Lecture 4: Model Free Control

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CS234 Reinforcement Learning.

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- Structure closely follows much of David Silver's Lecture 5. For additional reading please see SB Sections 5.2-5.4, 6.4, 6.5, 6.7

Refresh Your Knowledge 3. Piazza Poll

- Which of the following equations express a TD update?
  1. $V(s_t) = r(s_t, a_t) + \gamma \sum_{s'} p(s'|s_t, a_t) V(s')$
  2. $V(s_t) = (1-\alpha)V(s_t) + \alpha(r(s_t, a_t) + \gamma V(s_{t+1}))$
  3. $V(s_t) = (1-\alpha)V(s_t) + \alpha \sum_{i=t}^{H} r(s_i, a_i)$
  4. $V(s_t) = (1-\alpha)V(s_t) + \alpha \max_a (r(s_t, a) + \gamma V(s_{t+1}))$
  5. Not sure
- Bootstrapping is when
  1. Samples of (s,a,s') transitions are used to approximate the true expectation over next states
  2. An estimate of the next state value is used instead of the true next state value
  3. Used in Monte-Carlo policy evaluation
  4. Not sure

Refresh Your Knowledge 3. Piazza Poll

- Which of the following equations express a TD update?
  True. $V(s_t) = (1-\alpha)V(s_t) + \alpha(r(s_t, a_t) + \gamma V(s_{t+1}))$

- Bootstrapping is when
  An estimate of the next state value is used instead of the true next state value

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Class Structure

- Last time: Policy evaluation with no knowledge of how the world works (MDP model not given)
- This time: Control (making decisions) without a model of how the world works
- Next time: Generalization – Value function approximation

Evaluation to Control

- Last time: how good is a specific policy?
  - Given no access to the decision process model parameters
  - Instead have to estimate from data / experience
- Today: how can we learn a good policy?

Recall: Reinforcement Learning Involves

- Optimization
- Delayed consequences
- Exploration
- Generalization

Today: Learning to Control Involves

- Optimization: Goal is to identify a policy with high expected rewards (similar to Lecture 2 on computing an optimal policy given decision process models)
- Delayed consequences: May take many time steps to evaluate whether an earlier decision was good or not
- Exploration: Necessary to try different actions to learn what actions can lead to high rewards

Today: Model-free Control

- Generalized policy improvement
- Importance of exploration
- Monte Carlo control
- Model-free control with temporal difference (SARSA, Q-learning)
- Maximization bias

Model-free Control Examples

- Many applications can be modeled as a MDP: Backgammon, Go, Robot locomation, Helicopter flight, Robocup soccer, Autonomous driving, Customer ad selection, Invasive species management, Patient treatment
- For many of these and other problems either:
  - MDP model is unknown but can be sampled
  - MDP model is known but it is computationally infeasible to use directly, except through sampling

On and Off-Policy Learning

- On-policy learning
  - Direct experience
  - Learn to estimate and evaluate a policy from experience obtained from following that policy
- Off-policy learning
  - Learn to estimate and evaluate a policy using experience gathered from following a different policy

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Recall Policy Iteration

- Initialize policy π
- Repeat:
  - Policy evaluation: compute V^π
  - Policy improvement: update π

$$\pi'(s) = \arg\max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^\pi(s') = \arg\max_a Q^\pi(s,a)$$

- Now want to do the above two steps without access to the true dynamics and reward models
- Last lecture introduced methods for model-free policy evaluation

Model Free Policy Iteration

- Initialize policy π
- Repeat:
  - Policy evaluation: compute Q^π
  - Policy improvement: update π

MC for On Policy Q Evaluation

Initialize $N(s,a) = 0,\ G(s,a) = 0,\ Q^{\pi}(s,a) = 0,\ \forall s \in S,\ \forall a \in A$
Loop

- Using policy π sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$
- $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$
- For each **state,action** (s,a) visited in episode i
  - For **first or every** time t that (s,a) is visited in episode i
    - $N(s,a) = N(s,a) + 1,\ G(s,a) = G(s,a) + G_{i,t}$
    - Update estimate $Q^{\pi}(s,a) = G(s,a)/N(s,a)$

Model-free Generalized Policy Improvement

- Given an estimate $Q^{\pi_i}(s,a)\ \forall s,a$
- Update new policy

$$\pi_{i+1}(s) = \arg\max_a Q^{\pi_i}(s,a)$$

Model-free Policy Iteration

- Initialize policy π
- Repeat:
  - Policy evaluation: compute Q^π
  - Policy improvement: update π given Q^π

- May need to modify policy evaluation:
  - If π is deterministic, can't compute $Q(s, a)$ for any $a \neq \pi(s)$
- How to interleave policy evaluation and improvement?
  - Policy improvement is now using an estimated Q

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Policy Evaluation with Exploration

- Want to compute a model-free estimate of Q^π
- In general seems subtle
  - Need to try all (s, a) pairs but then follow π
  - Want to ensure resulting estimate Q^π is good enough so that policy improvement is a monotonic operator
- For certain classes of policies can ensure all (s,a) pairs are tried such that asymptotically Q^π converges to the true value

ϵ-greedy Policies

- Simple idea to balance exploration and exploitation
- Let $|A|$ be the number of actions
- Then an ϵ-greedy policy w.r.t. a state-action value $Q(s, a)$ is $\pi(a|s) =$

ϵ-greedy Policies

- Simple idea to balance exploration and exploitation
- Let $|A|$ be the number of actions
- Then an ϵ-greedy policy w.r.t. a state-action value $Q(s, a)$ is $\pi(a|s) =$
  - $\arg\max_a Q(s, a)$, w. prob $1 - \epsilon + \frac{\epsilon}{|A|}$
  - $a' \neq \arg\max Q(s, a)$ w. prob $\frac{\epsilon}{|A|}$

For Later Practice: MC for On Policy Q Evaluation

Initialize $N(s,a) = 0$, $G(s,a) = 0$, $Q^\pi(s,a) = 0$, $\forall s \in S$, $\forall a \in A$
Loop

- Using policy π sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$
- $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$
- For each **state,action** (s,a) visited in episode i
    - For **first or every** time t that (s,a) is visited in episode i
        - $N(s,a) = N(s,a) + 1$, $G(s,a) = G(s,a) + G_{i,t}$
        - Update estimate $Q^\pi(s,a) = G(s,a)/N(s,a)$

- Mars rover with new actions:
    - $r(-,a_1) = [\ 1\ 0\ 0\ 0\ 0\ 0\ +10]$, $r(-,a_2) = [\ 0\ 0\ 0\ 0\ 0\ 0\ +5]$, $\gamma = 1$.
- Assume current greedy $\pi(s) = a_1\ \forall s$, ϵ=.5
- Sample trajectory from ϵ-greedy policy
- Trajectory = (s_3, a_1, 0, s_2, a_2, 0, s_3, a_1, 0, s_2, a_2, 0, s_1, a_1, 1, terminal)
- First visit MC estimate of Q of each (s,a) pair?
- $Q^{\epsilon-\pi}(-,a_1) = [1\ 0\ 1\ 0\ 0\ 0\ 0]$, $Q^{\epsilon-\pi}(-,a_2) = [0\ 1\ 0\ 0\ 0\ 0\ 0]$

Monotonic ϵ-greedy Policy Improvement

Theorem

For any ϵ-greedy policy π_i, the ϵ-greedy policy w.r.t. Q^{π_i}, π_{i+1} is a monotonic improvement $V^{\pi_{i+1}} \geq V^{\pi_i}$

$$\begin{aligned} Q^{\pi_i}(s, \pi_{i+1}(s)) &= \sum_{a \in A} \pi_{i+1}(a|s) Q^{\pi_i}(s, a) \\ &= (\epsilon/|A|) \left[\sum_{a \in A} Q^{\pi_i}(s, a) \right] + (1 - \epsilon) \max_a Q^{\pi_i}(s, a) \end{aligned}$$

Monotonic ϵ-greedy Policy Improvement

Theorem

For any ϵ-greedy policy π_i, the ϵ-greedy policy w.r.t. Q^{π_i}, π_{i+1} is a monotonic improvement $V^{\pi_{i+1}} \geq V^{\pi_i}$

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\begin{aligned}
Q^{\pi_i}(s, \pi_{i+1}(s)) &= \sum_{a \in A} \pi_{i+1}(a|s) Q^{\pi_i}(s, a) \\
&= (\epsilon/|A|) \left[\sum_{a \in A} Q^{\pi_i}(s, a) \right] + (1-\epsilon) \max_a Q^{\pi_i}(s, a) \\
&= (\epsilon/|A|) \left[\sum_{a \in A} Q^{\pi_i}(s, a) \right] + (1-\epsilon) \max_a Q^{\pi_i}(s, a) \frac{1-\epsilon}{1-\epsilon} \\
&= (\epsilon/|A|) \left[\sum_{a \in A} Q^{\pi_i}(s, a) \right] + (1-\epsilon) \max_a Q^{\pi_i}(s, a) \sum_{a \in A} \frac{\pi_i(a|s) - \frac{\epsilon}{|A|}}{1-\epsilon} \\
&\geq \frac{\epsilon}{|A|} \left[\sum_{a \in A} Q^{\pi_i}(s, a) \right] + (1-\epsilon) \sum_{a \in A} \frac{\pi_i(a|s) - \frac{\epsilon}{|A|}}{1-\epsilon} Q^{\pi_i}(s, a) \\
&= \sum_{a \in A} \pi_i(a|s) Q^{\pi_i}(s, a) = V^{\pi_i}(s)
\end{aligned}
$$

- Therefore $V^{\pi_{i+1}} \geq V^{\pi}$ (from the policy improvement theorem)

Greedy in the Limit of Infinite Exploration (GLIE)

Definition of GLIE

- All state-action pairs are visited an infinite number of times

$$\lim_{i \to \infty} N_i(s, a) \to \infty$$

- Behavior policy (policy used to act in the world) converges to greedy policy
  $\lim_{i \to \infty} \pi(a|s) \to \arg\max_a Q(s, a)$ with probability 1

Greedy in the Limit of Infinite Exploration (GLIE)

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- Behavior policy (policy used to act in the world) converges to greedy policy
  $\lim_{i \to \infty} \pi(a|s) \to \arg\max_a Q(s, a)$ with probability 1

- A simple GLIE strategy is ϵ-greedy where ϵ is reduced to 0 with the following rate: $\epsilon_i = 1/i$

Monte Carlo Online Control / On Policy Improvement

Initialize $Q(s,a) = 0, N(s,a) = 0 \; \forall (s,a)$, Set $\epsilon = 1$, $k = 1$
$\pi_k = \epsilon$-greedy(Q) // Create initial ϵ-greedy policy
loop
  Sample k-th episode $(s_{k,1}, a_{k,1}, r_{k,1}, s_{k,2}, \ldots, s_{k,T})$ given π_k
  $G_{k,t} = r_{k,t} + \gamma r_{k,t+1} + \gamma^2 r_{k,t+2} + \cdots \gamma^{T_i - 1} r_{k,T_i}$
  for $t = 1, \ldots, T$ **do**
    if First visit to (s,a) in episode k **then**
      $N(s,a) = N(s,a) + 1$
      $Q(s_t, a_t) = Q(s_t, a_t) + \frac{1}{N(s,a)}(G_{k,t} - Q(s_t, a_t))$
    end if
  end for
  $k = k + 1$, $\epsilon = 1/k$
  $\pi_k = \epsilon$-greedy(Q) // Policy improvement
end loop

Poll. Check Your Understanding: MC for On Policy Control

- Mars rover with new actions:
  - $r(-, a_1) = [\ 1\ 0\ 0\ 0\ 0\ 0\ +10]$, $r(-, a_2) = [\ 0\ 0\ 0\ 0\ 0\ 0\ +5]$, $\gamma = 1$.
- Assume current greedy $\pi(s) = a_1\ \forall s$, ϵ=.5. $Q(s, a) = 0$ for all (s, a)
- Sample trajectory from ϵ-greedy policy
- Trajectory = (s_3, a_1, 0, s_2, a_2, 0, s_3, a_1, 0, s_2, a_2, 0, s_1, a_1, 1, terminal)
- First visit MC estimate of Q of each (s, a) pair?
- $Q^{\epsilon-\pi}(-, a_1) = [1\ 0\ 1\ 0\ 0\ 0\ 0]$

After this trajectory (Select all)

- $Q^{\epsilon-\pi}(-, a_2) = [0\ 0\ 0\ 0\ 0\ 0\ 0]$
- The new **greedy** policy would be: $\pi = [1$ tie 1 tie tie tie tie$]$
- The new **greedy** policy would be: $\pi = [1\ 2\ 1$ tie tie tie tie$]$
- If $\epsilon = 1/3$, the new $\pi(s_1) = a_1$ with prob $2/3$ else selects randomly.
- Not sure

Check Your Understanding: MC for On Policy Control

- Mars rover with new actions:
  - $r(-, a_1) = [\ 1\ 0\ 0\ 0\ 0\ 0\ +10]$, $r(-, a_2) = [\ 0\ 0\ 0\ 0\ 0\ 0\ +5]$, $\gamma = 1$.
- Assume current greedy $\pi(s) = a_1\ \forall s$, ϵ=.5
- Sample trajectory from ϵ-greedy policy
- Trajectory = (s_3, a_1, 0, s_2, a_2, 0, s_3, a_1, 0, s_2, a_2, 0, s_1, a_1, 1, terminal)
- First visit MC estimate of Q of each (s, a) pair?
- $Q^{\epsilon-\pi}(-, a_1) = [1\ 0\ 1\ 0\ 0\ 0\ 0]$, $Q^{\epsilon-\pi}(-, a_2) = [0\ 1\ 0\ 0\ 0\ 0\ 0]$
- What is $\pi(s) = \arg\max_a Q^{\epsilon-\pi}(s, a)\ \forall s$?
  π = [1 2 1 tie tie tie tie]
- What is new ϵ-greedy policy, if $k = 3$, $\epsilon = 1/k$
  With probability 2/3 choose $\pi(s)$ else choose randomly. As an example, $\pi(s_1) = a_1$ with prob (2/3) else randomly choose an action.

GLIE Monte-Carlo Control

Theorem

GLIE Monte-Carlo control converges to the optimal state-action value function $Q(s, a) \to Q^*(s, a)$

Model-free Policy Iteration

- Initialize policy π
- Repeat:
  - Policy evaluation: compute Q^{π}
  - Policy improvement: update π given Q^{π}

- What about TD methods?

Model-free Policy Iteration with TD Methods

- Use temporal difference methods for policy evaluation step
- Initialize policy π
- Repeat:
  - Policy evaluation: compute Q^{π} using temporal difference updating with ϵ-greedy policy
  - Policy improvement: Same as Monte carlo policy improvement, set π to ϵ-greedy (Q^{π})
- First consider SARSA, which is an on-policy algorithm.

General Form of SARSA Algorithm

Set initial ϵ-greedy policy π randomly, $t = 0$, initial state $s_t = s_0$
Take $a_t \sim \pi(s_t)$
Observe (r_t, s_{t+1})
loop
  Take action $a_{t+1} \sim \pi(s_{t+1})$ // Sample action from policy
  Observe (r_{t+1}, s_{t+2})
  Update Q given $(s_t, a_t, r_t, s_{t+1}, a_{t+1})$:

  Perform policy improvement:

  $t = t + 1$
end loop

General Form of SARSA Algorithm

Set initial ϵ-greedy policy π, $t = 0$, initial state $s_t = s_0$
Take $a_t \sim \pi(s_t)$ // Sample action from policy
Observe (r_t, s_{t+1})
loop
    Take action $a_{t+1} \sim \pi(s_{t+1})$
    Observe (r_{t+1}, s_{t+2})
    $Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(r_t + \gamma Q(s_{t+1}, a_{t+1}) - Q(s_t, a_t))$
    $\pi(s_t) = \arg\max_a Q(s_t, a)$ w.prob $1 - \epsilon$, else random
    $t = t + 1$
end loop

Worked Example: SARSA for Mars Rover

Set initial ϵ-greedy policy π, $t = 0$, initial state $s_t = s_0$
Take $a_t \sim \pi(s_t)$ // Sample action from policy
Observe (r_t, s_{t+1})
loop
  Take action $a_{t+1} \sim \pi(s_{t+1})$
  Observe (r_{t+1}, s_{t+2})
  $Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(r_t + \gamma Q(s_{t+1}, a_{t+1}) - Q(s_t, a_t))$
  $\pi(s_t) = \arg\max_a Q(s_t, a)$ w.prob $1 - \epsilon$, else random
  $t = t + 1$
end loop

- Initialize $\epsilon = 1/k$, $k = 1$, and $\alpha = 0.5$, $Q(-, a_1) = [\ 1\ 0\ 0\ 0\ 0\ 0\ +10]$, $Q(-, a_2) = [\ 1\ 0\ 0\ 0\ 0\ 0\ +5]$, $\gamma = 1$
- Assume starting state is s_6 and sample a_1

Worked Example: SARSA for Mars Rover

Set initial ϵ-greedy policy π, $t = 0$, initial state $s_t = s_0$
Take $a_t \sim \pi(s_t)$ // Sample action from policy
Observe (r_t, s_{t+1})
loop
    Take action $a_{t+1} \sim \pi(s_{t+1})$
    Observe (r_{t+1}, s_{t+2})
    $Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(r_t + \gamma Q(s_{t+1}, a_{t+1}) - Q(s_t, a_t))$
    $\pi(s_t) = \arg\max_a Q(s_t, a)$ w.prob $1 - \epsilon$, else random
    $t = t + 1$
end loop

- Initialize $\epsilon = 1/k$, $k = 1$, and $\alpha = 0.5$, $Q(-, a_1) = [\ 1\ 0\ 0\ 0\ 0\ 0\ +10]$, $Q(-, a_2) =[\ 1\ 0\ 0\ 0\ 0\ 0\ +5]$, $\gamma = 1$
- Tuple: $(s_6, a_1, 0, s_7, a_2, 5, s_7)$.
- $Q(s_6, a_1) = .5 * 0 + .5 * (0 + \gamma Q(s_7, a_2)) = 2.5$

SARSA Initialization

- Mars rover with new actions:
  - $r(-, a_1) = [\ 1\ 0\ 0\ 0\ 0\ 0\ +10]$, $r(-, a_2) = [\ 0\ 0\ 0\ 0\ 0\ 0\ +5]$, $\gamma = 1$.
- Initialize $\epsilon = 1/k$, $k = 1$, and $\alpha = 0.5$, $Q(-, a_1) = r(-, a_1)$, $Q(-, a_2) = r(-, a_2)$
- SARSA: $(s_6, a_1, 0, s_7, a_2, 5, s_7)$.
- Does how Q is initialized matter (initially? asymptotically?)? Asymptotically no, under mild condiditions, but at the beginning, yes

Convergence Properties of SARSA

Theorem

SARSA for finite-state and finite-action MDPs converges to the optimal action-value, $Q(s,a) \to Q^*(s,a)$, under the following conditions:

1. The policy sequence $\pi_t(a|s)$ satisfies the condition of GLIE
2. The step-sizes α_t satisfy the Robbins-Munro sequence such that

$$\sum_{t=1}^{\infty} \alpha_t = \infty$$
$$\sum_{t=1}^{\infty} \alpha_t^2 < \infty$$

- For ex. $\alpha_t = \frac{1}{T}$ satisfies the above condition.
- Would one want to use a step size choice that satisfies the above in practice? Likely not.

Q-Learning: Learning the Optimal State-Action Value

- SARSA is an **on-policy** learning algorithm
- SARSA estimates the value of the current **behavior** policy (policy using to take actions in the world)
- And then updates the policy trying to estimate
- Alternatively, can we directly estimate the value of π^* while acting with another behavior policy π_b?
- Yes! Q-learning, an **off-policy** RL algorithm
- Key idea: Maintain state-action Q estimates and use to bootstrap– use the value of the best future action
- Recall SARSA

$$Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha((r_t + \gamma Q(s_{t+1}, a_{t+1})) - Q(s_t, a_t))$$

- Q-learning:

$$Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha((r_t + \gamma \max_{a'} Q(s_{t+1}, a')) - Q(s_t, a_t))$$

Q-Learning with ϵ-greedy Exploration

Initialize $Q(s, a), \forall s \in S, a \in A$ $t = 0$, initial state $s_t = s_0$
Set π_b to be ϵ-greedy w.r.t. Q
loop
    Take $a_t \sim \pi_b(s_t)$ // Sample action from policy
    Observe (r_t, s_{t+1})
    $Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(r_t + \gamma \max_a Q(s_{t+1}, a) - Q(s_t, a_t))$
    $\pi(s_t) = \arg\max_a Q(s_t, a)$ w.prob $1 - \epsilon$, else random
    $t = t + 1$
end loop

Worked Example: ϵ-greedy Q-Learning Mars

Initialize $Q(s,a), \forall s \in S, a \in A$ $t = 0$, initial state $s_t = s_0$
Set π_b to be ϵ-greedy w.r.t. Q
loop
    Take $a_t \sim \pi_b(s_t)$ // Sample action from policy
    Observe (r_t, s_{t+1})
    $Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(r_t + \gamma \max_a Q(s_{t+1}, a) - Q(s_t, a_t))$
    $\pi(s_t) = \arg\max_a Q(s_t, a)$ w.prob $1 - \epsilon$, else random
    $t = t + 1$
end loop

- Initialize $\epsilon = 1/k$, $k = 1$, and $\alpha = 0.5$, $Q(-, a_1) = [\ 1\ 0\ 0\ 0\ 0\ 0\ +10]$, $Q(-, a_2) = [\ 1\ 0\ 0\ 0\ 0\ 0\ +5]$, $\gamma = 1$
- Like in SARSA example, start in s_6 and take a_1.

Worked Example: ϵ-greedy Q-Learning Mars

Initialize $Q(s, a), \forall s \in S, a \in A$ $t = 0$, initial state $s_t = s_0$
Set π_b to be ϵ-greedy w.r.t. Q
loop
  Take $a_t \sim \pi_b(s_t)$ // Sample action from policy
  Observe (r_t, s_{t+1})
  $Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(r_t + \gamma \max_a Q(s_{t+1}, a) - Q(s_t, a_t))$
  $\pi(s_t) = \arg\max_a Q(s_t, a)$ w.prob $1 - \epsilon$, else random
  $t = t + 1$
end loop

- Initialize $\epsilon = 1/k$, $k = 1$, and $\alpha = 0.5$, $Q(-, a_1) = [\ 1\ 0\ 0\ 0\ 0\ 0\ +10]$, $Q(-, a_2) = [\ 1\ 0\ 0\ 0\ 0\ 0\ +5]$, $\gamma = 1$
- Tuple: $(s_6, a_1, 0, s_7)$.
- $Q(s_6, a_1) = 0 + .5 * (0 + \gamma \max_{a'} Q(s_7, a') - 0) = .5*10 = 5$
- Recall that in the SARSA update we saw $Q(s_6, a_1) = 2.5$ because we used the actual action taken at s_7 instead of the max
- Does how Q is initialized matter (initially? asymptotically?)? Asymptotically no, under mild condiditions, but at the beginning, yes

Check Your Understanding: SARSA and Q-Learning

- SARSA: $Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(r_t + \gamma Q(s_{t+1}, a_{t+1}) - Q(s_t, a_t))$
- Q-Learning:
  $Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(r_t + \gamma \max_{a'} Q(s_{t+1}, a') - Q(s_t, a_t))$

Select all that are true

1. Both SARSA and Q-learning may update their policy after every step
2. If $\epsilon = 0$ for all time steps, and Q is initialized randomly, a SARSA Q state update will be the same as a Q-learning Q state update
3. Not sure

Q-Learning with ϵ-greedy Exploration

- What conditions are sufficient to ensure that Q-learning with ϵ-greedy exploration converges to optimal Q^*?
  Visit all (s, a) pairs infinitely often, and the step-sizes α_t satisfy the Robbins-Munro sequence. Note: the algorithm does not have to be greedy in the limit of infinite exploration (GLIE) to satisfy this (could keep ϵ large).
- What conditions are sufficient to ensure that Q-learning with ϵ-greedy exploration converges to optimal π^*?
  The algorithm is GLIE, along with the above requirement to ensure the Q value estimates converge to the optimal Q.

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- What conditions are sufficient to ensure that Q-learning with ϵ-greedy exploration converges to optimal π^*?
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Maximization Bias[1]

- Consider single-state MDP ($|S| = 1$) with 2 actions, and both actions have 0-mean **random** rewards, ($\mathbb{E}(r|a = a_1) = \mathbb{E}(r|a = a_2) = 0$).
- Then $Q(s, a_1) = Q(s, a_2) = 0 = V(s)$
- Assume there are prior samples of taking action a_1 and a_2
- Let $\hat{Q}(s, a_1), \hat{Q}(s, a_2)$ be the finite sample estimate of Q
- Use an unbiased estimator for Q: e.g. $\hat{Q}(s, a_1) = \frac{1}{n(s,a_1)} \sum_{i=1}^{n(s,a_1)} r_i(s, a_1)$
- Let $\hat{\pi} = \arg\max_a \hat{Q}(s, a)$ be the greedy policy w.r.t. the estimated $\hat{Q}$

[1] Example from Mannor, Simester, Sun and Tsitsiklis. Bias and Variance Approximation in Value Function Estimates. Management Science 2007

Maximization Bias[2] Proof

- Consider single-state MDP ($|S| = 1$) with 2 actions, and both actions have 0-mean random rewards, ($\mathbb{E}(r|a = a_1) = \mathbb{E}(r|a = a_2) = 0$).
- Then $Q(s, a_1) = Q(s, a_2) = 0 = V(s)$
- Assume there are prior samples of taking action a_1 and a_2
- Let $\hat{Q}(s, a_1), \hat{Q}(s, a_2)$ be the finite sample estimate of Q
- Use an unbiased estimator for Q: e.g. $\hat{Q}(s, a_1) = \frac{1}{n(s,a_1)} \sum_{i=1}^{n(s,a_1)} r_i(s, a_1)$
- Let $\hat{\pi} = \arg\max_a \hat{Q}(s, a)$ be the greedy policy w.r.t. the estimated $\hat{Q}$
- *Even though each estimate of the state-action values is unbiased*, the estimate of $\hat{\pi}$'s value $\hat{V}^{\hat{\pi}}$ can be biased:
  $\hat{V}^{\hat{\pi}}(s) = \mathbb{E}[\max \hat{Q}(s, a_1), \hat{Q}(s, a_2)]$
  $\geq \max[\mathbb{E}[\hat{Q}(s, a_1)], [\hat{Q}(s, a_2)]]$
  $= max[0, 0] = V^{\pi}$,
  where the inequality comes from Jensen's inequality.

[2] Example from Mannor, Simester, Sun and Tsitsiklis. Bias and Variance Approximation in Value Function Estimates. Management Science 2007

Double Q-Learning

- The greedy policy w.r.t. estimated Q values can yield a maximization bias during finite-sample learning
- Avoid using max of estimates as estimate of max of true values
- Instead split samples and use to create two independent unbiased estimates of $Q_1(s_1, a_i)$ and $Q_2(s_1, a_i)$ $\forall a$.
  - Use one estimate to select max action: $a^* = \arg\max_a Q_1(s_1, a)$
  - Use other estimate to estimate value of a^*: $Q_2(s, a^*)$
  - Yields unbiased estimate: $\mathbb{E}(Q_2(s, a^*)) = Q(s, a^*)$
- Why does this yield an unbiased estimate of the max state-action value?

  Using independent samples to estimate the value
- If acting online, can alternate samples used to update Q_1 and Q_2, using the other to select the action chosen
- Next slides extend to full MDP case (with more than 1 state)

Double Q-Learning

```
Initialize Q_1(s,a) and Q_2(s,a), ∀s ∈ S, a ∈ A t = 0, initial state s_t = s_0
loop
    Select a_t using ε-greedy π(s) = arg max_a Q_1(s_t,a) + Q_2(s_t,a)
    Observe (r_t, s_{t+1})
    if (with 0.5 probability) then
        Q_1(s_t,a_t) ← Q_1(s_t,a_t) + α(r_t + γ max_a Q_2(s_{t+1},a) − Q_1(s_t,a_t))
    else
        Q_2(s_t,a_t) ← Q_2(s_t,a_t) + α(r_t + γ max_a Q_1(s_{t+1},a) − Q_2(s_t,a_t))
    end if
    t = t + 1
end loop
```

Compared to Q-learning, how does this change the: memory requirements, computation requirements per step, amount of data required?

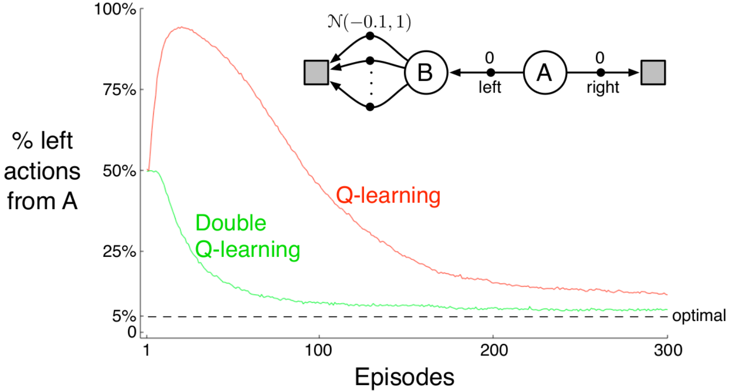
Double Q-Learning

Initialize $Q_1(s,a)$ and $Q_2(s,a)$,$\forall s \in S, a \in A$ $t=0$, initial state $s_t = s_0$
loop
    Select a_t using ϵ-greedy $\pi(s) = \arg\max_a Q_1(s_t,a) + Q_2(s_t,a)$
    Observe (r_t, s_{t+1})
    if (with 0.5 probability) **then**
        $Q_1(s_t,a_t) \leftarrow Q_1(s_t,a_t) + \alpha(r_t + \gamma \max_a Q_2(s_{t+1},a) - Q_1(s_t,a_t))$
    else
        $Q_2(s_t,a_t) \leftarrow Q_2(s_t,a_t) + \alpha(r_t + \gamma \max_a Q_1(s_{t+1},a) - Q_2(s_t,a_t))$
    end if
    $t = t+1$
end loop

Compared to Q-learning, how does this change the: memory requirements, computation requirements per step, amount of data required?

Doubles the memory, same computation requirements, data requirements are subtle– might reduce amount of exploration needed due to lower bias

Double Q-Learning (Figure 6.7 in Sutton and Barto 2018)



Due to the maximization bias, Q-learning spends much more time selecting suboptimal actions than double Q-learning.

What You Should Know

- Be able to implement MC on policy control and SARSA and Q-learning
- Compare them according to properties of how quickly they update, (informally) bias and variance, computational cost
- Define conditions for these algorithms to converge to the optimal Q and optimal π and give at least one way to guarantee such conditions are met.

Class Structure

- Last time: Policy evaluation with no knowledge of how the world works (MDP model not given)
- This time: Control (making decisions) without a model of how the world works
- **Next time: Generalization – Value function approximation**

Q-Learning with ϵ-greedy Exploration

Initialize $Q(s, a), \forall s \in S, a \in A$ $t = 0$, initial state $s_t = s_0$
Set π_b to be ϵ-greedy w.r.t. Q
loop
    Take $a_t \sim \pi_b(s_t)$ // Sample action from policy
    Observe (r_t, s_{t+1})
    Update Q given (s_t, a_t, r_t, s_{t+1}):

    Perform policy improvement: set π_b to be ϵ-greedy w.r.t. Q
    $t = t + 1$
end loop

Worked Example: SARSA and Q-learning

- Mars rover with new actions:
  - $r(-, a_1) = [\ 1\ 0\ 0\ 0\ 0\ 0\ +10]$, $r(-, a_2) = [\ 0\ 0\ 0\ 0\ 0\ 0\ +5]$, $\gamma = 1$.
- Assume current greedy $\pi(s) = a_1\ \forall s$, ϵ=.5,
- Sample trajectory from ϵ-greedy policy
- Trajectory = $(s_3, a_1, 0, s_2, a_2, 0, s_3, a_1, 0, s_2, a_2, 0, s_1, a_1, 1, \text{terminal})$
- New ϵ-greedy policy under MC, if $k = 3$, $\epsilon = 1/k$: with probability $2/3$ choose $\pi = [1\ 2\ 1\ \text{tie tie tie tie}]$, else choose randomly
- Q-learning updates? Initialize $\epsilon = 1/k$, $k = 1$, and $\alpha = 0.5$
- π is random with probability ϵ, else $\pi = [\ 1\ 1\ 1\ 2\ 1\ 2\ 1]$
- First tuple: $(s_3, a_1, 0, s_2)$.
- Do a Q-learning update and compute new ϵ-greedy π for (s_3, a_1):
  $Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(r_t + \gamma \max_a Q(s_{t+1}, a) - Q(s_t, a_t))$
  Update $Q(s_3, a_1) = 0$. $k = 2$

  New policy is random with probability $1/k$ else
  $\pi(s) = \arg\max Q(s_3, a) =$ tie between actions 1 and 2.

Check Your Understanding: SARSA and Updating

Set initial ϵ-greedy policy π, $t = 0$, initial state $s_t = s_0$
Take $a_t \sim \pi(s_t)$ // Sample action from policy
Observe (r_t, s_{t+1})
loop
  Take action $a_{t+1} \sim \pi(s_{t+1})$
  Observe (r_{t+1}, s_{t+2})
  $Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(r_t + \gamma Q(s_{t+1}, a_{t+1}) - Q(s_t, a_t))$
  $\pi(s_t) = \arg\max_a Q(s_t, a)$ w.prob $1 - \epsilon$, else random
  $t = t + 1$
end loop

Select all that are true

- SARSA may update its policy after every step
- It is best to update the policy after each step
- It is best to update Q at each step but sometimes better to update the policy less frequently
- Not sure

Off-Policy Control Using TD Methods

- In policy evaluation assume there was a behavior π_b used to act
- π_b determines the actual rewards received
- Now consider how to improve the behavior policy (policy improvement)
- Let behavior policy π_b be ϵ-greedy with respect to (w.r.t.) current estimate of the optimal $Q(s, a)$