Lecture 11: Fast Reinforcement Learning [1]

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CS234 Reinforcement Learning

Winter 2019

[1] With many slides from or derived from David Silver, Examples new

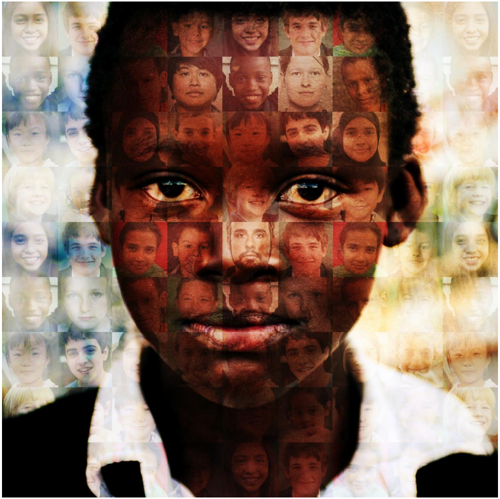
Class Structure

- Last time: Midterm
- **This time: Fast Learning**
- Next time: Fast Learning

Up Till Now

- Discussed optimization, generalization, delayed consequences

Teach Computers to Help Us



education
healthcare
consumer
marketing

Computational Efficiency and Sample Efficiency

Q learning

Computational Efficiency	Sample Efficiency
driving car at 60mph simulators	experience costly/hard to gather - education - students - patients - customers

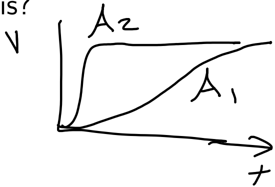
Algorithms Seen So Far

- How many steps did it take for DQN to learn a good policy for pong?

  5 millions

Evaluation Criteria

- How do we evaluate how "good" an algorithm is?
- If converges? $t \to \infty$
- If converges to optimal policy? $t \to \infty$
- How quickly reaches optimal policy?
- Mistakes make along the way?
- Will introduce different measures to evaluate RL algorithms



Settings, Frameworks & Approaches

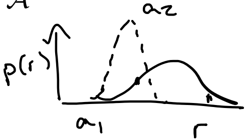
- Over next couple lectures will consider 2 settings, multiple frameworks, and approaches
- Settings: Bandits (single decisions), MDPs   tabular
- Frameworks: evaluation criteria for formally assessing the quality of a RL algorithm
- Approaches: Classes of algorithms for achieving particular evaluation criteria in a certain set
- Note: We will see that some approaches can achieve multiple frameworks in multiple settings

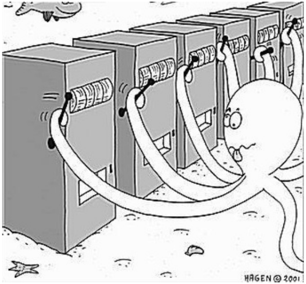
Today

- Setting: Introduction to multi-armed bandits
- Framework: Regret
- Approach: Optimism under uncertainty
- Framework: Bayesian regret
- Approach: Probability matching / Thompson sampling

Multiarmed Bandits

- Multi-armed bandit is a tuple of $(\mathcal{A}, \mathcal{R})$
- $\mathcal{A}$: known set of $\underline{m}$ actions (arms)
- $\mathcal{R}^a(r) = \mathbb{P}[r \mid a]$ is an unknown probability distribution over rewards
- At each step t the agent selects an action $a_t \in \mathcal{A}$
- The environment generates a reward $r_t \sim \mathcal{R}^{a_t}$
- Goal: Maximize cumulative reward $\sum_{\tau=1}^{t} r_\tau$





Regret

- **Action-value** is the mean reward for action a

$$\underline{Q(a)} = \mathbb{E}[r \mid a]$$

- **Optimal value** V^*

$$V^* = Q(a^*) = \underbrace{\max_{a \in \mathcal{A}} Q(a)}$$

arm selec

- **Regret** is the opportunity loss for one step

$Q(a^*) - Q(a_t)$

$$l_t = \mathbb{E}[V^* - Q(a_t)]$$

- **Total Regret** is the total opportunity loss

unknown unknown

$$L_t = \mathbb{E}[\sum_{\tau=1}^{t} V^* - Q(a_\tau)]$$

$r_t \sim P(r \mid a_t)$

- Maximize cumulative reward $\iff$ minimize total regret

Evaluating Regret

- **Count** $N_t(a)$ is expected number of selections for action a
- **Gap** Δ_a is the difference in value between action a and optimal action a^*, $\Delta_i = V^* - Q(a_i)$ $\Delta_{a^*} = 0$
- Regret is a function of gaps and counts

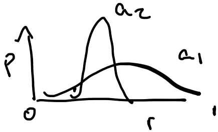
$$
\begin{aligned}
L_t &= \mathbb{E}\left[\sum_{\tau=1}^{t} V^* - Q(a_\tau)\right] \\
&= \sum_{a \in \mathcal{A}} \mathbb{E}[N_t(a)](V^* - Q(a)) \\
&= \sum_{a \in \mathcal{A}} \mathbb{E}[N_t(a)]\Delta_a
\end{aligned}
$$

- A good algorithm ensures small counts for large gap, but gaps are not known

Greedy Algorithm

- We consider algorithms that estimate $\hat{Q}_t(a) \approx Q(a)$
- Estimate the value of each action by Monte-Carlo evaluation

$$\hat{Q}_t(a) = \frac{1}{N_t(a)} \sum_{t=1}^{T} r_t \mathbb{1}(\underline{a_t = a})$$



- The **greedy** algorithm selects action with highest value

.2 a1
.5 a2

$$a_t^* = \arg\max_{a \in \mathcal{A}} \hat{Q}_t(a)$$

- Greedy can lock onto suboptimal action, forever

ϵ-Greedy Algorithm

- The ϵ-**greedy** algorithm proceeds as follows:
  - With probability $1 - \epsilon$ select $a_t = \arg\max_{a \in \mathcal{A}} \hat{Q}_t(a)$
  - With probability ϵ select a random action
- Always will be making a sub-optimal decision ϵ fraction of the time
  $(1 - 1/m)$
- Already used this in prior homeworks

Toy Example: Ways to Treat Broken Toes[1]

- Consider deciding how to best treat patients with broken toes
- Imagine have 3 possible options: (1) surgery (2) buddy taping the broken toe with another toe, (3) do nothing
- Outcome measure / reward is binary variable: whether the toe has healed (+1) or not healed (0) after 6 weeks, as assessed by x-ray

[1] Note:This is a made up example. This is not the actual expected efficacies of the various treatment options for a broken toe

Toy Example: Ways to Treat Broken Toes[1]

- Consider deciding how to best treat patients with broken toes
- Imagine have 3 common options: (1) surgery (2) surgical boot (3) buddy taping the broken toe with another toe
- Outcome measure is binary variable: whether the toe has healed (+1) or not (0) after 6 weeks, as assessed by x-ray
- Model as a multi-armed bandit with 3 arms, where each arm is a Bernoulli variable with an unknown parameter θ_i
- Check your understanding: what does a pull of an arm / taking an action correspond to? Why is it reasonable to model this as a multi-armed bandit instead of a Markov decision process?

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Toy Example: Ways to Treat Broken Toes[1]

- Imagine true (unknown) Bernoulli reward parameters for each arm (action) are
  - surgery: $Q(a^1) = \theta_1 = .95$
  - buddy taping: $Q(a^2) = \theta_2 = .9$
  - doing nothing: $Q(a^3) = \theta_3 = .1$

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Toy Example: Ways to Treat Broken Toes, Greedy[1]

- Imagine true (unknown) Bernoulli reward parameters for each arm (action) are
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  - buddy taping: $Q(a^2) = \theta_2 = .9$
  - doing nothing: $Q(a^3) = \theta_3 = .1$
- Greedy
  1. Sample each arm once
     - Take action a^1 ($r \sim$Bernoulli(0.95)), get $+1$, $\hat{Q}(a^1) = 1$
     - Take action a^2 ($r \sim$Bernoulli(0.90)), get $+1$, $\hat{Q}(a^2) = 1$
     - Take action a^3 ($r \sim$Bernoulli(0.1)), get 0, $\hat{Q}(a^3) = 0$
  2. What is the probability of greedy selecting each arm next? Assume ties are split uniformly.

     $p(a_1) = p(a_2) = 1/2$

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Toy Example: Ways to Treat Broken Toes, Optimism, Assessing Regret of Greedy

- True (unknown) Bernoulli reward parameters for each arm (action) are
  - surgery: $Q(a^1) = \theta_1 = .95$ ← optimal
  - buddy taping: $Q(a^2) = \theta_2 = .9$
  - doing nothing: $Q(a^3) = \theta_3 = .1$

$\underbrace{Q(a^*)}_{V^*} - Q(a_t)$

- Greedy

r	Action	Optimal Action	Regret	
1	a^1	a^1	0	init
1	a^2	a^1	$.95 - .9 = .05$	init
0	a^3	a^1	$.95 - .1 = .85$	init
	a^1	a^1	0	
	a^2	a^1	.05	

- Will greedy ever select a^3 again? If yes, why? If not, is this a problem?
  no

Toy Example: Ways to Treat Broken Toes, ϵ-**Greedy**[1]

- Imagine true (unknown) Bernoulli reward parameters for each arm (action) are
  - surgery: $Q(a^1) = \theta_1 = .95$
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- ϵ-greedy
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     - Take action a^2 ($r \sim$Bernoulli(0.90)), get $+1$, $\hat{Q}(a^2) = 1$
     - Take action a^3 ($r \sim$Bernoulli(0.1)), get 0, $\hat{Q}(a^3) = 0$
  2. Let $\epsilon = 0.1$
  3. What is the probability ϵ-greedy will pull each arm next? Assume ties are split uniformly.

$\frac{\epsilon}{3}$: a_1 or a_2 or a_3

$\frac{1-\epsilon}{2}$ a_1 or a_2

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  - doing nothing: $Q(a^3) = \theta_3 = .1$
- UCB1 (Auer, Cesa-Bianchi, Fischer 2002)

Action	Optimal Action	Regret
a^1	a^1	
a^2	a^1	
a^3	a^1	
a^1	a^1	
a^2	a^1	

- Will ϵ-greedy ever select a^3 again? If ϵ is fixed, how many times will each arm be selected? yes

Check Your Understanding

- **Count** $N_t(a)$ is expected number of selections for action a
- **Gap** Δ_a is the difference in value between action a and optimal action a^*, $\Delta_i = V^* - Q(a_i)$
- Regret is a function of gaps and counts

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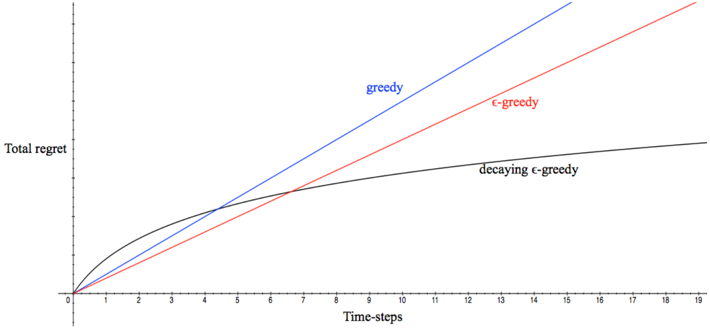
\begin{aligned}
L_t &= \mathbb{E}\left[\sum_{\tau=1}^{t} V^* - Q(a_\tau)\right] \\
&= \sum_{a \in \mathcal{A}} \mathbb{E}[N_t(a)](V^* - Q(a)) \\
&= \sum_{a \in \mathcal{A}} \mathbb{E}[N_t(a)]\Delta_a
\end{aligned}
$$

- A good algorithm ensures small counts for large gap, but gaps are not known
- Check your understanding: Does fixed $\epsilon = 0.1$ greedy have large regret ?

"Good": Sublinear or below regret



- **Explore forever**: have linear total regret T
- **Explore never**: have linear total regret
- Is it possible to achieve sublinear regret?

Types of Regret bounds

- **Problem independent**: Bound how regret grows as a function of T, the total number of time steps the algorithm operates for
- **Problem dependent**: Bound regret as a function of the number of times we pull each arm and the gap between the reward for the pulled arm a^*

a_1	a_2
$E[r]$	$E[r]$
1	.001
.53	.525

Lower Bound

- Use lower bound to determine how hard this problem is
- The performance of any algorithm is determined by similarity between optimal arm and other arms
- Hard problems have similar looking arms with different means
- This is described formally by the gap Δ_a and the similarity in distributions $D_{KL}(\mathcal{R}^a \| \mathcal{R}^{a^*})$ KL div
- Theorem (Lai and Robbins): Asymptotic total regret is at least logarithmic in number of steps

$$\lim_{t \to \infty} L_t \geq \underbrace{\log t}_{\text{\# regret}} \sum_{a|\Delta_a > 0} \frac{\Delta_a}{D_{KL}(\mathcal{R}^a \| \mathcal{R}^{a^*})}$$

- Promising in that lower bound is sublinear

Approach: Optimism in the Face of Uncertainty

Kaelbling 1993

- Choose actions that that might have a high value
- Why?
- Two outcomes: a_1

1) a_1 has high reward

2) ~~a_1 has~~ $r(a_1)$ with low reward information

Upper Confidence Bounds

- Estimate an upper confidence $U_t(a)$ for each action value, such that $Q(a) \leq \underline{U_t(a)}$ with high probability
- This depends on the number of times $N_t(a)$ action a has been selected
- Select action maximizing Upper Confidence Bound (UCB)

$$a_t = \arg\max_{a \in \mathcal{A}} [U_t(a)]$$

UCB init phase: pull each arm once, compute $U_t(a_t)$ $\forall a$

for $t = 1 \ldots$

$a_t = \arg\max_a U_t(a)$

$r \sim$ R reward distrib a_t

Update $U_t(a_t)$ & all other arms

Hoeffding's Inequality

if confidence bounds hold

$$U_t(a_t) = \hat{Q}(a_t) + \sqrt{\frac{1}{2n(a_t)} \log(t^2/\delta)}$$
$$> Q(a_t)$$

- Theorem (Hoeffding's Inequality): Let $X_1, \ldots, X_n$ be i.i.d. random variables in $[0,1]$, and let $\bar{X}_n = \frac{1}{n}\sum_{\tau=1}^{n} X_\tau$ be the sample mean. Then

constant

$$\mathbb{P}\left[\underbrace{\mathbb{E}[X]}_{\text{true mean}} > \underbrace{\bar{X}_n}_{\text{empirical mean}} + u\right] \leq \exp(-2nu^2) = \delta/t^2$$

samples

$$\exp(-2nu^2) = \delta/t^2$$
$$u = \sqrt{\frac{1}{2n} \log(t^2/\delta)}$$
$$\bar{X}_n + u \geq E[X] \quad \text{w.prob} \geq 1 - \delta/t^2$$
$$U_t(a_t) = \hat{Q}(a_t) + \sqrt{1/2n(a_t) \cdot \log(t^2/\delta)}$$

High Probability Regret Bound for UCB Multi-armed Bandit

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High Probability Regret Bound for UCB Multi-armed Bandit

$$Regret(UCB, T) = \sum_{t=1}^{T} Q(a^*) - Q(a_t)$$

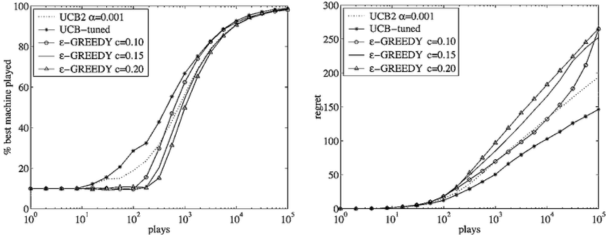
UCB Bandit Regret

- This leads to the UCB1 algorithm

$$a_t = \arg\max_{a \in \mathcal{A}} [\hat{Q}(a) + \sqrt{\frac{2 \log t}{N_t(a)}}]$$

- Theorem: The UCB algorithm achieves logarithmic asymptotic total regret

$$\lim_{t \to \infty} L_t \leq 8 \log t \sum_{a | \Delta_a > 0} \Delta_a$$



Toy Example: Ways to Treat Broken Toes, Thompson Sampling[1]

- True (unknown) parameters for each arm (action) are
  - surgery: $Q(a^1) = \theta_1 = .95$
  - buddy taping: $Q(a^2) = \theta_2 = .9$
  - doing nothing: $Q(a^3) = \theta_3 = .1$
- Optimism under uncertainty, UCB1 (Auer, Cesa-Bianchi, Fischer 2002)
  1. Sample each arm once

[1]Note:This is a made up example. This is not the actual expected efficacies of the various treatment options for a broken toe

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  2. Set $t = 3$, Compute upper confidence bound on each action

$$UCB(a) = \hat{Q}(a) + \sqrt{\frac{2 \log t}{N_t(a)}}$$

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$$UCB(a) = \hat{Q}(a) + \sqrt{\frac{2 \log t}{N_t(a)}}$$

  3. $t = 3$, Select action $a_t = \arg\max_a UCB(a)$,
  4. Observe reward 1
  5. Compute upper confidence bound on each action

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  2. Set $t = 3$, Compute upper confidence bound on each action
     $$UCB(a) = \hat{Q}(a) + \sqrt{\frac{2 \log t}{N_t(a)}}$$
  3. $t = t + 1$, Select action $a_t = \arg\max_a UCB(a)$,
  4. Observe reward 1
  5. Compute upper confidence bound on each action

Toy Example: Ways to Treat Broken Toes, Optimism, Assessing Regret

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Action	Optimal Action	Regret
a^1	a^1	
a^2	a^1	
a^3	a^1	
a^1	a^1	
a^2	a^1	

Check Your Understanding

- An alternative would be to always select the arm with the highest lower bound
- Why can this yield linear regret?
- Consider a two arm case for simplicity

Bayesian Bandits

- So far we have made no assumptions about the reward distribution $\mathcal{R}$
  - Except bounds on rewards
- **Bayesian bandits** exploit prior knowledge of rewards, $p[\mathcal{R}]$
- They compute posterior distribution of rewards $p[\mathcal{R} \mid h_t]$, where $h_t = (a_1, r_1, \ldots, a_{t-1}, r_{t-1})$
- Use posterior to guide exploration
  - Upper confidence bounds (Bayesian UCB)
  - Probability matching (Thompson Sampling)
- Better performance if prior knowledge is accurate

Regret and Bayesian Regret

- Frequentist regret assumes a true (unknown) set of parameters

$$Regret(\mathcal{A}, T; \theta) = \sum_{t=1}^{T} \mathbb{E}\left[Q(a^*) - Q(a_t) \leq \sum_{t=1}^{T} U_t(a_t) - Q(a_t) | \theta \right]$$

- Bayesian regret assumes there is a prior over parameters

$$BayesRegret(\mathcal{A}, T; \theta) =$$
$$\mathbb{E}_{\theta \sim p_\theta}\left[\sum_{t=1}^{T} \mathbb{E}\left[Q(a^*) - Q(a_t) \leq \sum_{t=1}^{T} U_t(a_t) - Q(a_t) | \theta \right] \right]$$

- *Note: Bayes regret and regret can be related using Markov inequality

Bayesian UCB Example: Independent Gaussians

- Assume reward distribution is Gaussian, $\mathcal{R}_a(r) = \mathcal{N}(r; \mu_a, \sigma_a^2)$
- Compute Gaussian posterior over μ_a and σ_a^2 (by Bayes law)

$$p[\mu_a, \sigma_a^2 \mid h_t] \propto p[\mu_a, \sigma_a^2] \prod_{t \mid a_t = a} \mathcal{N}(r_t; \mu_a, \sigma_a^2)$$

- Pick action that maximizes standard deviation of $Q(a)$

$$a_t = \arg\max_{a \in \mathcal{A}} \mu_a + c \frac{c\sigma_a}{\sqrt{N(a)}}$$

Probability Matching

- Assume have a parametric distribution over rewards for each arm
- **Probability matching** selects action a according to probability that a is the optimal action

$$\pi(a \mid h_t) = \mathbb{P}[Q(a) > Q(a'), \forall a' \neq a \mid h_t]$$

- Probability matching is optimistic in the face of uncertainty
  - Uncertain actions have higher probability of being max
- Can be difficult to compute analytically from posterior

Thompson sampling implements probability matching

- Thompson sampling:

$$\begin{aligned}\pi(a \mid h_t) &= \mathbb{P}[Q(a) > Q(a'), \forall a' \neq a \mid h_t] \\ &= \mathbb{E}_{\mathcal{R}|h_t}\left[\mathbb{1}(a = \arg\max_{a \in \mathcal{A}} Q(a))\right]\end{aligned}$$

Thompson sampling implements probability matching

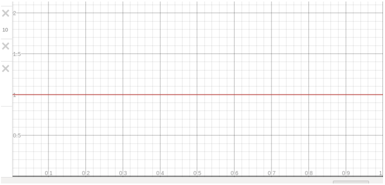
- Thompson sampling:

$$\begin{aligned}\pi(a \mid h_t) &= \mathbb{P}[Q(a) > Q(a'), \forall a' \neq a \mid h_t] \\ &= \mathbb{E}_{\mathcal{R}|h_t}\left[\mathbb{1}(a = \arg\max_{a \in \mathcal{A}} Q(a))\right]\end{aligned}$$

- Use Bayes law to compute posterior distribution $p[\mathcal{R} \mid h_t]$
- **Sample** a reward distribution $\mathcal{R}$ from posterior
- Compute action-value function $Q(a) = \mathbb{E}[\mathcal{R}_a]$
- Select action maximizing value on sample, $a_t = \arg\max_{a \in \mathcal{A}} Q(a)$
- Update posterior

Toy Example: Ways to Treat Broken Toes, Thompson Sampling

- True (unknown) Bernoulli parameters for each arm/action
- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- Thompson sampling:
- Place a prior over each arm's parameter. Here choose Beta(1,1) (Uniform)
  1. Sample a Bernoulli parameter given current prior over each arm Beta(1,1), Beta(1,1), Beta(1,1):



Toy Example: Ways to Treat Broken Toes, Thompson Sampling[2]

- True (unknown) Bernoulli parameters for each arm/action
- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- Thompson sampling:
- Place a prior over each arm's parameter. Here choose Beta(1,1)
  1. Sample a Bernoulli parameter given current prior over each arm Beta(1,1), Beta(1,1), Beta(1,1): 0.3 0.5 0.6
  2. Select $a = \arg\max_{a \in A} Q(a) = \arg\max_{ainA} \theta(a) =$

[2]Note:This is a made up example. This is not the actual expected efficacies of the various treatment options for a broken toe

Toy Example: Ways to Treat Broken Toes, Thompson Sampling

- True (unknown) Bernoulli parameters for each arm/action
- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- Thompson sampling:
- Place a prior over each arm's parameter. Here choose $\theta_i \sim$Beta(1,1)
  1. Per arm, sample a Bernoulli θ given prior: 0.3 0.5 0.6
  2. Select $a_t = \arg\max_{a \in A} Q(a) = \arg\max_{ainA} \theta(a) = 3$
  3. Observe the patient outcome's outcome: 0
  4. Update the posterior over the $Q(a_t) = Q(a^3)$ value for the arm pulled

Toy Example: Ways to Treat Broken Toes, Thompson Sampling

- True (unknown) Bernoulli parameters for each arm/action
- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- Thompson sampling:
- Place a prior over each arm's parameter. Here choose $\theta_i \sim$Beta(1,1)
  1. Sample a Bernoulli parameter given current prior over each arm Beta(1,1), Beta(1,1), Beta(1,1): 0.3 0.5 0.6
  2. Select $a_t = \arg\max_{a \in A} Q(a) = \arg\max_{a in A} \theta(a) = 3$
  3. Observe the patient outcome's outcome: 0
  4. Update the posterior over the $Q(a_t) = Q(a^1)$ value for the arm pulled
     - Beta(c_1, c_2) is the conjugate distribution for Bernoulli
     - If observe 1, $c_1 + 1$ else if observe 0 $c_2 + 1$
  5. New posterior over Q value for arm pulled is:
  6. New posterior $p(Q(a^3)) = p(\theta(a_3) = Beta(1, 2)$

Toy Example: Ways to Treat Broken Toes, Thompson Sampling

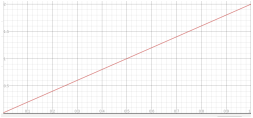
- True (unknown) Bernoulli parameters for each arm/action
- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- Thompson sampling:
- Place a prior over each arm's parameter. Here choose $\theta_i \sim$Beta(1,1)
  1. Sample a Bernoulli parameter given current prior over each arm Beta(1,1), Beta(1,1), Beta(1,1): 0.3 0.5 0.6
  2. Select $a_t = \arg\max_{a \in A} Q(a) = \arg\max_{ainA} \theta(a) = 1$
  3. Observe the patient outcome's outcome: 0
  4. New posterior $p(Q(a^1)) = p(\theta(a_1) = Beta(1,2)$

Toy Example: Ways to Treat Broken Toes, Thompson Sampling

- True (unknown) Bernoulli parameters for each arm/action
- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- Thompson sampling:
- Place a prior over each arm's parameter. Here choose $\theta_i \sim$Beta(1,1)
  1. Sample a Bernoulli parameter given current prior over each arm Beta(1,1), Beta(1,1), Beta(1,2): 0.7, 0.5, 0.3

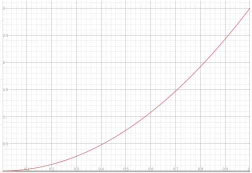
Toy Example: Ways to Treat Broken Toes, Thompson Sampling

- True (unknown) Bernoulli parameters for each arm/action
- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- Thompson sampling:
- Place a prior over each arm's parameter. Here choose $\theta_i \sim$Beta(1,1)
  1. Sample a Bernoulli parameter given current prior over each arm Beta(1,1), Beta(1,1), Beta(1,2): 0.7, 0.5, 0.3
  2. Select $a_t = \arg\max_{a \in A} Q(a) = \arg\max_{ainA} \theta(a) = 1$
  3. Observe the patient outcome's outcome: 1
  4. New posterior $p(Q(a^1)) = p(\theta(a_1) = Beta(2,1)$



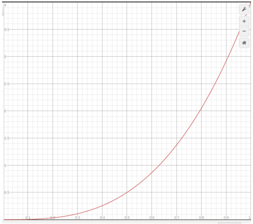
Toy Example: Ways to Treat Broken Toes, Thompson Sampling

- True (unknown) Bernoulli parameters for each arm/action
- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- Thompson sampling:
- Place a prior over each arm's parameter. Here choose $\theta_i \sim$Beta(1,1)
  1. Sample a Bernoulli parameter given current prior over each arm Beta(2,1), Beta(1,1), Beta(1,2): 0.71, 0.65, 0.1
  2. Select $a_t = \arg\max_{a \in A} Q(a) = \arg\max_{ainA} \theta(a) = 1$
  3. Observe the patient outcome's outcome: 1
  4. New posterior $p(Q(a^1)) = p(\theta(a_1) = Beta(3,1)$



Toy Example: Ways to Treat Broken Toes, Thompson Sampling

- True (unknown) Bernoulli parameters for each arm/action
- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- Thompson sampling:
- Place a prior over each arm's parameter. Here choose $\theta_i \sim$Beta(1,1)
  1. Sample a Bernoulli parameter given current prior over each arm Beta(2,1), Beta(1,1), Beta(1,2): 0.75, 0.45, 0.4
  2. Select $a_t = \arg\max_{a \in A} Q(a) = \arg\max_{ainA} \theta(a) = 1$
  3. Observe the patient outcome's outcome: 1
  4. New posterior $p(Q(a^1)) = p(\theta(a_1) = Beta(4, 1)$



Toy Example: Ways to Treat Broken Toes, Thompson Sampling vs Optimism

- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- How does the sequence of arm pulls compare in this example so far?

Optimism	TS	Optimal	Regret Optimism	Regret TS
a^1	a^3			
a^2	a^1			
a^3	a^1			
a^1	a^1			
a^2	a^1			

Toy Example: Ways to Treat Broken Toes, Thompson Sampling vs Optimism

- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- Incurred regret?

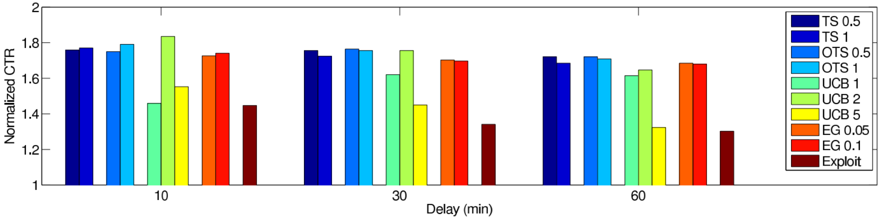
Optimism	TS	Optimal	Regret Optimism	Regret TS
a^1	a^3	a^1	0	0
a^2	a^1	a^1	0.05	
a^3	a^1	a^1	0.85	
a^1	a^1	a^1	0	
a^2	a^1	a^1	0.05	

Thompson sampling implements probability matching

- Thompson sampling(1929) achieves Lai and Robbins lower bound
- Bounds for optimism are tighter than for Thompson sampling
- But empirically Thompson sampling can be extremely effective

Thompson Sampling for News Article Recommendation (Chapelle and Li, 2010)

- Contextual bandit: input context which impacts reward of each arm, context sampled iid each step
- Arms = articles
- Reward = click (+1) on article ($Q(a)$=click through rate)



Bayesian Regret Bounds for Thompson Sampling

- Regret(UCB,T)

$$BayesRegret(TS, T) = E_{\theta \sim p_\theta}\left[\sum_{t=1}^{T} f^*(a^*) - f^*(a_t)\right]$$

- Posterior sampling has the same (ignoring constants) regret bounds as UCB

Optimistic Initialization

- Simple and practical idea: initialize Q(a) to high value
- Update action value by incremental Monte-Carlo evaluation
- Starting with $N(a) > 0$

$$\hat{Q}_t(a_t) = \hat{Q}_{t-1} + \frac{1}{N_t(a_t)}(r_t - \hat{Q}_{t-1})$$

- Encourages systematic exploration early on
- But can still lock onto suboptimal action
- Depends on how high initialize Q
- Check your understanding: What is the downside to initializing Q too high?

Greedy Bandit Algorithms and Optimistic Initialization

- **Greedy**: Linear total regret
- **Constant ϵ-greedy**: Linear total regret
- **Decaying ϵ-greedy**: Sublinear regret but schedule for decaying ϵ requires knowledge of gaps, which are unknown
- **Optimistic initialization**: Sublinear regret if initialize values sufficiently optimistically, else linear regret
- Check your understanding: why does fixed ϵ-greedy have linear regret? (Do a proof sketch)

Consider Montezuma's revenge

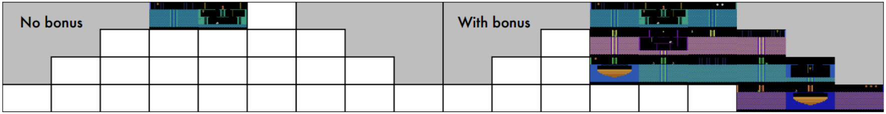


Figure 3: "Known world" of a DQN agent trained for 50 million frames with (**right**) and without (**left**) count-based exploration bonuses, in MONTEZUMA'S REVENGE.

- EB: move this to generalization and efficiency later on
- Bellemare et al. "Unifying Count-Based Exploration and Intrinsic Motivation"
- Enormously better than standard DQN with ϵ-greedy approach
- Uses principle of optimism under uncertainty which we will see today

Calculating UCB

- Pick a probability p that true value exceeds UCB
- Now solve for $U_t(a)$

$$\exp(-2N_t(a)U_t(a)^2) = p$$

$$U_t(a) = \sqrt{\frac{-\log p}{2N_t(a)}}$$

- Reduce p as we observe more rewards, e.q. $p = t^{-4}$
- Ensures we select optimal action as $t \to \infty$

$$U_t(a) = \sqrt{\frac{2\log t}{N_t(a)}}$$

UCB1

- This leads to the UCB1 algorithm

$$a_t = \arg\max_{a \in \mathcal{A}} [\hat{Q}(a) + \sqrt{\frac{2 \log t}{N_t(a)}}]$$