Lecture 3: Model-Free Policy Evaluation: Policy Evaluation Without Knowing How the World Works[1]

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CS234 Reinforcement Learning

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[1]Material builds on structure from David SIlver's Lecture 4: Model-Free Prediction. Other resources: Sutton and Barto Jan 1 2018 draft Chapter/Sections: 5.1; 5.5; 6.1-6.3

Today's Plan

- Last Time:
  - Markov reward / decision processes
  - Policy evaluation & control when have true model (of how the world works)
- **Today**
  - **Policy evaluation without known dynamics & reward models**
- Next Time:
  - Control when don't have a model of how the world works

This Lecture: Policy Evaluation

- Estimating the expected return of a particular policy if don't have access to true MDP models
- Dynamic programming
- Monte Carlo policy evaluation
  - Policy evaluation when don't have a model of how the world work
    - Given on-policy samples
- Temporal Difference (TD)
- Metrics to evaluate and compare algorithms

Recall

- Definition of Return, G_t (for a MRP)
  - Discounted sum of rewards from time step t to horizon

$$G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots$$

- Definition of State Value Function, $V^\pi(s)$
  - Expected return from starting in state s under policy π

$$V^\pi(s) = \mathbb{E}_\pi[G_t | s_t = s] = \mathbb{E}_\pi[r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots | s_t = s]$$

- Definition of State-Action Value Function, $Q^\pi(s,a)$
  - Expected return from starting in state s, taking action a and then following policy π

$$\begin{aligned} Q^\pi(s,a) &= \mathbb{E}_\pi[G_t | s_t = s, a_t = a] \\ &= \mathbb{E}_\pi[r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots | s_t = s, a_t = a] \end{aligned}$$

Dynamic Programming for Policy Evaluation

- Initialize $V_0^\pi(s) = 0$ for all s
- For $k = 1$ until convergence
  - For all s in S

$$V_k^\pi(s) = r(s, \pi(s)) + \gamma \sum_{s' \in S} p(s'|s, \pi(s)) V_{k-1}^\pi(s')$$

Dynamic Programming for Policy π, Value Evaluation

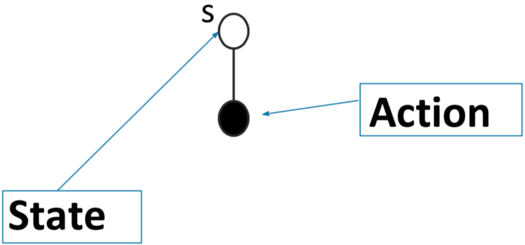
- Initialize $V_0^\pi(s) = 0$ for all s
- For $k = 1$ until convergence
  - For all s in S

$$V_k^\pi(s) = r(s, \pi(s)) + \gamma \sum_{s' \in S} p(s'|s, \pi(s)) V_{k-1}^\pi(s')$$

- $V_k^\pi(s)$ is exact value of k-horizon value of state s under policy π
- $V_k^\pi(s)$ is an estimate of infinite horizon value of state s under policy π

$$V^\pi(s) = \mathbb{E}_\pi[G_t|s_t = s] \approx \mathbb{E}_\pi[r_t + \gamma V_{k-1}|s_t = s]$$

Dynamic Programming Policy Evaluation $V^{\pi}(s) \leftarrow \mathbb{E}_{\pi}[r_t + \gamma V_{k-1} | s_t = s]$



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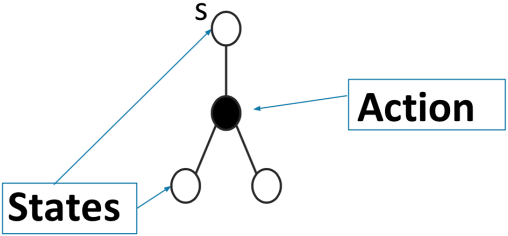

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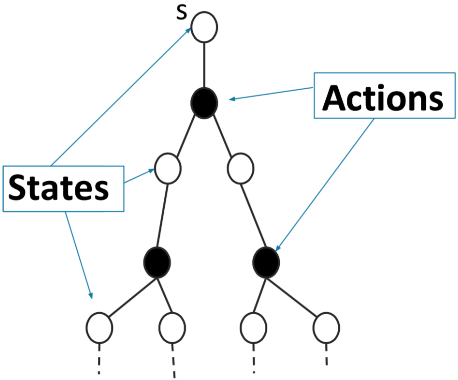

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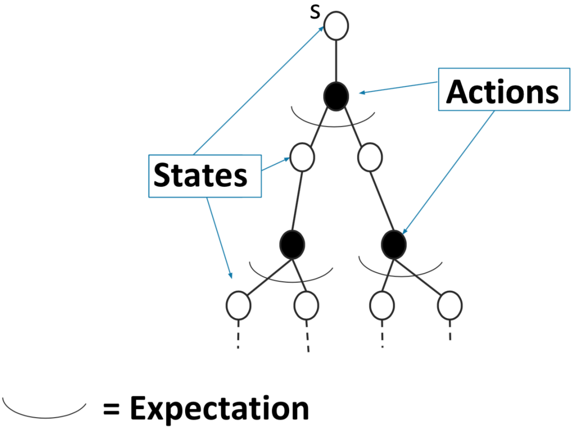

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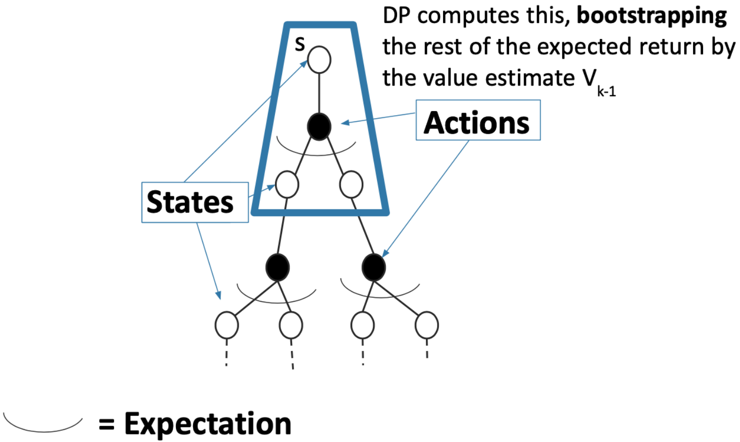
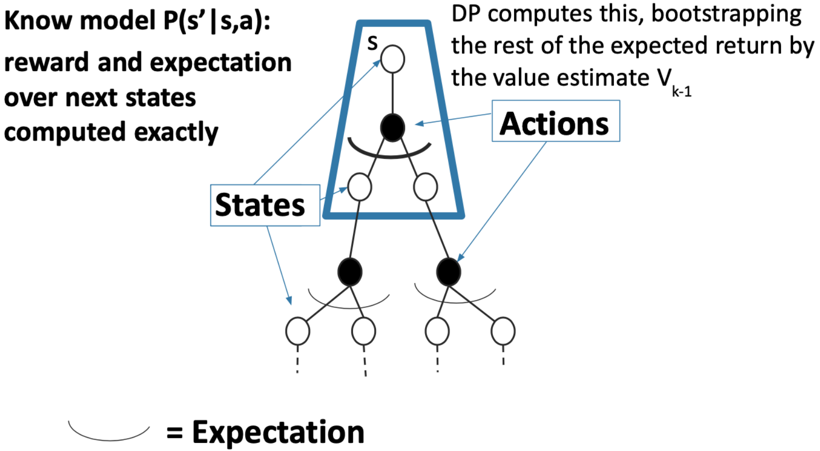


- Bootstrapping: Update for V uses an estimate

Dynamic Programming Policy Evaluation $V^\pi(s) \leftarrow \mathbb{E}_\pi[r_t + \gamma V_{k-1} | s_t = s]$



- Bootstrapping: Update for V uses an estimate

Policy Evaluation: $V^\pi(s) = \mathbb{E}_\pi[G_t|s_t = s]$

- $G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots$ in MDP M under policy π
- Dynamic Programming
  - $V^\pi(s) \approx \mathbb{E}_\pi[r_t + \gamma V_{k-1}|s_t = s]$
  - **Requires model of MDP** M
  - Bootstraps future return using value estimate
  - Requires Markov assumption: bootstrapping regardless of history
- What if don't know dynamics model P and/ or reward model R?
- **Today: Policy evaluation without a model**
  - Given data and/or ability to interact in the environment
  - Efficiently compute a good estimate of a policy π

This Lecture Overview: Policy Evaluation

- Dynamic Programming
- **Evaluating the quality of an estimator**
- **Monte Carlo policy evaluation**
    - Policy evaluation when don't know dynamics and/or reward model
        - Given on policy samples
- Temporal Difference (TD)
- Metrics to evaluate and compare algorithms

Monte Carlo (MC) Policy Evaluation

- $G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots$ in MDP M under policy π
- $V^\pi(s) = \mathbb{E}_{T \sim \pi}[G_t | s_t = s]$
  - Expectation over trajectories T generated by following π
- Simple idea: Value = mean return
- If trajectories are all finite, sample set of trajectories & average returns

Monte Carlo (MC) Policy Evaluation

- If trajectories are all finite, sample set of trajectories & average returns
- Does not require MDP dynamics/rewards
- No bootstrapping
- Does not assume state is Markov
- Can **only** be applied to episodic MDPs
  - Averaging over returns from a complete episode
  - Requires each episode to terminate

Monte Carlo (MC) On Policy Evaluation

- Aim: estimate $V^\pi(s)$ given episodes generated under policy π
  - $s_1, a_1, r_1, s_2, a_2, r_2, \ldots$ where the actions are sampled from π
- $G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots$ in MDP M under policy π
- $V^\pi(s) = \mathbb{E}_\pi[G_t | s_t = s]$
- MC computes empirical mean return
- Often do this in an incremental fashion
  - After each episode, update estimate of V^π

First-Visit Monte Carlo (MC) On Policy Evaluation

Initialize $N(s) = 0$, $G(s) = 0 \ \forall s \in S$

Loop

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$
- Define $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$ as return from time step t onwards in ith episode
- For each state s visited in episode i
  - For **first** time t that state s is visited in episode i
    - Increment counter of total first visits: $N(s) = N(s) + 1$
    - Increment total return $G(s) = G(s) + G_{i,t}$
    - Update estimate $V^\pi(s) = G(s)/N(s)$

Bias, Variance and MSE

- Consider a statistical model that is parameterized by θ and that determines a probability distribution over observed data $P(x|\theta)$
- Consider a statistic $\hat{\theta}$ that provides an estimate of θ and is a function of observed data x
  - E.g. for a Gaussian distribution with known variance, the average of a set of i.i.d data points is an estimate of the mean of the Gaussian
- Definition: the bias of an estimator $\hat{\theta}$ is:

$$Bias_\theta(\hat{\theta}) = \mathbb{E}_{x|\theta}[\hat{\theta}] - \theta$$

- Definition: the variance of an estimator $\hat{\theta}$ is:

$$Var(\hat{\theta}) = \mathbb{E}_{x|\theta}[(\hat{\theta} - \mathbb{E}[\hat{\theta}])^2]$$

- Definition: mean squared error (MSE) of an estimator $\hat{\theta}$ is:

$$MSE(\hat{\theta}) = Var(\hat{\theta}) + Bias_\theta(\hat{\theta})^2$$

First-Visit Monte Carlo (MC) On Policy Evaluation

Initialize $N(s) = 0$, $G(s) = 0 \ \forall s \in S$
Loop

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$
- Define $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$ as return from time step t onwards in ith episode
- For each state s visited in episode i
    - For **first** time t that state s is visited in episode i
        - Increment counter of total first visits: $N(s) = N(s) + 1$
        - Increment total return $G(s) = G(s) + G_{i,t}$
        - Update estimate $V^\pi(s) = G(s)/N(s)$

Properties:

- V^π estimator is an unbiased estimator of true $\mathbb{E}_\pi[G_t | s_t = s]$
- By law of large numbers, as $N(s) \rightarrow \infty$, $V^\pi(s) \rightarrow \mathbb{E}_\pi[G_t | s_t = s]$

Every-Visit Monte Carlo (MC) On Policy Evaluation

Initialize $N(s) = 0$, $G(s) = 0$ $\forall s \in S$
Loop

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$
- Define $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$ as return from time step t onwards in ith episode
- For each state s visited in episode i
  - For **every** time t that state s is visited in episode i
    - Increment counter of total first visits: $N(s) = N(s) + 1$
    - Increment total return $G(s) = G(s) + G_{i,t}$
    - Update estimate $V^\pi(s) = G(s)/N(s)$

Every-Visit Monte Carlo (MC) On Policy Evaluation

Initialize $N(s) = 0,\ G(s) = 0\ \forall s \in S$
Loop

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \dots, s_{i,T_i}$
- Define $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$ as return from time step t onwards in ith episode
- For each state s visited in episode i
  - For **every** time t that state s is visited in episode i
    - Increment counter of total first visits: $N(s) = N(s) + 1$
    - Increment total return $G(s) = G(s) + G_{i,t}$
    - Update estimate $V^\pi(s) = G(s)/N(s)$

Properties:

- V^π every-vist MC estimator is an **biased** estimator of V^π
- But consistent estimator and often has better MSE

Incremental Monte Carlo (MC) On Policy Evaluation

After each episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots$

- Define $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots$ as return from time step t onwards in ith episode
- For state s visited at time step t in episode i
  - Increment counter of total first visits: $N(s) = N(s) + 1$
  - Update estimate

$$V^\pi(s) = V^\pi(s)\frac{N(s)-1}{N(s)} + \frac{G_{i,t}}{N(s)} = V^\pi(s) + \frac{1}{N(s)}(G_{i,t} - V^\pi(s))$$

Incremental Monte Carlo (MC) On Policy Evaluation, Running Mean

Initialize $N(s) = 0$, $G(s) = 0 \ \forall s \in S$
Loop

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$
- Define $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$ as return from time step t onwards in ith episode
- For state s visited at time step t in episode i
  - Increment counter of total first visits: $N(s) = N(s) + 1$
  - Update estimate

$$V^\pi(s) = V^\pi(s) + \alpha(G_{i,t} - V^\pi(s))$$

- $\alpha = \frac{1}{N(s)}$: identical to every visit MC
- $\alpha > \frac{1}{N(s)}$: forget older data, helpful for non-stationary domains

Check Your Understanding: MC On Policy Evaluation

Initialize $N(s) = 0$, $G(s) = 0 \ \forall s \in S$
Loop

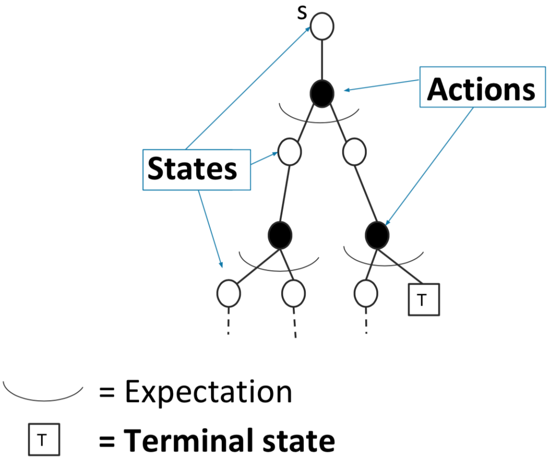
- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$
- $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$
- For each state s visited in episode i
  - For **first or every** time t that state s is visited in episode i
    - $N(s) = N(s) + 1$, $G(s) = G(s) + G_{i,t}$
    - Update estimate $V^\pi(s) = G(s)/N(s)$

Example:

- Mars rover: R = [1 0 0 0 0 0 +10] for any action
- $\pi(s) = a_1 \ \forall s$, $\gamma = 1$. any action from s_1 and s_7 terminates episode
- Trajectory = (s_3, a_1, 0, s_2, a_1, 0, s_2, a_1, 0, s_1, a_1, 1, terminal)
- First visit MC estimate of V of each state?
  V = [1 1 1 0 0 0 0]
- Every visit MC estimate of s_2?
  $V(s_2) = 1$

MC Policy Evaluation

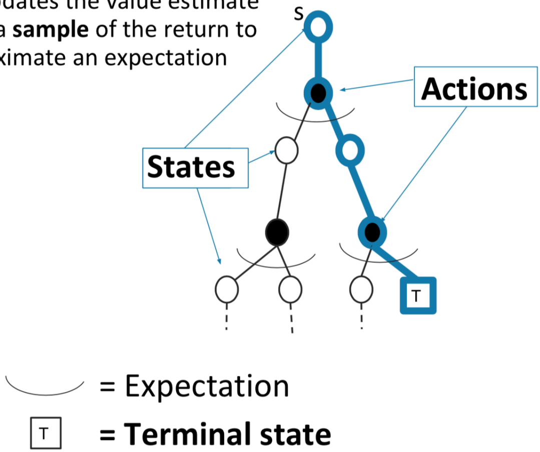
$$V^\pi(s) = V^\pi(s) + \alpha(G_{i,t} - V^\pi(s))$$



MC Policy Evaluation

$$V^\pi(s) = V^\pi(s) + \alpha(G_{i,t} - V^\pi(s))$$

MC updates the value estimate using a **sample** of the return to approximate an expectation



Monte Carlo (MC) Policy Evaluation Key Limitations

- Generally high variance estimator
  - Reducing variance can require a lot of data
- Requires episodic settings
  - Episode must end before data from that episode can be used to update the value function

Monte Carlo (MC) Policy Evaluation Summary

- Aim: estimate $V^\pi(s)$ given episodes generated under policy π
  - $s_1, a_1, r_1, s_2, a_2, r_2, \ldots$ where the actions are sampled from π
- $G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots$ in MDP M under policy π
- $V^\pi(s) = \mathbb{E}_\pi[G_t | s_t = s]$
- Simple: Estimates expectation by empirical average (given episodes sampled from policy of interest) or reweighted empirical average (importance sampling)
- Updates value estimate by using a **sample** of return to approximate the expectation
- No bootstrapping
- Converges to true value under some (generally mild) assumptions

This Lecture: Policy Evaluation

- Estimating the expected return of a particular policy if don't have access to true MDP models
- Dynamic programming
- Monte Carlo policy evaluation
  - Policy evaluation when don't have a model of how the world work
    - Given on-policy samples
- **Temporal Difference (TD)**
- Metrics to evaluate and compare algorithms

Temporal Difference Learning

- "If one had to identify one idea as central and novel to reinforcement learning, it would undoubtedly be temporal-difference (TD) learning." – Sutton and Barto 2017
- Combination of Monte Carlo & dynamic programming methods
- Model-free
- **Bootstraps and samples**
- Can be used in episodic or infinite-horizon non-episodic settings
- Immediately updates estimate of V after each (s, a, r, s') tuple

Temporal Difference Learning for Estimating V

- Aim: estimate $V^\pi(s)$ given episodes generated under policy π
- $G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots$ in MDP M under policy π
- $V^\pi(s) = \mathbb{E}_\pi[G_t | s_t = s]$
- Recall Bellman operator (if know MDP models)

$$B^\pi V(s) = r(s, \pi(s)) + \gamma \sum_{s' \in S} p(s'|s, \pi(s)) V(s')$$

- In incremental every-visit MC, update estimate using 1 sample of return (for the current ith episode)

$$V^\pi(s) = V^\pi(s) + \alpha(G_{i,t} - V^\pi(s))$$

- Insight: have an estimate of V^π, use to estimate expected return

$$V^\pi(s) = V^\pi(s) + \alpha([r_t + \gamma V^\pi(s_{t+1})] - V^\pi(s))$$

Temporal Difference [$TD(0)$] Learning

- Aim: estimate $V^\pi(s)$ given episodes generated under policy π
  - $s_1, a_1, r_1, s_2, a_2, r_2, \ldots$ where the actions are sampled from π
- Simplest TD learning: update value towards estimated value

$$V^\pi(s_t) = V^\pi(s_t) + \alpha(\underbrace{[r_t + \gamma V^\pi(s_{t+1})]}_{\text{TD target}} - V^\pi(s_t))$$

- TD error:

$$\delta_t = r_t + \gamma V^\pi(s_{t+1}) - V^\pi(s_t)$$

- Can immediately update value estimate after (s, a, r, s') tuple
- Don't need episodic setting

Temporal Difference $[TD(0)]$ Learning Algorithm

Input: α
Initialize $V^\pi(s) = 0$, $\forall s \in S$
Loop

- Sample **tuple** (s_t, a_t, r_t, s_{t+1})
- $V^\pi(s_t) = V^\pi(s_t) + \alpha(\underbrace{[r_t + \gamma V^\pi(s_{t+1})]}_{\text{TD target}} - V^\pi(s_t))$

Check Your Understanding: TD Learning

Input: α
Initialize $V^\pi(s) = 0$, $\forall s \in S$
Loop

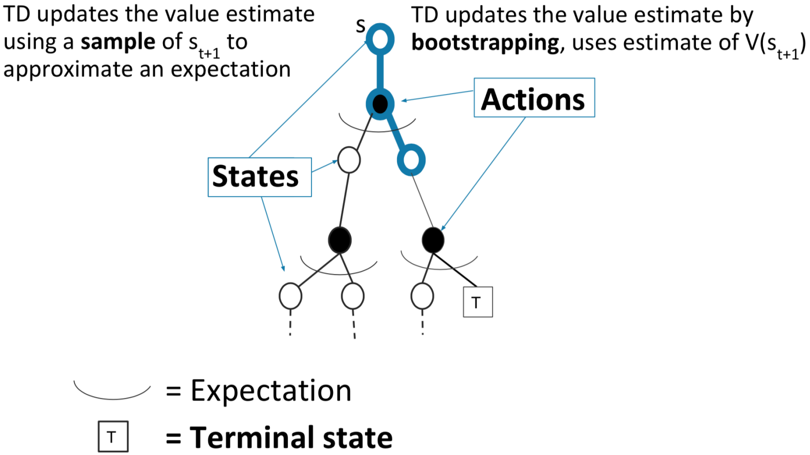
- Sample **tuple** (s_t, a_t, r_t, s_{t+1})
- $V^\pi(s_t) = V^\pi(s_t) + \alpha(\underbrace{[r_t + \gamma V^\pi(s_{t+1})]}_{\text{TD target}} - V^\pi(s_t))$

Example:

- Mars rover: R = [1 0 0 0 0 0 +10] for any action
- $\pi(s) = a_1$ $\forall s$, $\gamma = 1$. any action from s_1 and s_7 terminates episode
- Trajectory = $(s_3, a_1, 0, s_2, a_1, 0, s_2, a_1, 0, s_1, a_1, 1, \text{terminal})$
- First visit MC estimate of V of each state? [1 1 1 0 0 0 0]
- Every visit MC estimate of V of $s2$? 1
- TD estimate of all states (init at 0) with $\alpha = 1$?
  V = [1 0 0 0 0 0 0 0]

Temporal Difference Policy Evaluation

$$V^\pi(s_t) = V^\pi(s_t) + \alpha([r_t + \gamma V^\pi(s_{t+1})] - V^\pi(s_t))$$



This Lecture: Policy Evaluation

- Estimating the expected return of a particular policy if don't have access to true MDP models
- Dynamic programming
- Monte Carlo policy evaluation
    - Policy evaluation when don't have a model of how the world work
        - Given on-policy samples
        - Given off-policy samples
- Temporal Difference (TD)
- **Metrics to evaluate and compare algorithms**

Check Your Understanding: For Dynamic Programming, MC and TD Methods, Which Properties Hold?

- Usable when no models of current domain
- Handles continuing (non-episodic) domains
- Handles Non-Markovian domains
- Converges to true value in limit [1]
- Unbiased estimate of value

[1] For tabular representations of value function. More on this in later lectures

Some Important Properties to Evaluate Policy Evaluation Algorithms

- Usable when no models of current domain
  - DP: No   MC: Yes   TD: Yes
- Handles continuing (non-episodic) domains
  - DP: Yes   MC: No   TD: Yes
- Handles Non-Markovian domains
  - DP: No   MC: Yes   TD: No
- Converges to true value in limit [2]
  - DP: Yes   MC: Yes   TD: Yes
- Unbiased estimate of value
  - DP: NA   MC: Yes   TD: No

[2] For tabular representations of value function. More on this in later lectures

Some Important Properties to Evaluate Model-free Policy Evaluation Algorithms

- Bias/variance characteristics
- Data efficiency
- Computational efficiency

Bias/Variance of Model-free Policy Evaluation Algorithms

- Return G_t is an unbiased estimate of $V^\pi(s_t)$
- TD target $[r_t + \gamma V^\pi(s_{t+1})]$ is a biased estimate of $V^\pi(s_t)$
- But often much lower variance than a single return G_t
- Return function of multi-step sequence of random actions, states & rewards
- TD target only has one random action, reward and next state
- MC
  - Unbiased
  - High variance
  - Consistent (converges to true) even with function approximation
- TD
  - Some bias
  - Lower variance
  - TD(0) converges to true value with tabular representation
  - TD(0) does not always converge with function approximation

s_1	s_2	s_3	s_4	s_5	s_6	s_7
$R(s_1) = +1$ *Okay Field Site*	$R(s_2) = 0$	$R(s_3) = 0$	$R(s_4) = 0$ 	$R(s_5) = 0$	$R(s_6) = 0$	$R(s_7) = +10$ *Fantastic Field Site*

- Mars rover: R $= [\ 1\ 0\ 0\ 0\ 0\ 0\ +10]$ for any action
- $\pi(s) = a_1\ \forall s$, $\gamma = 1$. any action from s_1 and s_7 terminates episode
- Trajectory $= (s_3, a_1, 0, s_2, a_1, 0, s_2, a_1, 0, s_1, a_1, 1,$ terminal)
- First visit MC estimate of V of each state? $[1\ 1\ 1\ 0\ 0\ 0\ 0]$
- Every visit MC estimate of V of s_2? 1
- TD estimate of all states (init at 0) with $\alpha = 1$ is $[1\ 0\ 0\ 0\ 0\ 0\ 0]$
- TD(0) only uses a data point (s, a, r, s') once
- Monte Carlo takes entire return from s to end of episode

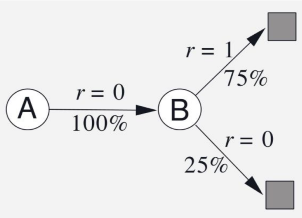
Batch MC and TD

- Batch (Offline) solution for finite dataset
  - Given set of K episodes
  - Repeatedly sample an episode from K
  - Apply MC or TD(0) to the sampled episode
- What do MC and TD(0) converge to?

AB Example: (Ex. 6.4, Sutton & Barto, 2018

- Two states A, B with $\gamma = 1$
- Given 8 episodes of experience:
  - $A, 0, B, 0$
  - $B, 1$ (observed 6 times)
  - $B, 0$
- What are $V(A), V(B)$?

AB Example: (Ex. 6.4, Sutton & Barto, 2018)



- Two states A, B with $\gamma = 1$
- Given 8 episodes of experience:
  - $A, 0, B, 0$
  - $B, 1$ (observed 6 times)
  - $B, 0$
- $V(B) = 0.75$ by TD or MC
- What about $V(A)$?

Batch MC and TD: Converges

- Monte Carlo in batch setting converges to min MSE (mean squared error)
  - Minimize loss with respect to observed returns
  - In AB example, $V(A) = 0$
- TD(0) converges to DP policy V^π for the MDP with the maximum likelihood model estimates
  - Maximum likelihood Markov decision process model

$$\hat{P}(s'|s,a) = \frac{1}{N(s,a)} \sum_{k=1}^{K} \sum_{t=1}^{L_k-1} \mathbb{1}(s_{k,t} = s, a_{k,t} = a, s_{k,t+1} = s')$$

$$\hat{r}(s,a) = \frac{1}{N(s,a)} \sum_{k=1}^{K} \sum_{t=1}^{L_k-1} \mathbb{1}(s_{k,t} = s, a_{k,t} = a) r_{t,k}$$

  - Compute V^π using this model
  - In AB example, $V(A) = 0.75$

Some Important Properties to Evaluate Model-free Policy Evaluation Algorithms

- Data efficiency & Computational efficiency
- In simplest TD, use (s, a, r, s') once to update $V(s)$
  - $O(1)$ operation per update
  - In an episode of length L, $O(L)$
- In MC have to wait till episode finishes, then also $O(L)$
- MC can be more data efficient than simple TD
- But TD exploits Markov structure
  - If in Markov domain, leveraging this is helpful

Alternative: Certainty Equivalence V^π MLE MDP Model Estimates

- Model-based option for policy evaluation without true models
- After each (s, a, r, s') tuple
  - Recompute maximum likelihood MDP model for (s, a)

$$\hat{P}(s'|s,a) = \frac{1}{N(s,a)} \sum_{k=1}^{K} \sum_{t=1}^{L_k - 1} \mathbb{1}(s_{k,t} = s, a_{k,t} = a, s_{k,t+1} = s')$$

$$\hat{r}(s,a) = \frac{1}{N(s,a)} \sum_{k=1}^{K} \sum_{t=1}^{L_k - 1} \mathbb{1}(s_{k,t} = s, a_{k,t} = a) r_{t,k}$$

  - Compute V^π using MLE MDP [3] (e.g. see method from lecture 2)

[3] Requires initializing for all (s, a) pairs

Alternative: Certainty Equivalence V^π MLE MDP Model Estimates

- Model-based option for policy evaluation without true models
- After each (s, a, r, s') tuple
  - Recompute maximum likelihood MDP model for (s, a)

$$\hat{P}(s'|s,a) = \frac{1}{N(s,a)} \sum_{k=1}^{K} \sum_{t=1}^{L_k - 1} 1(s_{k,t} = s, a_{k,t} = a, s_{k,t+1} = s')$$

$$\hat{r}(s,a) = \frac{1}{N(s,a)} \sum_{k=1}^{K} \sum_{t=1}^{L_k - 1} 1(s_{k,t} = s, a_{k,t} = a) r_{t,k}$$

  - Compute V^π using MLE MDP [4] (e.g. see method from lecture 2)
- Cost: Updating MLE model and MDP planning at each update ($O(|S|^3)$ for analytic matrix solution, $O(|S|^2|A|)$ for iterative methods)
- Very data efficient and very computationally expensive
- Consistent
- Can also easily be used for off-policy evaluation

s_1	s_2	s_3	s_4	s_5	s_6	s_7
$R(s_1) = +1$ *Okay Field Site*	$R(s_2) = 0$	$R(s_3) = 0$	$R(s_4) = 0$ 	$R(s_5) = 0$	$R(s_6) = 0$	$R(s_7) = +10$ *Fantastic Field Site*

- Mars rover: R = [1 0 0 0 0 0 +10] for any action
- $\pi(s) = a_1 \ \forall s$, $\gamma = 1$. any action from s_1 and s_7 terminates episode
- Trajectory = $(s_3, a_1, 0, s_2, a_1, 0, s_2, a_1, 0, s_1, a_1, 1, \text{terminal})$
- First visit MC estimate of V of each state? [1 1 1 0 0 0 0]
- Every visit MC estimate of V of s_2? 1
- TD estimate of all states (init at 0) with $\alpha = 1$ is [1 0 0 0 0 0 0]
- What is the certainty equivalent estimate?
  $\hat{r} = [1\ 0\ 0\ 0\ 0\ 0\ 0]$,
  $\hat{p}(terminate|s_1, a_1) = \hat{p}(s_1|s_2, a_1) = \hat{p}(s_2|s_3, a_1) = 1$,
  V = [1 1 1 0 0 0 0]

Some Important Properties to Evaluate Policy Evaluation Algorithms

- Robustness to Markov assumption
- Bias/variance characteristics
- Data efficiency
- Computational efficiency

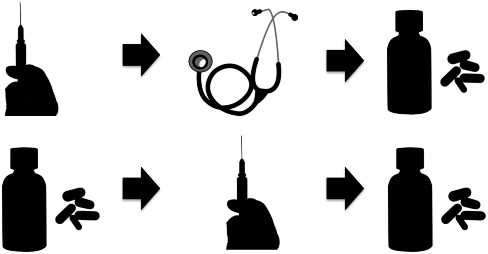
Summary: Policy Evaluation

- Dynamic Programming
- Monte Carlo policy evaluation
  - Policy evaluation when we don't have a model of how the world works
    - Given on policy samples
    - Given off policy samples
- Temporal Difference (TD)
- Metrics to evaluate and compare algorithms

This Lecture: Policy Evaluation

- Estimating the expected return of a particular policy if don't have access to true MDP models
- Dynamic programming
- Monte Carlo policy evaluation
  - Policy evaluation when don't have a model of how the world work
    - Given on-policy samples
    - **Given off-policy samples**
- Temporal Difference (TD)
- Metrics to evaluate and compare algorithms

MC Off Policy Evaluation



- Sometimes trying actions out is costly or high stakes
- Would like to use old data about policy decisions and their outcomes to estimate the potential value of an alternate policy

Monte Carlo (MC) Off Policy Evaluation

- Aim: estimate value of policy π_1, $V^{\pi_1}(s)$, given episodes generated under behavior policy π_2
  - $s_1, a_1, r_1, s_2, a_2, r_2, \ldots$ where the actions are sampled from π_2
- $G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots$ in MDP M under policy π
- $V^\pi(s) = \mathbb{E}_\pi[G_t | s_t = s]$
- Have data from a different policy, behavior policy π_2
- If π_2 is stochastic, can often use it to estimate the value of an alternate policy (formal conditions to follow)
- Again, no requirement that have a model nor that state is Markov

Monte Carlo (MC) Off Policy Evaluation: Distribution Mismatch

- Distribution of episodes & resulting returns differs between policies

Importance Sampling

- Goal: estimate the expected value of a function $f(x)$ under some probability distribution $p(x)$, $\mathbb{E}_{x \sim p}[f(x)]$
- Have data $x_1, x_2, \ldots, x_n$ sampled from distribution $q(s)$
- Under a few assumptions, we can use samples to obtain an unbiased estimate of $\mathbb{E}_{x \sim q}[f(x)]$

$$\mathbb{E}_{x \sim q}[f(x)] = \int_x q(x) f(x)$$

Importance Sampling (IS) for Policy Evaluation

- Let h_j be episode j (history) of states, actions and rewards

$$h_j = (s_{j,1}, a_{j,1}, r_{j,1}, s_{j,2}, a_{j,2}, r_{j,2}, \ldots, s_{j,L_j(terminal)})$$

Importance Sampling (IS) for Policy Evaluation

- Let h_j be episode j (history) of states, actions and rewards

$$h_j = (s_{j,1}, a_{j,1}, r_{j,1}, s_{j,2}, a_{j,2}, r_{j,2}, \ldots, s_{j,L_j(terminal)})$$

$$\begin{aligned} p(h_j|\pi, s = s_{j,1}) =& p(a_{j,1}|s_{j,1})p(r_{j,1}|s_{j,1}, a_{j,1})p(s_{j,2}|s_{j,1}, a_{j,1}) \\ & p(a_{j,2}|s_{j,2})p(r_{j,2}|s_{j,2}, a_{j,2})p(s_{j,3}|s_{j,2}, a_{j,2}) \ldots \\ =& \prod_{t=1}^{L_j-1} p(a_{j,t}|s_{j,t})p(r_{j,t}|s_{j,t}, a_{j,t})p(s_{j,t+1}|s_{j,t}, a_{j,t}) \\ =& \prod_{t=1}^{L_j-1} \pi(a_{j,t}|s_{j,t})p(r_{j,t}|s_{j,t}, a_{j,t})p(s_{j,t+1}|s_{j,t}, a_{j,t}) \end{aligned}$$

Importance Sampling (IS) for Policy Evaluation

- Let h_j be episode j (history) of states, actions and rewards, where the actions are sampled from π_2

$$h_j = (s_{j,1}, a_{j,1}, r_{j,1}, s_{j,2}, a_{j,2}, r_{j,2}, \ldots, s_{j,L_j(terminal)})$$

$$V^{\pi_1}(s) \approx \sum_{j=1}^{n} \frac{p(h_j|\pi_1, s)}{p(h_j|\pi_2, s)} G(h_j)$$

Importance Sampling for Policy Evaluation

- Aim: estimate $V^{\pi_1}(s)$ given episodes generated under policy π_2
  - $s_1, a_1, r_1, s_2, a_2, r_2, \ldots$ where the actions are sampled from π_2
- Have access to $G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots$ in MDP M under policy π_2
- Want $V^{\pi_1}(s) = \mathbb{E}_{\pi_1}[G_t | s_t = s]$
- IS = Monte Carlo estimate given off policy data
- Model-free method
- Does not require Markov assumption
- Under some assumptions, unbiased & consistent estimator of V^{π_1}
- Can be used when agent is interacting with environment to estimate value of policies different than agent's control policy
- More later this quarter about batch learning

Today's Plan

- Last Time:
  - Markov reward / decision processes
  - Policy evaluation & control when have true model (of how the world works)
- Today
  - Policy evaluation when don't have a model of how the world works
- **Next Time**:
  - **Control when don't have a model of how the world works**