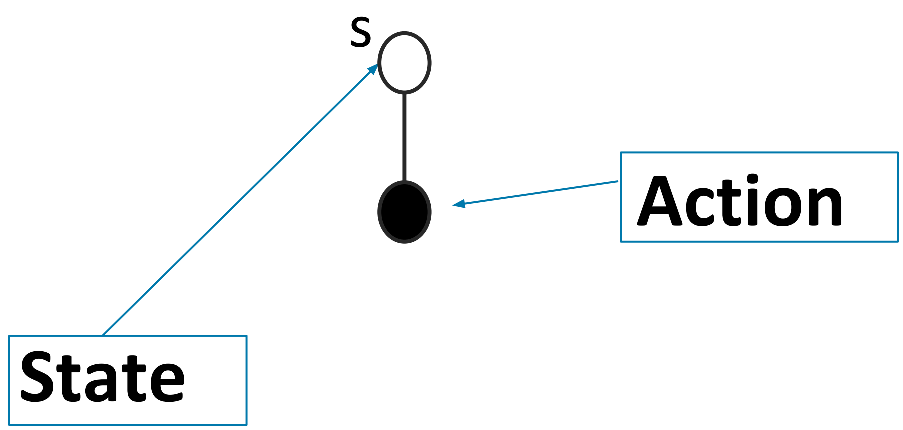
- Initialize $V_0(s) = 0$ for all s
- For i=1 until convergence*
  - For all s in S:

$$V_k^\pi(s) = R^\pi(s) + \gamma \sum_{s' \in S} P^\pi(s'|s) V_{k-1}^\pi(s')$$

- In finite horizon case, $V_k^\pi(s)$ is exact value of *k*-horizon value of state s under policy π
- In infinite horizon case, $V_k^\pi(s)$ is an estimate of infinite horizon value of state s
  - $\mathbf{V^\pi(s) = \mathbb{E}_\pi [G_t | s_t = s] \cong \mathbb{E}_\pi [r_t + \gamma V_{i-1} | s_t = s]}$

# Dynamic Programming Policy Evaluation

$$V^{\pi}(s) \leftarrow \mathbb{E}_{\pi}[r_t + \gamma V_{i-1} | s_t = s]$$



# Dynamic Programming Policy Evaluation

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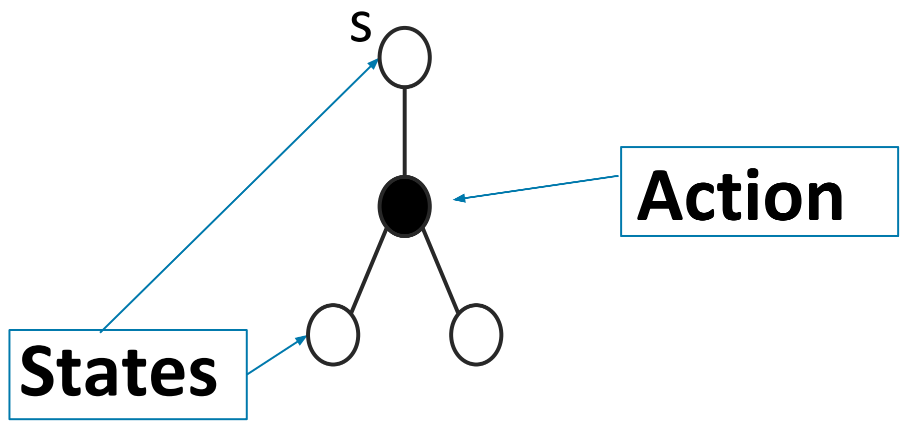

# Dynamic Programming Policy Evaluation

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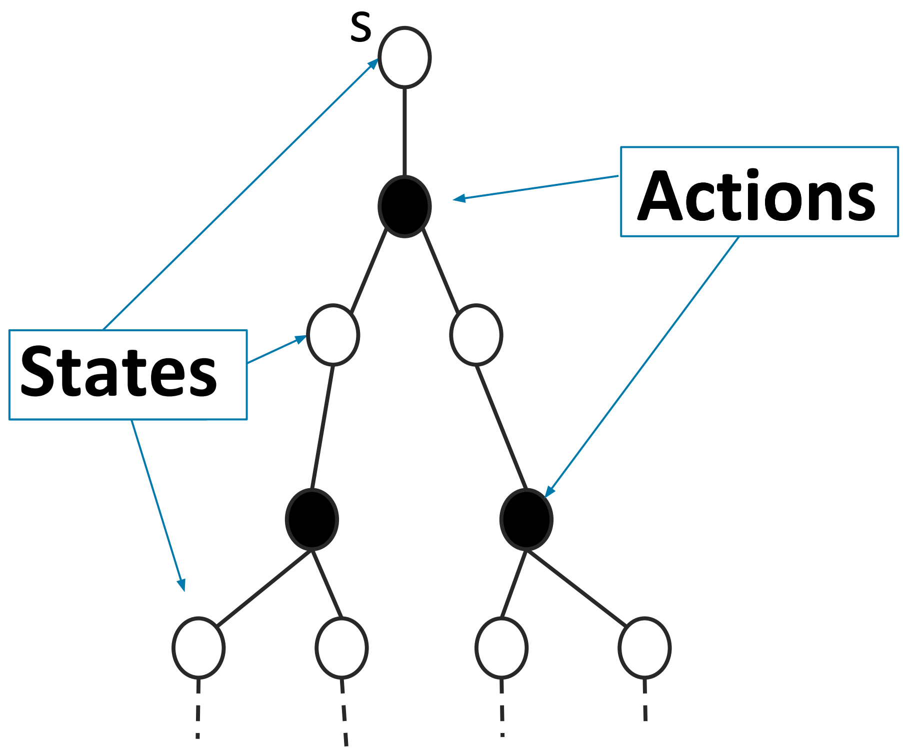
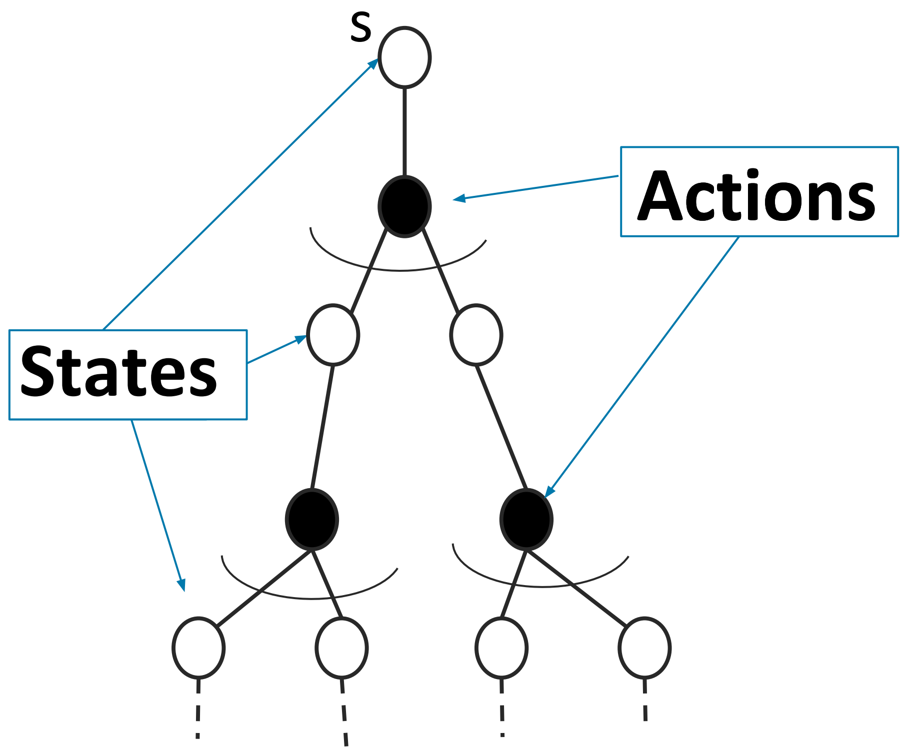


# Dynamic Programming Policy Evaluation

$$V^{\pi}(s) \leftarrow \mathbb{E}_{\pi}[r_t + \gamma V_{i-1} | s_t = s]$$

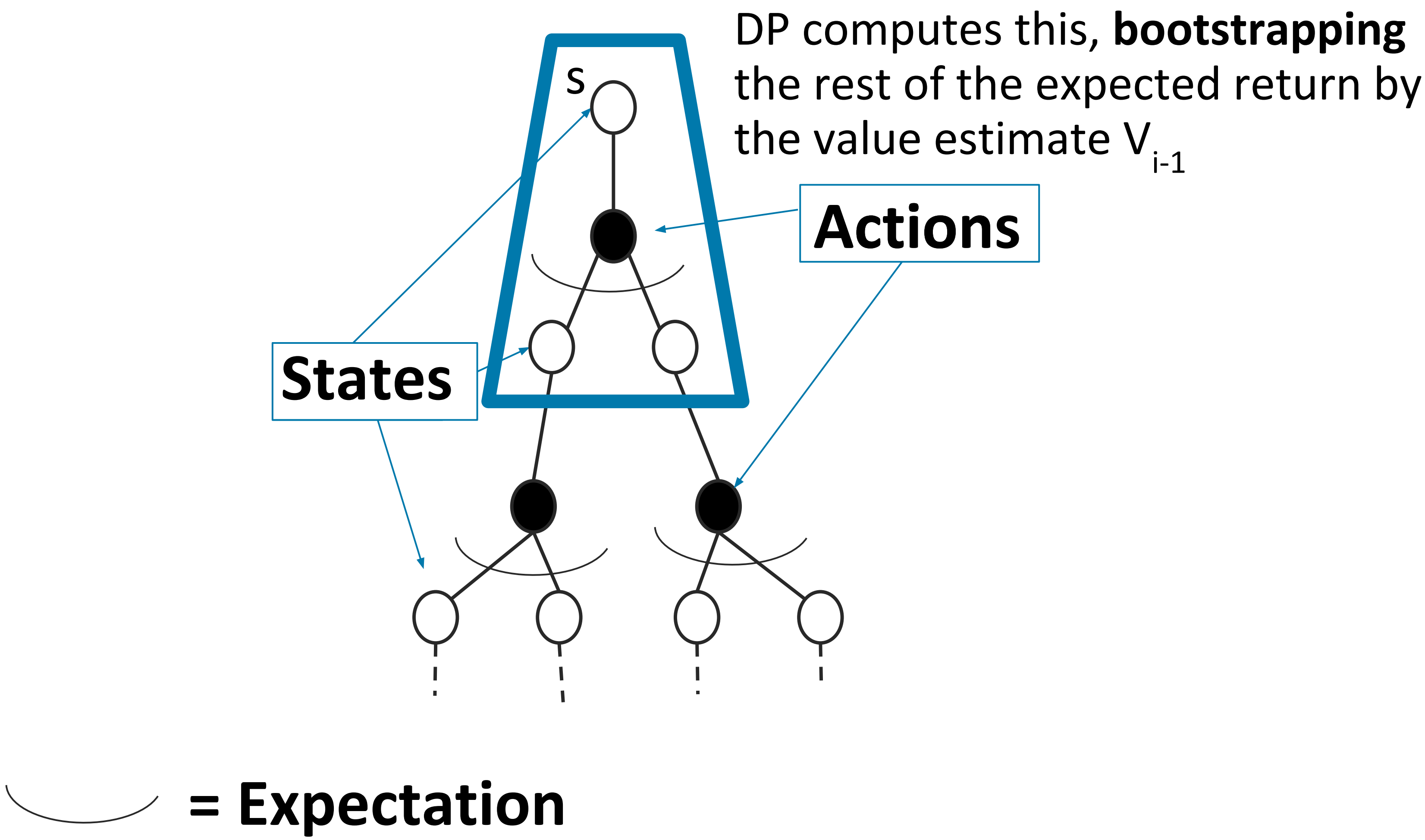




**= Expectation**

# Dynamic Programming Policy Evaluation

$$V^{\pi}(s) \leftarrow \mathbb{E}_{\pi}[r_t + \gamma \mathbf{V_{i-1}} \,|\, s_t = s]$$




- **Bootstrapping**: Update for V uses an estimate

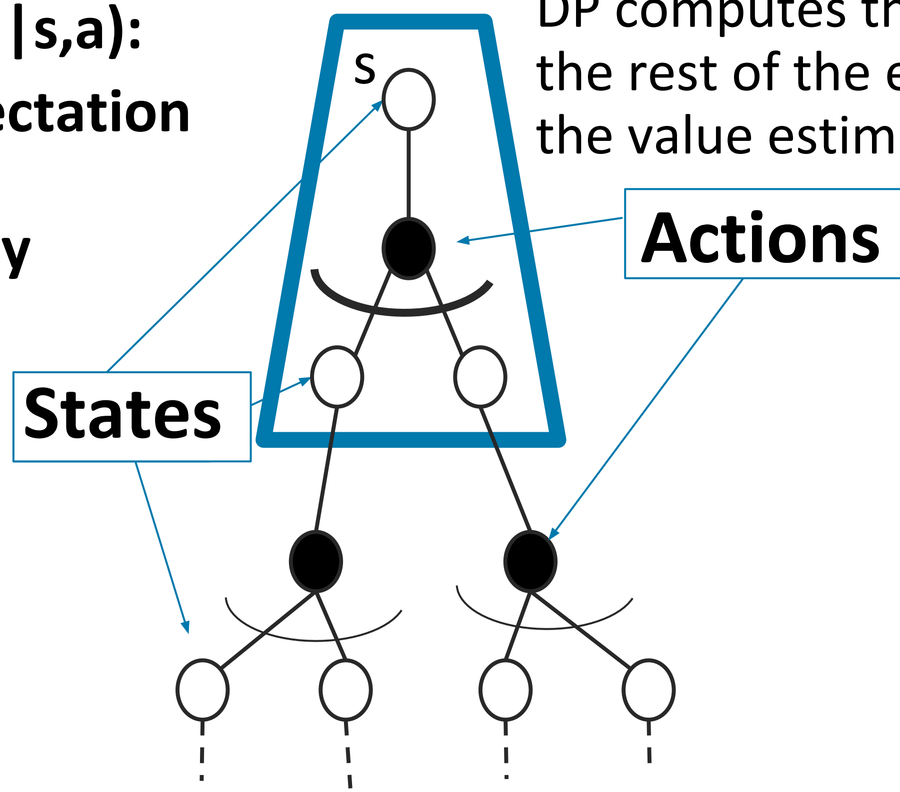
# Dynamic Programming Policy Evaluation

$$V^{\pi}(s) \leftarrow \mathbb{E}_{\pi}[r_t + \gamma V_{i-1} | s_t = s]$$

**Know model P(s'|s,a):**

**reward and expectation over next states computed exactly**

DP computes this, bootstrapping the rest of the expected return by the value estimate $V_{i-1}$




**= Expectation**

- Bootstrapping: Update for V uses an estimate

# Policy Evaluation: $V^\pi(s) = \mathbb{E}_\pi [G_t | s_t = s]$

- $G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \ldots$ in MDP M under a policy π
- Dynamic programming
  - $V^\pi(s) \cong \mathbb{E}_\pi[r_t + \gamma V_{i-1} | s_t = s]$
  - Requires model of MDP M
  - Bootstraps future return using value estimate
- What if don't know how the world works?
  - Precisely, don't know dynamics model P or reward model R
- **Today: Policy evaluation without a model**
  - Given data and/or ability to interact in the environment
  - Efficiently compute a good estimate of a policy π

# This Lecture: Policy Evaluation

- Dynamic programming
- **Monte Carlo policy evaluation**
  - Policy evaluation when don't have a model of how the world work
    - Given on policy samples
    - Given off policy samples
- Temporal Difference (TD)
- Axes to evaluate and compare algorithms

# Monte Carlo (MC) Policy Evaluation

- $G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \ldots$ in MDP M under a policy π
- $V^{\pi}(s) = \mathbb{E}_{\tau \sim \pi} [G_t | s_t = s]$
  - Expectation over trajectories τ generated by following π

# Monte Carlo (MC) Policy Evaluation

- $G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \ldots$ in MDP M under a policy π
- $V^\pi(s) = \mathbb{E}_{\tau \sim \pi}[G_t | s_t = s]$
  - Expectation over trajectories τ generated by following π

- Simple idea: Value = mean return
- If trajectories are all finite, sample a bunch of trajectories and average returns
- By law of large numbers, average return converges to mean

# Monte Carlo (MC) Policy Evaluation

- If trajectories are all finite, sample a bunch of trajectories and average returns

- Does not require MDP dynamics / rewards
- No bootstrapping
- Does not assume state is Markov
- Can only be applied to episodic MDPs
  - Averaging over returns from a complete episode
  - Requires each episode to terminate

# Monte Carlo (MC) On Policy Evaluation

- Aim: estimate $V^{\pi}(s)$ given episodes generated under policy $\pi$
  - $s_1, a_1, r_1, s_2, a_2, r_2, \ldots$ where the actions are sampled from $\pi$
- $G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \ldots$ in MDP M under a policy $\pi$
- $V^{\pi}(s) = \mathbb{E}_{\pi} [G_t | s_t = s]$
- MC computes empirical mean return
- Often do this in an incremental fashion
  - After each episode, update estimate of $V^{\pi}$

# First-Visit Monte Carlo (MC) On Policy Evaluation

11 10 8 17 4

1 2 1 3 4

t 1 2 3 4 5

- After each episode *i* $= s_{i1}, a_{i1}, r_{i1}, s_{i2}, a_{i2}, r_{i2}, \ldots$
  - Define $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \ldots$ as return from time step *t* onwards in *i*-th episode
  - For each state *s* visited in episode *i*
    - For **first** time *t* state *s* is visited in episode *i*
      - Increment counter of total first visits $N(s) = N(s) + 1$
      - Increment total return $S(s) = S(s) + G_{i,t}$
      - Update estimate $V^{\pi}(s) = S(s) / N(s)$
- By law of large numbers, as $N(s) \rightarrow \infty$, $V^{\pi}(s) \rightarrow \mathbb{E}_{\pi} [G_t | s_t = s]$

# Every-Visit Monte Carlo (MC) On Policy Evaluation

- After each episode $i = s_{i1}, a_{i1}, r_{i1}, s_{i2}, a_{i2}, r_{i2}, \ldots$
  - Define $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \ldots$ as return from time step *t* onwards in *i*-th episode
  - For each state *s* visited in episode *i*
    - For **every** time *t* state *s* is visited in episode *i*
      - Increment counter of total visits $N(s) = N(s) + 1$
      - Increment total return $S(s) = S(s) + G_{i,t}$
      - Update estimate $V^\pi(s) = S(s) / N(s)$

2/2

- As $N(s) \to \infty$, $V^\pi(s) \to \mathbb{E}_\pi [G_t | s_t = s]$

# Incremental Monte Carlo (MC) On Policy Evaluation

- After each episode $i$ = $s_{i1}$, $a_{i1}$, $r_{i1}$, $s_{i2}$, $a_{i2}$, $r_{i2}$, …
  - Define $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \ldots$ as return from time step $t$ onwards in $i$-th episode
  - For state $s$ visited at time step $t$ in episode $i$
    - Increment counter of total visits N(s) = N(s) + 1
    - Update estimate

$$V^\pi(s) = V^\pi(s)\frac{N(s)-1}{N(s)} + \frac{G_{it}}{N(s)} = V^\pi(s) + \frac{1}{N(s)}(G_{it} - V^\pi(s))$$

# Incremental Monte Carlo (MC) On Policy Evaluation Running Mean

- After each episode $i$ = $s_{i1}$, $a_{i1}$, $r_{i1}$, $s_{i2}$, $a_{i2}$, $r_{i2}$, …
  - Define $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \ldots$ as return from time step $t$ onwards in $i$-th episode
  - For state $s$ visited at time step $t$ in episode $i$
    - Increment counter of total visits N(s) = N(s) + 1
    - Update estimate

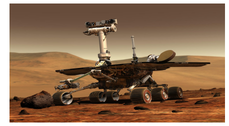
$$V^\pi(s) = V^\pi(s) + \alpha(G_{it} - V^\pi(s))$$

$V^\pi(s_2) = 1$

$\alpha = \frac{1}{N(s)}$ : identical to every visit MC

$V^\pi(s_2) = 1 + \alpha(1 - 1)$

$\alpha > \frac{1}{N(s)}$ : forget older data, helpful for nonstationary domains

| S1 | S2 | S3 | S4 | S5 | S6 | S7 |
|---|---|---|---|---|---|---|
| Okay Field Site +1 | | |  | | | Fantastic Field Site +10 |

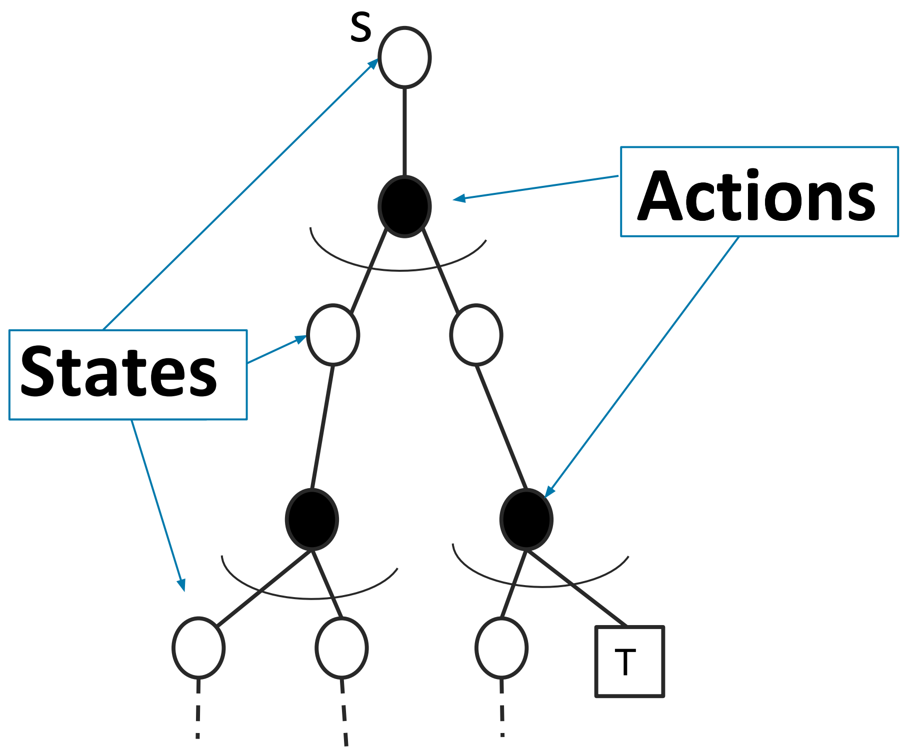
$S_T$

- Policy: TryLeft (TL) in all states, use $\gamma$=1, S1 and S7 transition to terminal upon any action $V^\pi(s) = 0 \ \forall s$ $\alpha = 1$
- Start in state S3, take TryLeft, get r=0, go to S2
- Start in state S2, take TryLeft, get r=0, go to S2
- Start in state S2, take TryLeft, get r=0, go to S1
- Start in state S1, take TryLeft, get r=+1, go to terminal
- Trajectory = (S3,TL,0,S2,TL,0,S2,TL,0,S1,TL,1, terminal)
- First visit MC estimate of V of each state? $[1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0]$
- Every visit MC estimate of S2? $[1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0]$

$$V^\pi(s) = V^\pi(s) + \alpha(G_{i,t}(s) - V^\pi(s))$$

# MC Policy Evaluation

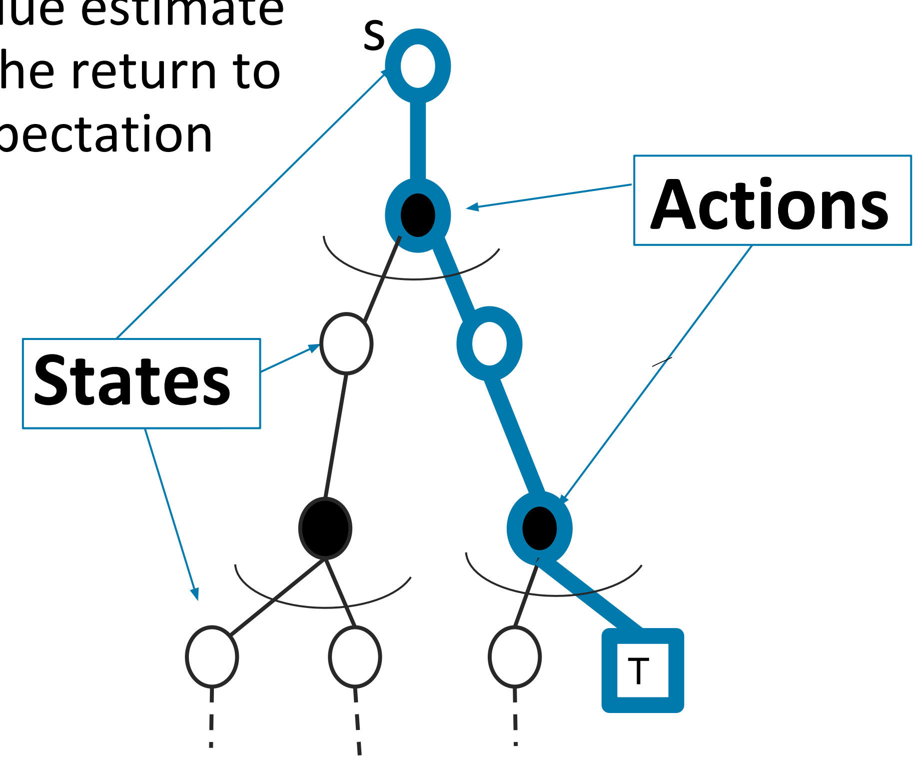
$$V^{\pi}(s) = V^{\pi}(s) + \alpha(G_{it} - V^{\pi}(s))$$

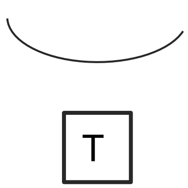



# MC Policy Evaluation

$$V^\pi(s) = V^\pi(s) + \alpha(G_{it} - V^\pi(s))$$

MC updates the value estimate using a **sample** of the return to approximate an expectation

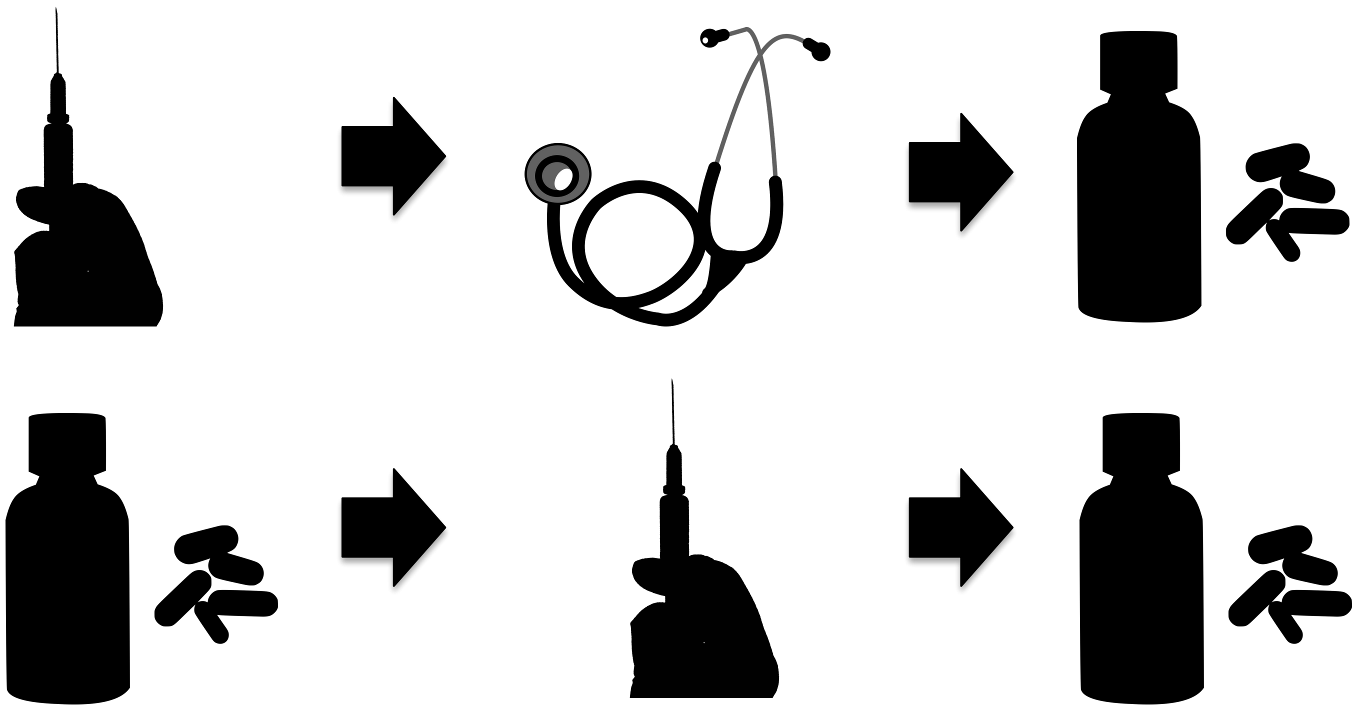


= Expectation

**= Terminal state**

# MC Off Policy Evaluation



- Sometimes trying actions out is costly or high stakes
- Would like to use old data about policy decisions and their outcomes to estimate the potential value of an alternate policy

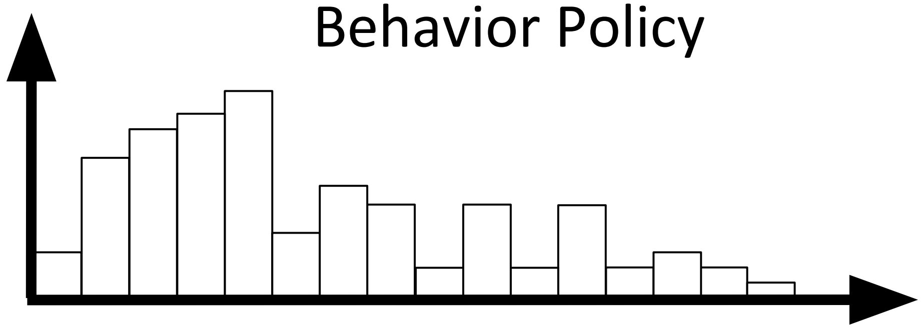
# Monte Carlo (MC) **Off Policy** Evaluation

- Aim: estimate given episodes generated under policy $\pi_1$
  - $s_1, a_1, r_1, s_2, a_2, r_2, \ldots$ where the actions are sampled from $\pi_1$
- $G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \ldots$ in MDP M under a policy $\pi$
- $V^\pi(s) = \mathbb{E}_\pi [G_t | s_t = s]$
- Have data from another policy
- If $\pi_1$ is stochastic can often use it to estimate the value of an alternate policy (formal conditions to follow)
- Again, no requirement for model nor that state is Markov

# Monte Carlo (MC) **Off Policy** Evaluation: Distribution Mismatch

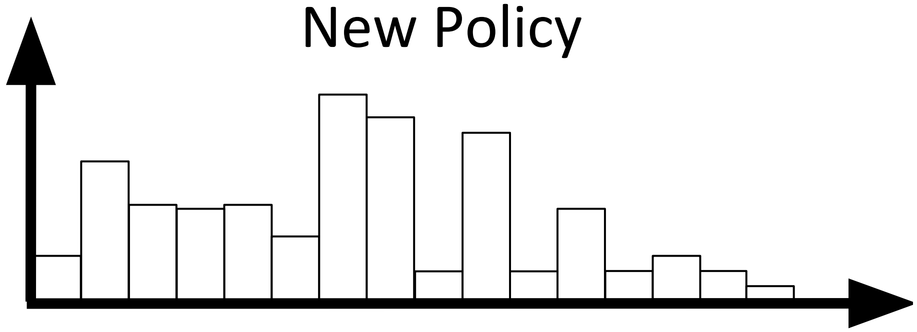
- Distribution of episodes & resulting returns differs between policies

Behavior Policy



Returns

New Policy



Returns

# Bias, Variance and MSE

- Consider a statistical model that is parameterized by $\theta$ and that determines a probability distribution over observed data $P(x|\theta)$.
- Consider a statistic $\hat{\theta}$ that provides an estimate of $\theta$ and is a function of observed data $x$.
  - E.g. for a Gaussian distribution with known variance, the average of a set of data points is an estimate of the mean of the Gaussian.
- Definition: the bias of an estimator $\hat{\theta}$ is:

$$Bias_\theta(\hat{\theta}) = \mathbb{E}_{x|\theta}[\hat{\theta}] - \theta \quad = 0 \quad \text{unbiased} \tag{1}$$

- Definition: the variance of an estimator $\hat{\theta}$ is:

$$Var(\hat{\theta}) = \mathbb{E}_{x|\theta}\left[(\hat{\theta} - \mathbb{E}[\hat{\theta}])^2\right] \tag{2}$$

- Definition: mean squared error (MSE) of an estimator $\hat{\theta}$ is:

$$MSE(\hat{\theta}) = Var(\hat{\theta}) + Bias_\theta(\hat{\theta})^2 \tag{3}$$

# Importance Sampling

assume can evaluate $f(x_i)$

evaluate $q(x_i)$

- Goal: estimate the expected value of a function $f(x)$ under some probability distribution $\cancel{p}\,q(x)$, $\mathbb{E}_{x\sim \cancel{p}\,q}[f(x)]$
- Have data $x_1, x_2, \ldots, x_n$ sampled from distribution $\cancel{q}\,p(s)$ $P$
- Under a few assumptions, can use samples to obtain an unbiased estimate of $\mathbb{E}_{x\sim q}[f(x)]$

$$
\begin{aligned}
\mathbb{E}_{x\sim q}[f(x)] &= \int_x q(x)f(x)\,dx \\
&= \int_x q(x)f(x)\cdot\frac{p(x)}{p(x)}\,dx \\
&= \int_x p(x)\left[\frac{q(x)}{p(x)}f(x)\right]dx \\
&= E_{x\sim p}\left[q(x)/p(x)\,f(x)\right] \\
&\simeq \frac{1}{N}\sum_{i=1}^{N}\frac{q(x_i)}{p(x_i)}f(x_i)
\end{aligned}
$$

$f(x_i)q(x_i) > 0$ then $p(x_i) > 0$

# Importance Sampling for Policy Evaluation

- Aim: estimate $V^{\pi_1}$ given episodes generated under policy $\pi_2$
  - $s_1$, $a_1$, $r_1$, $s_2$, $a_2$, $r_2$, … where the actions are sampled from $\pi_2$
- Have access to $G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \ldots$ in MDP M under a policy $\pi_2$
- Want $V^{\pi_1}(s) = E_{\pi_1}[G_t | s_t = s]$
- Have data from another policy
- If $\pi_2$ is stochastic can often use it to estimate the value of an alternate policy (formal conditions to follow)
- Again, no requirement for model nor that state is Markov

# Importance Sampling (IS) for Policy Evaluation

trajectory

- Let h be a particular episode (history) of states, actions and rewards

$$h = (s_1, a_1, r_1, s_2, a_2, r_2, \ldots, s_{terminal})$$

# Probability of a Particular Episode

- Let h be a particular episode (history) of states, actions and rewards

$$h = (s_1, a_1, r_1, s_2, a_2, r_2, \ldots, s_{terminal})$$

$$\begin{aligned}
p(h_j|\pi, s) &= p(a_{j1}|s_{j1})p(r_{j1}|s_{j1}, a_{j1})p(s_{j2}|s_{j1}, a_{j1})p(a_{j2}|s_{j2}) \ldots \\
&= \prod_{t=1}^{L_j-1} p(a_{j,t}|s_{j,t})p(r_{j,t}|s_{j,t}, a_{j,t})p(s_{j,t+1}|s_{j,t}, a_{j,t}) \\
&= \prod_{t=1}^{L_j-1} \underbrace{\pi(a_{j,t}|s_{j,t})}_{\text{policy}}\underset{\uparrow \\ \text{reward model}}{p(r_{j,t}|s_{j,t}, a_{j,t})}\underbrace{p(s_{j,t+1}|s_{j,t}, a_{j,t})}_{\text{dynamics}}
\end{aligned}$$

$$V^{\pi_1}(s) = E_{h \sim \pi_1}[G(s)]$$

# Importance Sampling (IS) for Policy Evaluation

- Let h be a particular episode (history) of states, actions and rewards

$$h = (s_1, a_1, r_1, s_2, a_2, r_2, \ldots, s_{terminal})$$

$$V^{\pi_1}(s) \approx \frac{1}{N}\sum_{j=1}^{N} \frac{p(h_j|\pi_1, s)}{p(h_j|\pi_2, s)} G(h_j) \quad h_j \sim \pi_2$$

$$= \frac{1}{N}\sum_{j=1}^{N} \left( \frac{\prod_t \pi_1(a_t|s_t)\, p(r_t|a_t, s_t)\, p(s_{t+1}|a_t, s_t)}{\prod_t \pi_2(a_t|s_t)\, p(r_t|a_t, s_t)\, p(s_{t+1}|a_t, s_t)} \right) G(h_j)$$

$$= \frac{1}{N}\sum_{j=1} \left( \frac{\prod_t \pi_1(a_t|s_t)}{\pi_2(a_t|s_t)} \right) G(h_j)$$

# Importance Sampling for Policy Evaluation

- Aim: estimate $V^{\pi_1}$ given episodes generated under policy $\pi_2$
  - $s_1, a_1, r_1, s_2, a_2, r_2, \ldots$ where the actions are sampled from $\pi_2$
- Have access to $G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \ldots$ in MDP M under a policy $\pi_2$
- Want $V^{\pi_1}(s) = E_{\pi_1}[G_t | s_t = s]$
- IS = Monte Carlo estimate given off policy data
- Model-free method
- Does not require Markov assumption
- Under some assumptions, unbiased & consistent estimator of $V^{\pi_1}$
- Can be used when agent is interacting with environment to estimate value of policies different than agent's control policy
- More later this quarter about batch learning

# Monte Carlo (MC) Policy Evaluation Summary

- Aim: estimate $V^{\pi}(s)$ given episodes generated under policy $\pi$
  - $s_1, a_1, r_1, s_2, a_2, r_2, \ldots$ where the actions are sampled from $\pi$
- $G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \ldots$ in MDP M under a policy $\pi$
- $V^{\pi}(s) = \mathbb{E}_{\pi}[G_t | s_t = s]$
- Simple: Estimates expectation by empirical average (given episodes sampled from policy of interest) or reweighted empirical average (importance sampling)
- Updates value estimate by using a **sample** of return to approximate the expectation
- No bootstrapping
- Converges to true value under some (generally mild) assumptions

# Monte Carlo (MC) Policy Evaluation Key Limitations

- Generally high variance estimator
  - Reducing variance can require a lot of data
- Requires episodic settings
  - Episode must end before data from that episode can be used to update the value function

# This Lecture: Policy Evaluation

- Dynamic programming
- Monte Carlo policy evaluation
  - Policy evaluation when don't have a model of how the world work
    - Given on policy samples
    - Given off policy samples
- **Temporal Difference (TD)**
- Axes to evaluate and compare algorithms

# Temporal Difference Learning

- "If one had to identify one idea as central and novel to reinforcement learning, it would undoubtedly be temporal-difference (TD) learning." -- *Sutton and Barto 2017*
- Combination of Monte Carlo & dynamic programming methods
- Model-free
- **Bootstraps and samples**
- Can be used in episodic or infinite-horizon non-episodic settings
  - Immediately updates estimate of V after each (s,a,r,s') tuple

update $V(s)$

# Temporal Difference Learning for Estimating V

- Aim: estimate $V^{\pi}(s)$ given episodes generated under policy π
- $G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \ldots$ in MDP M under a policy π
- $V^{\pi}(s) = \mathbb{E}_{\pi}[G_t | s_t = s]$
- Recall Bellman operator (if know MDP models)

sampling

$$B^{\pi}V(s) = R^{\pi}(s) + \gamma \sum_{s' \in S} P^{\pi}(s'|s)V(s')$$

bootstrapping

- In incremental every-visit MC, update estimate using 1 sample of return (for the current $i^{th}$ episode)

$$V^{\pi}(s_{it}) = V^{\pi}(s_{it}) + \alpha(G_{it} - V^{\pi}(s_{it}))$$

- Insight: have an estimate of $V^{\pi}$, use to estimate expected return

$$V^{\pi}(s_t) = V^{\pi}(s_t) + \alpha\left([r_t + \gamma V^{\pi}(s_{t+1})] - V^{\pi}(s_t)\right)$$

# Temporal Difference [TD(0)] Learning

- Aim: estimate $V^{\pi}(s)$ given episodes generated under policy $\pi$
  - $s_1, a_1, r_1, s_2, a_2, r_2, \ldots$ where the actions are sampled from $\pi$

(s, a, r, s')

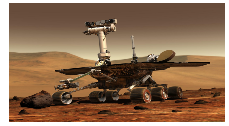
- Simplest TD learning: update value towards estimated value

$$V^{\pi}(s_t) = V^{\pi}(s_t) + \alpha\left(\underbrace{[r_t + \gamma V^{\pi}(s_{t+1})]}_{\text{TD target}} - V^{\pi}(s_t)\right)$$

- TD error:

$$\delta_t = r_t + \gamma V^{\pi}(s_{t+1}) - V^{\pi}(s_t))$$

- **Can immediately update value estimate after (s,a,r,s') tuple**
- **Don't need episodic setting**

| S1 | S2 | S3 | S4 | S5 | S6 | S7 |
|---|---|---|---|---|---|---|
| Okay Field Site +1 | | |  | | | Fantastic Field Site +10 |

- Policy: TryLeft (TL) in all states, use $\gamma$=1, S1 and S7 transition to terminal upon any action
- Start in state S3, take TryLeft, get r=0, go to S2
- Start in state S2, take TryLeft, get r=0, go to S2
- Start in state S2, take TryLeft, get r=0, go to S1
- Start in state S1, take TryLeft, get r=+1, go to terminal
- Trajectory = (S3,TL,0,S2,TL,0,S2,TL,0,S1,TL,1, terminal)
- First visit MC estimate of all states? [1 1 1 0 0 0 0]
- Every visit MC estimate of S2? 1
- TD estimate of all states (init at 0) with alpha = 1?

$V^\pi(s) = 0 \ \forall s$ initially

[1 0 0 0 0 0 0]

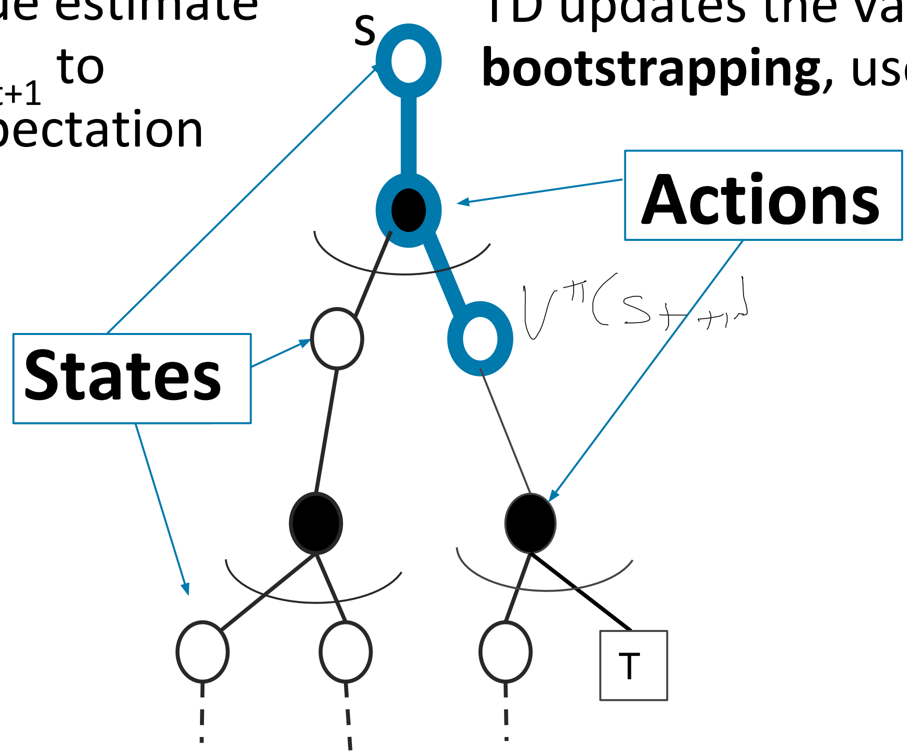
$$V^\pi(s_t) = V^\pi(s_t) + \alpha\left(r_t + \gamma V^\pi(s_{t+1}) - V^\pi(s_t)\right)$$

# Temporal Difference Policy Evaluation

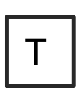
$$V^\pi(s_t) = V^\pi(s_t) + \alpha\left([r_t + \gamma V^\pi(s_{t+1})] - V^\pi(s_t)\right)$$

TD updates the value estimate using a **sample** of $s_{t+1}$ to approximate an expectation

TD updates the value estimate by **bootstrapping**, uses estimate of $V(s_{t+1})$




 = Expectation

 **= Terminal state**

# This Lecture: Policy Evaluation

- Dynamic programming
- Monte Carlo policy evaluation
  - Policy evaluation when don’t have a model of how the world work
    - Given on policy samples
    - Given off policy samples
- Temporal Difference (TD)
- **Axes to evaluate and compare algorithms**

# Some Important Properties to Evaluate Policy Evaluation Algorithms

- Usable when no models of current domain
  - DP: No MC: Yes TD: Yes
- Handles continuing (non-episodic) domains
  - DP: Yes MC: No TD: Yes
- Handles Non-Markovian domains
  - DP: No MC: Yes TD: No
- Converges to true value in limit*
  - DP: Yes MC: Yes TD: Yes

tabular
mild cond on $\alpha$

- Unbiased estimate of value
  - DP: NA MC: Yes TD: No

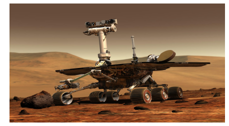
* For tabular representations of value function. More on this in later lectures

# Some Important Properties to Evaluate Model-free Policy Evaluation Algorithms

- Bias/variance characteristics
- Data efficiency
- Computational efficiency

# Bias/Variance of Model-free Policy Evaluation Algorithms

- Return $G_t$ is an unbiased estimate of $V^\pi(s_t)$
- TD target $[r_t + \gamma V^\pi(s_{t+1})]$ is a biased estimate of $V^\pi(s_t)$
- But often much lower variance than a single return $G_t$
- Return function of multi-step seq. of random actions, states & rewards
- TD target only has one random action, reward and next state
- MC
  - Unbiased
  - High variance
  - Consistent (converges to true) even with function approximation
- TD
  - Some bias
  - Lower variance
  - TD(0) converges to true value with tabular representation
  - TD(0) does not always converge with function approximation

| S1 | S2 | S3 | S4 | S5 | S6 | S7 |
| --- | --- | --- | --- | --- | --- | --- |
| Okay Field Site +1 | | |  | | | Fantastic Field Site +10 |

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- Trajectory = (S3,TL,0,S2,TL,0,S2,TL,0,S1,TL,1, terminal)
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- First visit MC estimate of all states? [1 1 1 0 0 0 0]
- Every visit MC estimate of S2? 1
- TD estimate of all states (init at 0) [1 0 0 0 0 0 0] with alpha = 1
- TD(0) only uses a data point (s,a,r,s') once
- Monte Carlo takes entire return from s to end of episode

# Batch MC and TD

- Batch (Offline) solution for finite dataset
  - Given set of K episodes
  - Repeatedly sample an episode from K
  - Apply MC or TD(0) to that episode
- What do MC and TD(0) converge to?

# AB Example: (Ex.6.4, Sutton & Barto, 2018)

- Two states A, B; $\gamma$=1; 8 episodes of experience

| | |
|---|---|
| A, 0, B, 0 | B, 1 |
| B, 1 | B, 1 |
| B, 1 | B, 1 |
| B, 1 | B, 0 |

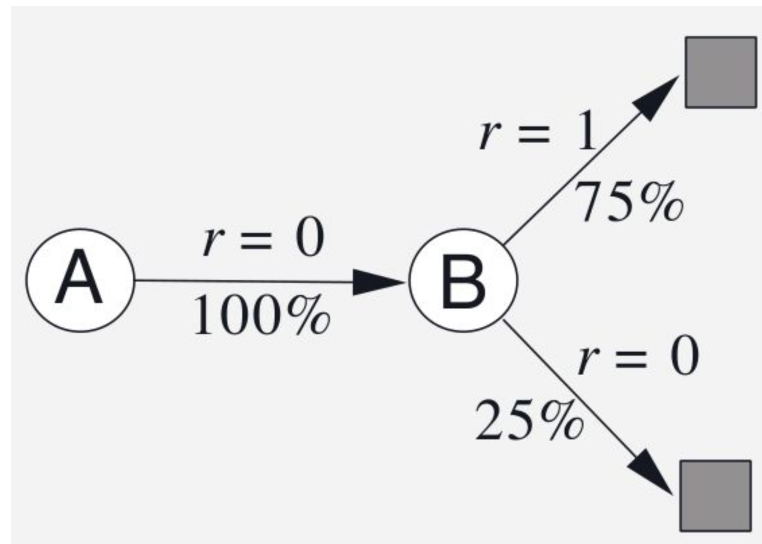
What is V(A), V(B)?

# AB Example: (Ex.6.4, Sutton & Barto, 2018)

- Two states A, B; $\gamma$=1; 8 episodes of experience

  A, 0, B, 0      B, 1<br>
  B, 1      B, 1<br>
  B, 1      B, 1<br>
  B, 1      B, 0

  What is V(A), V(B)?




- V(B) = .75 (by TD or MC)
- V(A)? 0 MC

TD: $V^\pi(A) = V^\pi(A) + \alpha(0 + \gamma V^\pi(B))$

$= .75$

# Batch MC and TD: Converges

- Monte Carlo in batch setting converges to min MSE (mean squared error)
  - Minimize loss with respect to observed returns
  - In AB example, V(A) = 0
- TD(0) converges to DP policy $V^{\pi}$ for the MDP with the maximum likelihood model estimates
  - Maximum likelihood Markov decision process model

$$\hat{P}(s'|s,a) = \frac{1}{N(s,a)} \sum_{k=1}^{K} \sum_{t=1}^{L_k-1} \mathbb{1}(s_{k,t} = s, a_{k,t} = a, s_{k,t+1} = s')$$

$$\hat{r}(s,a) = \frac{1}{N(s,a)} \sum_{k=1}^{K} \sum_{t=1}^{L_k-1} \mathbb{1}(s_{k,t} = s, a_{k,t} = a) r_{t,k}$$

  - Compute $V^{\pi}$ using this model
  - In AB example, V(A) = 0.75

# Some Important Properties to Evaluate Model-free Policy Evaluation Algorithms

- Data efficiency & Computational efficiency
- In simplest TD, use (s,a,r,s’) once to update V(s)
  - O(1) operation per update
  - In an episode of length L, O(L)
- In MC have to wait till episode finishes, then also O(L)
- MC can be more data efficient than simple TD
- But TD exploits Markov structure
  - If in Markov domain, leveraging this is helpful

# Alternative: Certainty Equivalence $V^\pi$ MLE MDP Model Estimates

- Model-based option for policy evaluation without true models
- After each (s,a,r,s') tuple
  - Recompute maximum likelihood MDP model for (s,a)

$$\hat{P}(s'|s,a) = \frac{1}{N(s,a)} \sum_{k=1}^{K} \sum_{t=1}^{L_k-1} \mathbb{1}(s_{k,t} = s, a_{k,t} = a, s_{k,t+1} = s')$$

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  - Compute $V^\pi$ using MLE MDP* (e.g. see method from lecture 2)
- *Requires initializing for all (s,a) pairs

important for control

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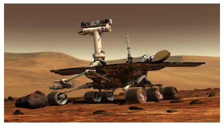
  - Compute $V^\pi$ using MLE MDP* (e.g. see method from lecture 2)
- *Requires initializing for all (s,a) pairs
- Cost: Updating MLE model and MDP planning at each update ($O(|S|^3$ for analytic matrix soln, $O(|S|^2|A|)$ for iterative methods)
- Very data efficient and very computationally expensive
- Consistent
- Can also easily be used for off-policy evaluation

| S1 | S2 | S3 | S4 | S5 | S6 | S7 |
|---|---|---|---|---|---|---|
| Okay Field Site +1 | | |  | | | Fantastic Field Site +10 |

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- TD estimate of all states (init at 0) [1 0 0 0 0 0 0] with alpha = 1
- TD(0) only uses a data point (s,a,r,s') once
- Monte Carlo takes entire return from s to end of episode
- What is certainty equivalent estimate?

# Some Important Properties to Evaluate Policy Evaluation Algorithms

- Robustness to Markov assumption
- Bias/variance characteristics
- Data efficiency
- Computational efficiency

# Summary: Policy Evaluation

- Dynamic programming
- Monte Carlo policy evaluation
  - Policy evaluation when don't have a model of how the world work
    - Given on policy samples
    - Given off policy samples
- Temporal Difference (TD)
- Axes to evaluate and compare algorithms

# Class Structure

- Last Time:
  - Markov reward / decision processes
  - Policy evaluation & control when have true model (of how the world works)
- Today:
  - Policy evaluation when don't have a model of how the world works
- **Next time:**
  - **Control when don't have a model of how the world works**