Lecture 12: Fast Reinforcement Learning [1]

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CS234 Reinforcement Learning

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[1] With some slides derived from David Silver

Class Structure

- Last time: Fast Learning (Bandits and regret)
- **This time: Fast Learning (Bayesian bandits to MDPs)**
- Next time: Fast Learning & Exploration

Settings, Frameworks & Approaches

- Over next couple lectures will consider 2 settings, multiple frameworks, and approaches
- Settings: Bandits (single decisions), MDPs
- Frameworks: evaluation criteria for formally assessing the quality of a RL algorithm
- Approaches: Classes of algorithms for achieving particular evaluation criteria in a certain set
- Note: We will see that some approaches can achieve multiple frameworks in multiple settings

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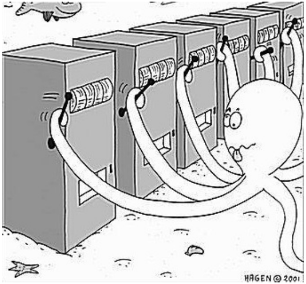
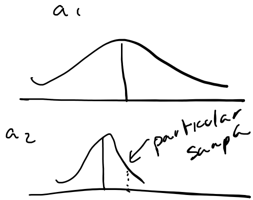
Recall: Multiarmed Bandits

- Multi-armed bandit is a tuple of $(\mathcal{A}, \mathcal{R})$
- $\mathcal{A}$: known set of m actions (arms)
- $\mathcal{R}^a(r) = \mathbb{P}[r \mid a]$ is an unknown probability distribution over rewards
- At each step t the agent selects an action $a_t \in \mathcal{A}$
- The environment generates a reward $r_t \sim \mathcal{R}^{a_t}$
- Goal: Maximize cumulative reward $\sum_{\tau=1}^{t} r_\tau$



Regret



$Q(a_1) > Q(a_2)$

- **Action-value** is the mean reward for action a

$$Q(a) = \mathbb{E}[r \mid a]$$

- **Optimal value** V^*

$$V^* = Q(a^*) = \max_{a \in \mathcal{A}} Q(a)$$

- **Regret** is the opportunity loss for one step

$$l_t = \mathbb{E}[V^* - Q(a_t)]$$

- **Total Regret** is the total opportunity loss

$$L_t = \mathbb{E}[\sum_{\tau=1}^{t} V^* - Q(a_\tau)]$$

- Maximize cumulative reward $\iff$ minimize total regret

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Approach: Optimism Under Uncertainty

1993 Kaelbling (MIT)

- Estimate an upper confidence $U_t(a)$ for each action value, such that $Q(a) \leq U_t(a)$ with high probability
- This depends on the number of times $N_t(a)$ action a has been selected
- Select action maximizing Upper Confidence Bound (UCB)

Hoeffding inequality

$$a_t = \arg\max_{a \in \mathcal{A}} [U_t(a)]$$

2 things could happen
- either $a_t = a^*$ regret of 0
- or $a_t \neq a^*$ $U_t(a_t)$ decrease

UCB Bandit Regret

- UCB

$$a_t = \arg\max_{a \in \mathcal{A}} \hat{Q}(a) + \sqrt{\frac{2 \log t}{N_t(a)}}$$

- Theorem: The UCB algorithm achieves logarithmic asymptotic total regret

$$\lim_{t \to \infty} L_t \leq 8 \log t \sum_{a | \Delta_a > 0} \Delta_a$$

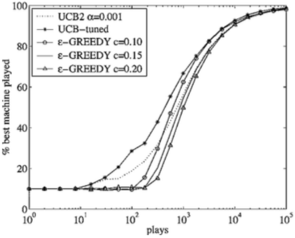
times we act

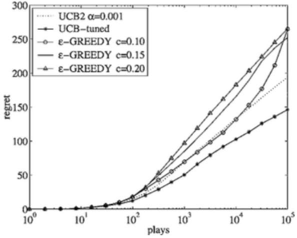
gaps

$\Delta_a = Q(a^*) - Q(a)$

problem-dep bound

related ~~but~~ diff to bound from last time





Toy Example: Ways to Treat Broken Toes, Optimism[1]

- True (unknown) parameters for each arm (action) are
  - surgery: $Q(a^1) = \theta_1 = .95$
  - buddy taping: $Q(a^2) = \theta_2 = .9$
  - doing nothing: $Q(a^3) = \theta_3 = .1$
- Optimism under uncertainty, UCB1 (Auer, Cesa-Bianchi, Fischer 2002)
  1. Sample each arm once

[1] Note:This is a made up example. This is not the actual expected efficacies of the various treatment options for a broken toe

Toy Example: Ways to Treat Broken Toes, Optimism[1]

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- UCB1 (Auer, Cesa-Bianchi, Fischer 2002)

empirical estimate

  1. Sample each arm once
     - Take action a^1 ($r \sim$Bernoulli(0.95)), get $+1$, $\hat{Q}(a^1) = 1$
     - Take action a^2 ($r \sim$Bernoulli(0.90)), get $+1$, $\hat{Q}(a^2) = 1$
     - Take action a^3 ($r \sim$Bernoulli(0.1)), get 0, $\hat{Q}(a^3) = 0$

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Toy Example: Ways to Treat Broken Toes, Optimism[1]

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     - Take action a^3 ($r \sim$Bernoulli(0.1)), get 0, $\hat{Q}(a^3) = 0$
  2. Set $t = 3$, Compute upper confidence bound on each action

total arm pulls

particular arm

$$UCB(a) = \hat{Q}(a) + \sqrt{\frac{2 \log t}{N_t(a)}}$$

$$UCB(a_3) = \sqrt{\frac{2 \log 3}{1}}$$

$$UCB(a_1) = 1 + \sqrt{2 \log 3 / 1} = UCB(a_2)$$

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  2. Set $t = 3$, Compute upper confidence bound on each action

$$UCB(a) = \hat{Q}(a) + \sqrt{\frac{2 \log t}{N_t(a)}}$$

  3. $t = 3$, Select action $a_t = \arg\max_a UCB(a)$,
  4. Observe reward 1
  5. Compute upper confidence bound on each action

Check Your Understanding

- True (unknown) parameters for each arm (action) are
  - surgery: $Q(a^1) = \theta_1 = .95$
  - buddy taping: $Q(a^2) = \theta_2 = .9$
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  1. Sample each arm once
     - Take action a^1 ($r \sim$Bernoulli(0.95)), get $+1$, $\hat{Q}(a^1) = 1$
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     - Take action a^3 ($r \sim$Bernoulli(0.1)), get 0, $\hat{Q}(a^3) = 0$
  2. Set $t = 3$, Compute upper confidence bound on each action

$$UCB(a) = \hat{Q}(a) + \sqrt{\frac{2 \log t}{N_t(a)}}$$

$UCB(a_1) = UCB(a_2) = 1 + \sqrt{\frac{2 \log 3}{1}}$

$UCB(a_3) = \sqrt{\frac{2 \log 3}{1}}$

  3. $t = t + 1$, Select action $a_t = 1$, Observe reward 1 a_1
  4. Compute upper confidence bound on each action
  5. Assume ties are evenly split. Prob of each arm if using ϵ-greedy (with ϵ=0.1)? If using UCB? ϵ $.1/|A|$

Toy Example: Ways to Treat Broken Toes, Optimism, Assessing Regret

pull a_1 got a 1

$UCB(a_1) = 1 + \sqrt{\frac{2\log 4}{2}}$ $UCB(a_2) = 1 + \sqrt{\frac{2\log 4}{1}}$

- True (unknown) parameters for each arm (action) are
  - surgery: $Q(a^1) = \theta_1 = .95$
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  - doing nothing: $Q(a^3) = \theta_3 = .1$

$UCB(a_3) = \sqrt{\frac{2\log 4}{1}}$

- UCB1 (Auer, Cesa-Bianchi, Fischer 2002)

Action	Optimal Action	Regret
a^1	a^1	0
a^2	a^1	.05 = .95 - .9
a^3	a^1	.85 = .95 - .1
a^1	a^1	0
a^2	a^1	.05

Check Your Understanding

- An alternative would be to always select the arm with the highest lower bound
- Why can this yield linear regret?
- Consider a two arm case for simplicity

Greedy Bandit Algorithms and Optimistic Initialization

- Simple optimism under uncertainty approach
  - Pretend already observed one pull of each arm, and saw some optimistic reward
  - Average these fake pulls and rewards in when computing average empirical reward
- Comparing regret results:
- **Greedy**: Linear total regret
- **Constant ϵ-greedy**: Linear total regret
- **Decaying ϵ-greedy**: Sublinear regret if can use right schedule for decaying ϵ, but that requires knowledge of gaps, which are unknown
- **Optimistic initialization**: Sublinear regret if initialize values sufficiently optimistically, else linear regret

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Bayesian Bandits

- So far we have made no assumptions about the reward distribution $\mathcal{R}$
  - Except bounds on rewards $\mathcal{R} = [0, 1)$
- **Bayesian bandits** exploit prior knowledge of rewards, $p[\mathcal{R}]$
- They compute posterior distribution of rewards $p[\mathcal{R} \mid h_t]$, where $h_t = (a_1, r_1, \ldots, a_{t-1}, r_{t-1})$
- Use posterior to guide exploration
  - Upper confidence bounds (Bayesian UCB)
  - Probability matching (Thompson Sampling)
- Better performance if prior knowledge is accurate

Short Refresher / Review on Bayesian Inference

- In Bayesian view, we start with a prior over the unknown parameters
  - Here the unknown distribution over the rewards for each arm
- Given observations / data about that parameter, update our uncertainty over the unknown parameters using Bayes Rule

Short Refresher / Review on Bayesian Inference

- In Bayesian view, we start with a prior over the unknown parameters
  - Here the unknown distribution over the rewards for each arm
- Given observations / data about that parameter, update our uncertainty over the unknown parameters using Bayes Rule
- For example, let the reward of arm i be a probability distribution that depends on parameter ϕ_i (unknown)
- Our initial prior over ϕ_i is $p(\phi_i)$
- We pull arm i and observe reward r_{i1}
- Then we can use this to update our estimate over ϕ_i as

Bayes rule

data evidence

prior

$$p(\phi_i|r_{i1}) = \frac{p(r_{i1}|\phi_i)p(\phi_i)}{p(r_{i1})} = \frac{p(r_{i1}|\phi_i)p(\phi_i)}{\int_{\phi_i} p(r_{i1}|\phi_i)p(\phi_i)d\phi_i}$$

- In Bayesian view, we start with a prior over the unknown parameters
- Given observations / data about that parameter, update our uncertainty over the unknown parameters using Bayes Rule

data likelihood   prior

$$p(\phi_i|r_{i1}) = \frac{p(r_{i1}|\phi_i)p(\phi_i)}{\int_{\phi_i} p(r_{i1}|\phi_i)p(\phi_i)d\phi_i}$$

- In general computing this update may be tricky to do exactly with no additional structure on the form of the prior and data likelihood

Short Refresher / Review on Bayesian Inference: Conjugate

- In Bayesian view, we start with a prior over the unknown parameters
- Given observations / data about that parameter, update our uncertainty over the unknown parameters using Bayes Rule

Gaussian   conjugate   Gaussian

$$p(\phi_i|r_{i1}) = \frac{p(r_{i1}|\phi_i)p(\phi_i)}{\int_{\phi_i} p(r_{i1}|\phi_i)p(\phi_i)d\phi_i}$$

- In general computing this update may be tricky
- But sometimes can be done analytically
- If the parametric representation of the prior and posterior is the same, the prior and model are called **conjugate**.
- For example, exponential families have conjugate priors

Short Refresher / Review on Bayesian Inference: Bernoulli

- Consider a bandit problem where the reward of an arm is a binary outcome $\{0, 1\}$ sampled from a Bernoulli with parameter θ
  - E.g. Advertisement click through rate, patient treatment succeeds/fails, ...
- The Beta distribution $Beta(\alpha, \beta)$ is conjugate for the Bernoulli distribution

prior

$$p(\theta|\alpha, \beta) = \theta^{\alpha-1}(1-\theta)^{\beta-1}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$$

where $\Gamma(x)$ is the Gamma family.

Short Refresher / Review on Bayesian Inference: Bernoulli

arm with mean $= \theta$

- Consider a bandit problem where the reward of an arm is a binary outcome $\{0,1\}$ sampled from a Bernoulli with parameter θ
  - E.g. Advertisement click through rate, patient treatment succeeds/fails, ...
- The Beta distribution $Beta(\alpha, \beta)$ is conjugate for the Bernoulli distribution

$$\overbrace{p(\theta|\alpha,\beta)}^{\text{prior}} = \theta^{\alpha-1}(1-\theta)^{\beta-1}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$$

where $\Gamma(x)$ is the Gamma family.

- Assume the prior over θ is a $Beta(\alpha, \beta)$ as above
- Then after observed a reward $r \in \{0,1\}$ then updated posterior over θ is $Beta(r+\alpha, 1-r+\beta)$

observe $r=1$ $Beta(\alpha+1, \beta)$

$r=0$ $Beta(\alpha, \beta+1)$

Framework: Regret and Bayesian Regret

- How do we evaluate performance in the Bayesian setting?
- Frequentist regret assumes a true (unknown) set of parameters

$$Regret(\mathcal{A}, T; \theta) = \sum_{t=1}^{T} \mathbb{E}\left[Q(a^*) - Q(a_t)\right]$$

- Bayesian regret assumes there is a prior over parameters

$$BayesRegret(\mathcal{A}, T; \theta) = \mathbb{E}_{\theta \sim p_\theta}\left[\sum_{t=1}^{T} \mathbb{E}\left[Q(a^*) - Q(a_t) | \theta\right]\right]$$

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Approach: Probability Matching

~1929

- Assume we have a parametric distribution over rewards for each arm
- **Probability matching** selects action *a* according to probability that *a* is the optimal action

prior pulls & reward outcomes

$$\pi(a \mid h_t) = \mathbb{P}[Q(a) > Q(a'), \forall a' \neq a \mid h_t]$$

- Probability matching is optimistic in the face of uncertainty
  - Uncertain actions have higher probability of being max
- Can be difficult to compute probability that an action is optimal analytically from posterior
- Somewhat incredibly, a simple approach implements probability matching

Thompson Sampling

θ_1 sample .9

$Q(a_1) = E[\theta_1] = .9$

Bernoulli $p(\theta_i)$ $i = 1 : |A|$

$p(\theta_i) = Beta(1, 1)$

1: Initialize prior over each arm a, $p(\mathcal{R}_a)$
2: **loop**
3: For each arm a **sample** a reward distribution $\mathcal{R}_a$ from posterior
4: Compute action-value function $Q(a) = \mathbb{E}[\mathcal{R}_a]$
5: $a_t = \arg\max_{a \in \mathcal{A}} Q(a)$ $\leftarrow$
6: Observe reward r
7: Update posterior $p(\mathcal{R}_a | r)$ using Bayes law
8: **end loop**

Thompson Sampling Implements Probability Matching

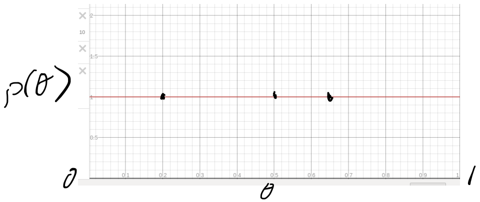
prob matching

$$
\begin{aligned}
\pi(a \mid h_t) &= \mathbb{P}[Q(a) > Q(a'), \forall a' \neq a \mid h_t] \\
&= \mathbb{E}_{\mathcal{R}|h_t}\left[\mathbb{1}(a = \arg\max_{a \in \mathcal{A}} Q(a))\right]
\end{aligned}
$$

Toy Example: Ways to Treat Broken Toes, Thompson Sampling

- True (unknown) Bernoulli parameters for each arm/action
- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- Thompson sampling:
- Place a prior over each arm's parameter. Here choose Beta(1,1) (Uniform) $p(\theta_1) = Beta(1,1)$
  1. Sample a Bernoulli parameter given current prior over each arm Beta(1,1), Beta(1,1), Beta(1,1):
     0.3 0.5 0.6

Beta(1,1)



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- Place a prior over each arm's parameter. Here choose Beta(1,1)
  1. Sample a Bernoulli parameter given current prior over each arm Beta(1,1), Beta(1,1), Beta(1,1): 0.3 0.5 0.6
  2. Select $a = \arg\max_{a \in A} Q(a) = \arg\max_{a \in A} \theta(a) = 3$

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Toy Example: Ways to Treat Broken Toes, Thompson Sampling

- True (unknown) Bernoulli parameters for each arm/action
- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- Thompson sampling:
- Place a prior over each arm's parameter. Here choose $\theta_i \sim$Beta(1,1)
  1. Per arm, sample a Bernoulli θ given prior: 0.3 0.5 0.6
  2. Select $a_t = \arg\max_{a \in A} Q(a) = \arg\max_{a \in A} \theta(a) = 3$
  3. Observe the patient outcome's outcome: 0
  4. Update the posterior over the $Q(a_t) = Q(a^3)$ value for the arm pulled

$p(\theta_3 \mid r=0)$

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  2. Select $a_t = \arg\max_{a \in A} Q(a) = \arg\max_{a \in A} \theta(a) = 3$
  3. Observe the patient outcome's outcome: 0
  4. Update the posterior over the $Q(a_t) = Q(a^3)$ value for the arm pulled
     - Beta(c_1, c_2) is the conjugate distribution for Bernoulli
     - If observe 1, $c_1 + 1$ else if observe 0 $c_2 + 1$
  5. New posterior over Q value for arm pulled is:
  6. New posterior $p(Q(a^3)) = p(\theta(a^3)) = Beta(1, 2)$

Toy Example: Ways to Treat Broken Toes, Thompson Sampling

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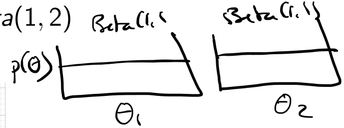
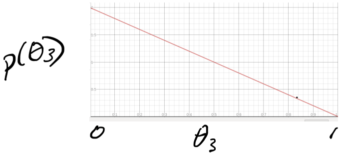
  1. Sample a Bernoulli parameter given current prior over each arm Beta(1,1), Beta(1,1), Beta(1,1): 0.3 0.5 0.6
  2. Select $a_t = \arg\max_{a \in A} Q(a) = \arg\max_{a \in A} \theta(a) = \cancel{1}\ 3$
  3. Observe the patient outcome's outcome: 0
  4. New posterior $p(Q(a_{\cancel{1}\,3})) = p(\theta(a_{\cancel{1}\,3})) = Beta(1, 2)$





Toy Example: Ways to Treat Broken Toes, Thompson Sampling

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- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
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- Place a prior over each arm's parameter. Here choose $\theta_i \sim$Beta(1,1)
  1. Sample a Bernoulli parameter given current prior over each arm Beta(1,1), Beta(1,1), Beta(1,2): 0.7, 0.5, 0.3 $\quad max \to a_1$

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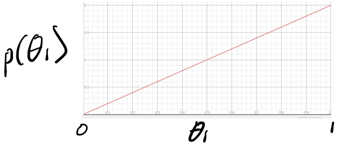
  1. Sample a Bernoulli parameter given current prior over each arm Beta(1,1), Beta(1,1), Beta(1,2): 0.7, 0.5, 0.3
  2. Select $a_t = \arg\max_{a \in A} Q(a) = \arg\max_{a \in A} \theta(a) = 1$
  3. Observe the patient outcome's outcome: 1
  4. New posterior $p(Q(a^1)) = p(\theta(a^1) = Beta(2, 1)$

← #r = 0s + 1

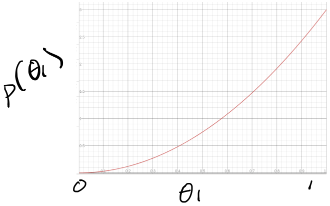
↑ #r = 1s + 1

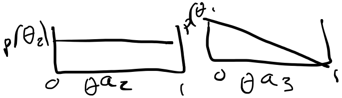


Beta(2,1)
Beta(1,1)
Beta(1,2)

Toy Example: Ways to Treat Broken Toes, Thompson Sampling

- True (unknown) Bernoulli parameters for each arm/action
- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- Thompson sampling:
- Place a prior over each arm's parameter. Here choose $\theta_i \sim$Beta(1,1)
  1. Sample a Bernoulli parameter given current prior over each arm Beta(2,1), Beta(1,1), Beta(1,2): 0.71, 0.65, 0.1
  2. Select $a_t = \arg\max_{a \in A} Q(a) = \arg\max_{a \in A} \theta(a) = 1$
  3. Observe the patient outcome's outcome: 1
  4. New posterior $p(Q(a^1)) = p(\theta(a^1) = Beta(3,1)$





Toy Example: Ways to Treat Broken Toes, Thompson Sampling

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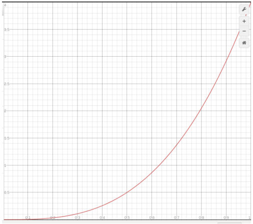
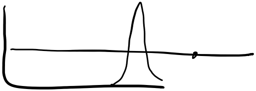
- Thompson sampling:
- Place a prior over each arm's parameter. Here choose $\theta_i \sim$Beta(1,1)
  1. Sample a Bernoulli parameter given current prior over each arm Beta(2,1), Beta(1,1), Beta(1,2): 0.75, 0.45, 0.4
  2. Select $a_t = \arg\max_{a \in A} Q(a) = \arg\max_{a \in A} \theta(a) = 1$
  3. Observe the patient outcome's outcome: 1
  4. New posterior $p(Q(a^1)) = p(\theta(a^1) = Beta(4,1)$



Toy Example: Ways to Treat Broken Toes, Thompson Sampling vs Optimism

.7



- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- How does the sequence of arm pulls compare in this example so far?

Optimism	TS	Optimal
a^1	a^3	a^1
a^2	a^1	a^1
a^3	a^1	a^1
a^1	a^1	a^1
a^2	a^1	a^1

Toy Example: Ways to Treat Broken Toes, Thompson Sampling vs Optimism

- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- Incurred (frequentist) regret?

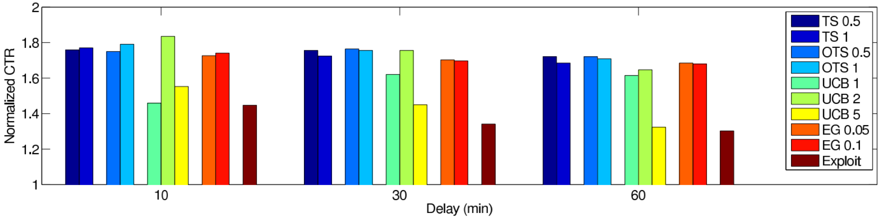
Optimism	TS	Optimal	Regret Optimism	Regret TS
a^1	a^3	a^1	0	.85
a^2	a^1	a^1	0.05	0
a^3	a^1	a^1	0.85	0
a^1	a^1	a^1	0	0
a^2	a^1	a^1	0.05	0

Thompson sampling implements probability matching

- Thompson sampling(1929) achieves Lai and Robbins lower bound
- Bounds for optimism are tighter than for Thompson sampling
- But empirically Thompson sampling can be extremely effective

Thompson Sampling for News Article Recommendation (Chapelle and Li, 2010)

- Contextual bandit: input context which impacts reward of each arm, context sampled iid each step
- Arms = articles
- Reward = click (+1) on article ($Q(a)$=click through rate)



Bayesian Regret Bounds for Thompson Sampling

- Regret(UCB,T)

$$BayesRegret(TS, T) = E_{\theta \sim p_\theta}\left[\sum_{t=1}^{T} Q(a^*) - Q(a_t) | \theta\right]$$

- Posterior sampling has the same (ignoring constants) regret bounds as UCB

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Framework: Probably Approximately Correct

- Theoretical regret bounds specify how regret grows with T
- Could be making lots of little mistakes or infrequent large ones
- May care about bounding the number of non-small errors
- More formally, probably approximately correct (PAC) results state that the algorithm will choose an action a whose value is ϵ-optimal ($Q(a) \geq Q(a^*) - \epsilon$) with probability at least $1 - \delta$ on all but a polynomial number of steps
- Polynomial in the problem parameters (# actions, ϵ, δ, etc)
- Most PAC algorithms based on optimism or Thompson sampling

Toy Example: Probably Approximately Correct and Regret

- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- Let $\epsilon = 0.05$.
- O = Optimism, TS = Thompson Sampling: W/in $\epsilon = I(Q(a_t) \geq Q(a^*) - \epsilon)$

counting mistakes

O	TS	Optimal	O Regret	O W/in ϵ	TS Regret	TS W/in ϵ
a^1	a^3	a^1	0	Y	0.85	N
a^2	a^1	a^1	0.05	Y	0	Y
a^3	a^1	a^1	0.85	N	0	Y
a^1	a^1	a^1	0	Y	0	Y
a^2	a^1	a^1	0.05	Y	0	Y

Toy Example: Probably Approximately Correct and Regret

- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- Let $\epsilon = 0.05$.
- O = Optimism, TS = Thompson Sampling: W/in $\epsilon = I(Q(a_t) \geq Q(a^*) - \epsilon)$

O	TS	Optimal	O Regret	O W/in ϵ	TS Regret	TS W/in ϵ
a^1	a^3	a^1	0	Y	0.85	N
a^2	a^1	a^1	0.05	Y	0	Y
a^3	a^1	a^1	0.85	N	0	Y
a^1	a^1	a^1	0	Y	0	Y
a^2	a^1	a^1	0.05	Y	0	Y

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Fast RL in Markov Decision Processes

tabular MDPs

- Very similar set of frameworks and approaches are relevant for fast learning in reinforcement learning
- Frameworks
  - Regret
  - Bayesian regret
  - Probably approximately correct (PAC)
- Approaches
  - Optimism under uncertainty
  - Probability matching / Thompson sampling
- Framework: Probably approximately correct

Fast RL in Markov Decision Processes

- Very similar set of frameworks and approaches are relevant for fast learning in reinforcement learning
- Frameworks
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  - **Optimism under uncertainty**
  - Probability matching / Thompson sampling
- Framework: Probably approximately correct

Optimistic Initialization: Model-Free RL

- Initialize action-value function Q(s,a) optimistically (for ex. $\frac{r_{max}}{1-\gamma}$)
  - where $r_{max} = \max_a \max_s R(s, a)$
  - Check your understanding: why is that value guaranteed to be optimistic?
- Run favorite model-free RL algorithm
  - Monte-Carlo control
  - Sarsa
  - Q-learning ...
- Encourages systematic exploration of states and actions

Optimistic Initialization: Model-Free RL

- Initialize action-value function Q(s,a) optimistically (for ex. $\frac{r_{max}}{1-\gamma}$)
  - where $r_{max} = \max_a \max_s R(s, a)$
- Run model-free RL algorithm: MC control, Sarsa, Q-learning ...
- In general the above have no guarantees on performance, but may work better than greedy or ϵ-greedy approaches
- Even-Dar and Mansour (NeurIPS 2002) proved that

  $\alpha = .1$

  $\frac{1}{.1^T} \simeq 10^T$

  - If run Q-learning with learning rates α_i on time step i,
  - If initialize $V(s) = \frac{r_{max}}{(1-\gamma)\prod_{i=1}^{T} \alpha_i}$ where α_i is the learning rate on step i and T is the number of samples need to learn a near optimal Q
  - Then greedy-only Q-learning is PAC
- Recent work (Jin, Allen-Zhu, Bubeck, Jordan NeurIPS 2018) proved that (much less) optimistically initialized Q-learning has good (though not tightest) regret bounds

Approaches to Model-based Optimism for Provably Efficient RL

1. Be very optimistic until confident that empirical estimates are close to true (dynamics/reward) parameters (Brafman & Tennenholtz JMLR 2002)
2. Be optimistic given the information have
    - Compute confidence sets on dynamics and reward models, or
    - Add reward bonuses that depend on experience / data

- We will focus on the last class of approaches

Model-Based Interval Estimation with Exploration Bonus (MBIE-EB)

(Strehl and Littman, J of Computer & Sciences 2008)

$r_{max} = 1$ $\quad\quad$ $r \in (0,1)$ bounded

Given ϵ, δ, m

$n_{sa}(s,a) = 0 \ \forall s \ \forall a \quad n(s,a,s') = 0 \ \forall s, \forall a, \forall s' \quad rc(s,a) = 0 \ \forall s \ \forall a$

$\beta = \frac{1}{1-\gamma}\sqrt{2\log(|S||A|2m/\delta)}$

$t = 0$, s_t = initial state

$\tilde{Q}_t(s,a) = 1/(1-\gamma) \ \forall s, \forall a$

loop

$\quad a_t = \arg\max_{a \in \mathcal{A}} \tilde{Q}(s_t, a)$

$\quad$ Observe reward r_t and state s_{t+1}

$\quad n_{sa}(s,a){+}{+} \quad n_{sas'}(s,a,s'){+}{+} \quad rc(s,a) = rc(s,a) + r_t$

$\quad \hat{R}(s,a) = rc(s,a)/n_{sa}(s,a) \quad \hat{T}(s'|s,a) = n(s,a,s')/n(s,a) \quad \forall s,a$

$\quad$ **while** not converged **do**

$$\tilde{Q}(s,a) = \hat{R}(s,a) + \gamma \sum_{s'} \hat{T}(s'|s,a) \max_{a'} \tilde{Q}(s',a') + \underbrace{\beta/\sqrt{n_{sa}(s,a)}}_{\text{reward bonus}}$$

$\quad$ **end while** $\quad \forall s,a$ s.t. $n_{sa}(s,a) = 0$ $\quad \tilde{Q}(s,a) = 1/(1-\gamma)$

end loop

Model-Based Interval Estimation with Exploration Bonus (MBIE-EB)

(Strehl and Littman, J of Computer & Sciences 2008)

Given ϵ, δ, m

$\beta = \frac{1}{1-\gamma}\sqrt{0.5 \ln(2|S||A|m/\delta}$

$n_{sas}(s, a, s') = 0 \; s \in S,\; a \in A,\; s' \in S$

$rc(s, a) = 0,\; n_{sa}(s, a) = 0,\; \tilde{Q}(s, a) = 1/(1-\gamma) \; \forall \; s \in S,\; a \in A$

$t = 0,\; s_t = s_{init}$

loop

  $a_t = \arg\max_{a \in \mathcal{A}} Q(s_t, a)$

  Observe reward r_t and state s_{t+1}

  $n_{sa}(s_t, a_t) = n(s_t, a_t) + 1,\; n_{sas}(s_t, a_t, s_{t+1}) = n_{sas}(s_t, a_t, s_{t+1}) + 1$

  $rc(s_t, a_t) = \frac{rc(s_t, a_t) n_{sa}(s_t, a_t) + r_t}{(n_{sa}(s_t, a_t) + 1)}$

  $\hat{R}(s, a) = \frac{rc(s_t, a_t)}{n(s_t, a_t)}$ and $\hat{T}(s'|s, a) = \frac{n_{sas}(s_t, a_t, s')}{n_{sa}(s_t, a_t)} \; \forall s' \in S$

  while not converged **do**

    $\tilde{Q}(s, a) = \hat{R}(s, a) + \gamma \sum_{s'} \hat{T}(s'|s, a) \max_{a'} \tilde{Q}(s', a') + \underbrace{\frac{\beta}{\sqrt{n_{sa}(s, a)}}}_{\text{reward bonus}} \; \forall \; s \in S,\; a \in A$

  end while

end loop

Framework: PAC for MDPs

- For a given ϵ and δ, A RL algorithm $\mathcal{A}$ is PAC if on all but N steps, the action selected by algorithm $\mathcal{A}$ on time step t, a_t, is ϵ-close to the optimal action, where N is a polynomial function of $(\underbrace{|S|, |A|, \gamma, \epsilon, \delta}_{})$
- Is this true for all algorithms? No

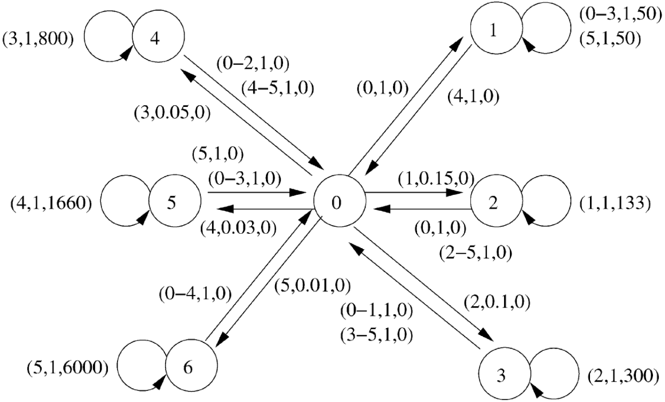
MBIE-EB is a PAC RL Algorithm

Theorem 2. *Suppose that* ϵ *and* δ *are two real numbers between* 0 *and* 1 *and* $M = \langle S, A, T, \mathcal{R}, \gamma \rangle$ *is any MDP. There exists an input* $m = m(\frac{1}{\epsilon}, \frac{1}{\delta})$*, satisfying* $m(\frac{1}{\epsilon}, \frac{1}{\delta}) = O\left(\frac{|S|}{\epsilon^2(1-\gamma)^4} + \frac{1}{\epsilon^2(1-\gamma)^4} \ln \frac{|S||A|}{\epsilon(1-\gamma)\delta}\right)$*, and* $\beta = (1/(1-\gamma))\sqrt{\ln(2|S||A|m/\delta)/2}$ *such that if MBIE-EB is executed on MDP* M*, then the following holds. Let* $\mathcal{A}_t$ *denote MBIE-EB's policy at time* t *and* s_t *denote the state at time* t*. With probability at least* $1-\delta$*,* $V_M^{\mathcal{A}_t}(s_t) \geqslant V_M^*(s_t) - \epsilon$ *is true for all but* $\underbrace{O\left(\frac{|S||A|}{\epsilon^3(1-\gamma)^6}\left(|S| + \ln \frac{|S||A|}{\epsilon(1-\gamma)\delta}\right) \ln \frac{1}{\delta} \ln \frac{1}{\epsilon(1-\gamma)}\right)}$ *timesteps* t*.*

A Sufficient Set of Conditions to Make a RL Algorithm PAC

- Strehl, A. L., Li, L., Littman, M. L. (2006). Incremental model-based learners with formal learning-time guarantees. In Proceedings of the Twenty-Second Conference on Uncertainty in Artificial Intelligence (pp. 485-493)

MBIE-EB Empirically: 6 Arms Simulation



MBIE-EB Empirically: 6 Arms Results