Lecture 5: Value Function Approximation

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The value function approximation structure for today closely follows much of David Silver's Lecture 6.

Refresh Your Knowledge 4

- The basic idea of TD methods are to make state-next state pairs fit the constraints of the Bellman equation on average (question by: Phil Thomas)
  1. True
  2. False
  3. Not sure
- In tabular MDPs, if using a decision poicy that visits all states an infinite number of times, and in each state randomly selects an action, then (select all)
  1. Q-learning will converge to the optimal Q-values
  2. SARSA will converge to the optimal Q-values
  3. Q-learning is learning off-policy
  4. SARSA is learning off-policy
  5. Not sure
- A TD error > 0 can occur even if the current $V(s)$ is correct $\forall s$: [select all]
  1. False
  2. True if the MDP has stochastic state transitions
  3. True if the MDP has deterministic state transitions
  4. True if $\alpha > 0$
  5. Not sure

rewards assumed determin.

TD error : $\alpha [r + \gamma V(s') - V(s)]$

$r + \gamma \sum p(s'|s,a) V(s')$

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Class Structure

- Last time: Control (making decisions) without a model of how the world works
- **This time: Finishing up maximization bias and Value function approximation**
- Next time: Deep reinforcement learning

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Maximization Bias[1]

- Consider single-state MDP ($|S| = 1$) with 2 actions, and both actions have 0-mean **random** rewards, ($\mathbb{E}(r|a = a_1) = \mathbb{E}(r|a = a_2) = 0$).
- Then $Q(s, a_1) = Q(s, a_2) = 0 = V(s)$
- Assume there are prior samples of taking action a_1 and a_2
- Let $\hat{Q}(s, a_1), \hat{Q}(s, a_2)$ be the finite sample estimate of Q
- Use an unbiased estimator for Q: e.g. $\hat{Q}(s, a_1) = \frac{1}{n(s,a_1)} \sum_{i=1}^{n(s,a_1)} r_i(s, a_1)$
- Let $\hat{\pi} = \arg\max_a \hat{Q}(s, a)$ be the greedy policy w.r.t. the estimated $\hat{Q}$

[1] Example from Mannor, Simester, Sun and Tsitsiklis. Bias and Variance Approximation in Value Function Estimates. Management Science 2007

Maximization Bias[2] Proof

- Consider single-state MDP ($|S| = 1$) with 2 actions, and both actions have 0-mean random rewards, ($\mathbb{E}(r|a = a_1) = \mathbb{E}(r|a = a_2) = 0$).
- Then $Q(s, a_1) = Q(s, a_2) = 0 = V(s)$
- Assume there are prior samples of taking action a_1 and a_2
- Let $\hat{Q}(s, a_1), \hat{Q}(s, a_2)$ be the finite sample estimate of Q
- Use an unbiased estimator for Q: e.g. $\hat{Q}(s, a_1) = \frac{1}{n(s,a_1)} \sum_{i=1}^{n(s,a_1)} r_i(s, a_1)$
- Let $\hat{\pi} = \arg\max_a \hat{Q}(s, a)$ be the greedy policy w.r.t. the estimated $\hat{Q}$
- *Even though each estimate of the state-action values is unbiased*, the estimate of $\hat{\pi}$'s value $\hat{V}^{\hat{\pi}}$ can be biased:

[2] Example from Mannor, Simester, Sun and Tsitsiklis. Bias and Variance Approximation in Value Function Estimates. Management Science 2007

Double Q-Learning

- The greedy policy w.r.t. estimated Q values can yield a maximization bias during finite-sample learning
- Avoid using max of estimates as estimate of max of true values
- Instead split samples and use to create two independent unbiased estimates of $Q_1(s_1, a_i)$ and $Q_2(s_1, a_i)$ $\forall a$.
  - Use one estimate to select max action: $a^* = \arg\max_a Q_1(s_1, a)$
  - Use other estimate to estimate value of a^*: $Q_2(s, a^*)$
  - Yields unbiased estimate: $\mathbb{E}(Q_2(s, a^*)) = Q(s, a^*)$
- Why does this yield an unbiased estimate of the max state-action value?
- If acting online, can alternate samples used to update Q_1 and Q_2, using the other to select the action chosen
- Next slides extend to full MDP case (with more than 1 state)

Double Q-Learning

```
Initialize Q_1(s,a) and Q_2(s,a), ∀s ∈ S, a ∈ A t = 0, initial state s_t = s_0
loop
    Select a_t using ε-greedy π(s) = arg max_a Q_1(s_t, a) + Q_2(s_t, a)
    Observe (r_t, s_{t+1})
    if (with 0.5 probability) then
        Q_1(s_t, a_t) ← Q_1(s_t, a_t) + α(r_t + γ max_a Q_2(s_{t+1}, a) − Q_1(s_t, a_t))
    else
        Q_2(s_t, a_t) ← Q_2(s_t, a_t) + α(r_t + γ max_a Q_1(s_{t+1}, a) − Q_2(s_t, a_t))
    end if
    t = t + 1
end loop
```

Compared to Q-learning, how does this change the: memory requirements, 2x
computation requirements per step, amount of data required?

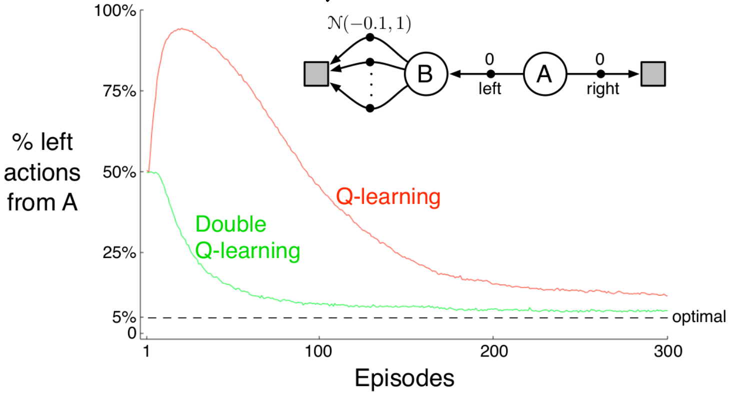
Double Q-Learning

Initialize $Q_1(s, a)$ and $Q_2(s, a)$,$\forall s \in S, a \in A$ $t = 0$, initial state $s_t = s_0$
loop
  Select a_t using ϵ-greedy $\pi(s) = \arg\max_a Q_1(s_t, a) + Q_2(s_t, a)$
  Observe (r_t, s_{t+1})
  if (with 0.5 probability) **then**
    $Q_1(s_t, a_t) \leftarrow Q_1(s_t, a_t) + \alpha(r_t + \gamma \max_a Q_2(s_{t+1}, a) - Q_1(s_t, a_t))$
  else
    $Q_2(s_t, a_t) \leftarrow Q_2(s_t, a_t) + \alpha(r_t + \gamma \max_a Q_1(s_{t+1}, a) - Q_2(s_t, a_t))$
  end if
  $t = t + 1$
end loop

Compared to Q-learning, how does this change the: memory requirements, computation requirements per step, amount of data required?

Double Q-Learning (Figure 6.7 in Sutton and Barto 2018)

Gaussian μ variance



Due to the maximization bias, Q-learning spends much more time selecting suboptimal actions than double Q-learning.

Finishing Up Last Time: Model-Free Control

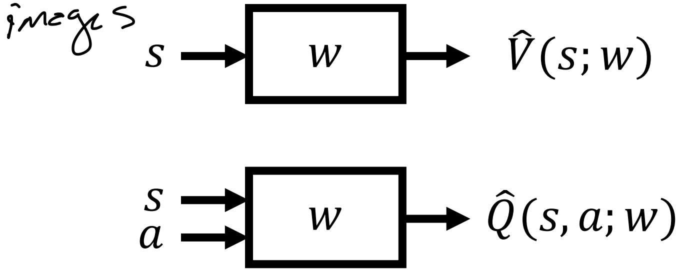
- Last time: how to learn a good policy from experience
- So far, have been assuming we can represent the value function or state-action value function as a vector/ matrix
  - Tabular representation
- Many real world problems have enormous state and/or action spaces
- Tabular representation is insufficient

Today: Focus on Generalization

- Optimization
- Delayed consequences
- Exploration
- **Generalization**

Value Function Approximation (VFA)

- Represent a (state-action/state) value function with a parameterized function instead of a table



- For finite action spaces, often represent the Q function as a vector: takes s as input and outputs a vector with one value for each action $[Q(s, a_1)Q(s, a_2)\ldots]$.

Motivation for VFA

- Don't want to have to explicitly store or learn for every single state a
  - Dynamics or reward model
  - Value
  - State-action value
  - Policy
- Want more compact representation that generalizes across state or states and actions
- When is this possible / a reasonable thing to hope for?

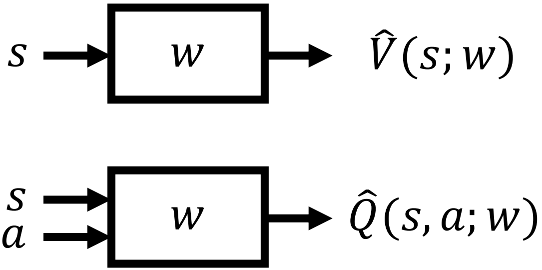
smoothness
structure

Benefits of Generalization

- Reduce memory needed to store $(P, R)/V/Q/\pi$
- Reduce computation needed to compute $(P, R)/V/Q/\pi$
- Reduce experience needed to find a good $P, R/V/Q/\pi$

Value Function Approximation (VFA)

- Represent a (state-action/state) value function with a parameterized function instead of a table



- **Which function approximator?**

Function Approximators

- Many possible function approximators including
  - Linear combinations of features
  - Neural networks
  - Decision trees
  - Nearest neighbors
  - Fourier/ wavelet bases
- In this class we will focus on function approximators that are differentiable (Why?)
- Two very popular classes of differentiable function approximators
  - Linear feature representations (Today)
  - Neural networks (Next lecture)

Review: Gradient Descent

- Consider a function $J(\boldsymbol{w})$ that is a differentiable function of a parameter vector $\boldsymbol{w}$
- Goal is to find parameter $\boldsymbol{w}$ that minimizes J
- The gradient of $J(\boldsymbol{w})$ is

$$\nabla J(w) = \left[\frac{\partial J}{\partial w_1} \quad \cdots \quad \frac{\partial J}{\partial w_n} \right]$$

$$w = w - \Delta w$$

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Value Function Approximation for Policy Evaluation with an Oracle

- First assume we could query any state s and an oracle would return the true value for $V^\pi(s)$
- The objective was to find the best approximate representation of V^π given a particular parameterized function

Stochastic Gradient Descent

- Goal: Find the parameter vector $\boldsymbol{w}$ that minimizes the loss between a true value function $V^\pi(s)$ and its approximation $\hat{V}(s;\boldsymbol{w})$ as represented with a particular function class parameterized by $\boldsymbol{w}$.
- Generally use mean squared error and define the loss as

$$J(\boldsymbol{w}) = \mathbb{E}_{\underset{\curvearrowleft s \sim \pi}{\pi}}[(\underbrace{V^\pi(s)}_{\text{true } V} - \hat{V}(s;\boldsymbol{w}))^2]$$

- Can use gradient descent to find a local minimum

$$\Delta \boldsymbol{w} = -\frac{1}{2}\alpha \nabla_{\boldsymbol{w}} J(\boldsymbol{w}) = -\alpha (V^\pi - \hat{V}) \nabla_w \hat{V}(s,w)$$

- Stochastic gradient descent (SGD) uses a finite number of (often one) samples to compute an approximate gradient:

$$\nabla_w J(w) = \nabla_w E_\pi [V^\pi(s) - \hat{V}(s,w)]^2$$
$$= E_\pi [2(V^\pi(s) - \hat{V}(s;w))] \nabla_w \hat{V}(s,w)$$

- Expected SGD is the same as the full gradient update

Model Free VFA Policy Evaluation

MC TD

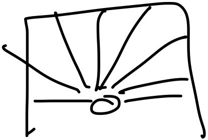
- Don't actually have access to an oracle to tell true $V^{\pi}(s)$ for any state s
- Now consider how to do model-free value function approximation for prediction / evaluation / policy evaluation without a model

Model Free VFA Prediction / Policy Evaluation

- Recall model-free policy evaluation (Lecture 3)
  - Following a fixed policy π (or had access to prior data)
  - Goal is to estimate V^π and/or Q^π
- Maintained a lookup table to store estimates V^π and/or Q^π
- Updated these estimates after each episode (Monte Carlo methods) or after each step (TD methods)
- **Now: in value function approximation, change the estimate update step to include fitting the function approximator**

Feature Vectors

- Use a feature vector to represent a state s



$$\mathbf{x}(s) = \begin{pmatrix} x_1(s) \\ x_2(s) \\ \cdots \\ x_n(s) \end{pmatrix}$$

Linear Value Function Approximation for Prediction With An Oracle

- Represent a value function (or state-action value function) for a particular policy with a weighted linear combination of features

$$\hat{V}(s; \mathbf{w}) = \sum_{j=1}^{n} x_j(s) w_j = \mathbf{x}(s)^T \mathbf{w}$$

$$\nabla_\omega \hat{V}(s;\omega) = x(s)$$

- Objective function is

$$J(\mathbf{w}) = \mathbb{E}_\pi[(V^\pi(s) - \hat{V}(s; \mathbf{w}))^2]$$

- Recall weight update is

$$\nabla J \quad (V^\pi(s) - \hat{V}(s;\omega)) \nabla_\omega \hat{V}(s;\omega)$$

$$\Delta \mathbf{w} = -\frac{1}{2} \alpha \nabla_{\mathbf{w}} J(\mathbf{w})$$

- Update is: $\Delta \omega = -\frac{1}{2} \alpha \, (V^\pi(s) - x(s)^T \omega) \, \vec{x}(s)$

- Update = - step-size $\times$ prediction error $\times$ feature value

Monte Carlo Value Function Approximation

episodic

- Return G_t is an unbiased but noisy sample of the true expected return $V^\pi(s_t)$
- Therefore can reduce MC VFA to doing supervised learning on a set of (state,return) pairs: $\langle s_1, G_1\rangle, \langle s_2, G_2\rangle, \ldots, \langle s_T, G_T\rangle$
  - Substitute G_t for the true $V^\pi(s_t)$ when fit function approximator
- Concretely when using linear VFA for policy evaluation

$V^\pi(s)$

$$\begin{aligned}
\Delta \mathbf{w} &= \alpha(G_t - \hat{V}(s_t; \mathbf{w}))\nabla_{\mathbf{w}} \hat{V}(s_t; \mathbf{w}) \\
&= \alpha(G_t - \hat{V}(s_t; \mathbf{w}))\mathbf{x}(s_t) \\
&= \alpha(G_t - \mathbf{x}(s_t)^T \mathbf{w})\mathbf{x}(s_t)
\end{aligned}$$

- Note: G_t may be a very noisy estimate of true return

MC Linear Value Function Approximation for Policy Evaluation

1. Initialize $\mathbf{w} = \mathbf{0}$, $k = 1$
2. **loop**
3.   Sample k-th episode $(s_{k,1}, a_{k,1}, r_{k,1}, s_{k,2}, \ldots, s_{k,L_k})$ given π
4.   **for** $t = 1, \ldots, L_k$ **do**
5.   **if** First visit to (s) in episode k **then**
6. $G_t(s) = \sum_{j=t}^{L_k} r_{k,j}$
7.   Update weights:

$$w = \alpha \left(G_t(s) - x(s_t)^T w \right) x(s_t)$$

8.   **end if**
9.   **end for**
10. $k = k + 1$
11. **end loop**

Baird (1995)-Like Example with MC Policy Evaluation[3]

$s_1 = [2\ 0\ 0\ 0\ 0\ 0\ 0\ 1]$

$s_2 = [0\ 2\ 0\ 0\ 0\ 0\ 0\ 1]$

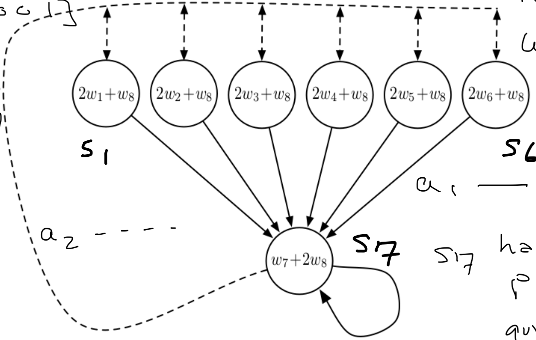
$\vdots$

$s_6\ [0\ 0\ 0\ 0\ 0\ 0\ \ 1]$

$s_7\ [0\ 0\ 0\ 0\ 0\ 0\ 1\ 2]$

rewards 0

$w_0 = [1\ 1\ 1\ 1\ 1\ 1\ 1]$



s_7 has small prob going to terminal state

- MC update: $\Delta \boldsymbol{w} = \alpha(G_t - \boldsymbol{x}(s_t)^T \boldsymbol{w})\boldsymbol{x}(s_t)$
- Small prob s_7 goes to terminal state, $\boldsymbol{x}(s_7)^T = [0\ 0\ 0\ 0\ 0\ 0\ 1\ 2]$

Consider $s_1\ a_1\ 0\ s_7\ a_1\ 0\ s_7\ a_1\ 0\ T$

$G(s_1) = 0 \qquad w = w - \alpha[0 - x(s_1)w]\,x(s_1)$

$x(s_1) \cdot w = 2+1 = 3 \qquad w = w - \alpha\, 3\, [2\ 0\ 0\ 0\ 0\ 0\ 0\ 1]$

Convergence Guarantees for Linear Value Function Approximation for Policy Evaluation: Preliminaries

- For infinite horizon, the Markov Chain defined by a MDP with a particular policy will eventually converge to a probability distribution over states $d(s)$
- $d(s)$ is called the stationary distribution over states of π
- $\sum_s d(s) = 1$
- $d(s)$ satisfies the following balance equation:

$$d(s') = \sum_s \sum_a \pi(a|s)p(s'|s,a)d(s)$$

Convergence Guarantees for Linear Value Function Approximation for Policy Evaluation[4]

- Define the mean squared error of a linear value function approximation for a particular policy π relative to the true value as

$$MSVE(\boldsymbol{w}) = \sum_{s \in S} d(s)(V^{\pi}(s) - \hat{V}^{\pi}(s; \boldsymbol{w}))^2$$

- where
  - $d(s)$: stationary distribution of π in the true decision process
  - $\hat{V}^{\pi}(s; \boldsymbol{w}) = \boldsymbol{x}(s)^T \boldsymbol{w}$, a linear value function approximation

[4] Tsitsiklis and Van Roy. An Analysis of Temporal-Difference Learning with Function Approximation. 1997.https://web.stanford.edu/ bvr/pubs/td.pdf

Convergence Guarantees for Linear Value Function Approximation for Policy Evaluation[1]

- Define the mean squared error of a linear value function approximation for a particular policy π relative to the true value as

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- where
  - $d(s)$: stationary distribution of π in the true decision process
  - $\hat{V}^\pi(s; \boldsymbol{w}) = \boldsymbol{x}(s)^T \boldsymbol{w}$, a linear value function approximation
- Monte Carlo policy evaluation with VFA converges to the weights $\boldsymbol{w}_{MC}$ which has the minimum mean squared error possible:

$$MSVE(\boldsymbol{w}_{MC}) = \min_{\boldsymbol{w}} \sum_{s \in S} d(s)(V^\pi(s) - \hat{V}^\pi(s; \boldsymbol{w}))^2$$

[1] Tsitsiklis and Van Roy. An Analysis of Temporal-Difference Learning with Function Approximation. 1997.https://web.stanford.edu/ bvr/pubs/td.pdf

Recall: Temporal Difference Learning w/ Lookup Table

- Uses bootstrapping and sampling to approximate V^π
- Updates $V^\pi(s)$ after each transition (s, a, r, s'):

$$V^\pi(s) \quad = \quad V^\pi(s) + \alpha(r + \gamma V^\pi(s') - V^\pi(s))$$

- Target is $r + \gamma V^\pi(s')$, a biased estimate of the true value $V^\pi(s)$
- Represent value for each state with a separate table entry

Temporal Difference (TD(0)) Learning with Value Function Approximation

- Uses bootstrapping and sampling to approximate true V^π
- Updates estimate $V^\pi(s)$ after each transition (s, a, r, s'):

$$V^\pi(s) \;=\; V^\pi(s) + \alpha(r + \gamma V^\pi(s') - V^\pi(s))$$

- Target is $r + \gamma V^\pi(s')$, a biased estimate of the true value $V^\pi(s)$
- In value function approximation, target is $r + \gamma \hat{V}^\pi(s'; \boldsymbol{w})$, a biased and approximated estimate of the true value $V^\pi(s)$
- 3 forms of approximation:

  - sampling
  - bootstrapping
  - VFA

Temporal Difference (TD(0)) Learning with Value Function Approximation

$V^{\pi}(s)$

- In value function approximation, target is $r + \gamma \hat{V}^{\pi}(s'; \boldsymbol{w})$, a biased and approximated estimate of the true value $V^{\pi}(s)$
- Can reduce doing TD(0) learning with value function approximation to supervised learning on a set of data pairs:
  - $\langle s_1, r_1 + \gamma \hat{V}^{\pi}(s_2; \boldsymbol{w})\rangle, \langle s_2, r_2 + \gamma \hat{V}(s_3; \boldsymbol{w})\rangle, \ldots$
- Find weights to minimize mean squared error

$$J(\boldsymbol{w}) = \mathbb{E}_{\pi}[(r_j + \gamma \hat{V}^{\pi}(s_{j+1}, \boldsymbol{w}) - \hat{V}(s_j; \boldsymbol{w}))^2]$$

Temporal Difference (TD(0)) Learning with Value Function Approximation

- In value function approximation, target is $r + \gamma \hat{V}^\pi(s'; \boldsymbol{w})$, a biased and approximated estimate of the true value $V^\pi(s)$
- Supervised learning on a different set of data pairs: $\langle s_1, r_1 + \gamma \hat{V}^\pi(s_2; \boldsymbol{w}) \rangle, \langle s_2, r_2 + \gamma \hat{V}(s_3; \boldsymbol{w}) \rangle, \ldots$
- In linear TD(0)

$$
\begin{aligned}
\Delta \boldsymbol{w} &= \alpha(\overbrace{r + \gamma \hat{V}^\pi(s'; \boldsymbol{w})}^{V^\pi(s)} - \hat{V}^\pi(s; \boldsymbol{w})) \nabla_{\boldsymbol{w}} \hat{V}^\pi(s; \boldsymbol{w}) \\
&= \alpha(r + \gamma \hat{V}^\pi(s'; \boldsymbol{w}) - \hat{V}^\pi(s; \boldsymbol{w})) \boldsymbol{x}(s) \\
&= \alpha(\underbrace{r + \gamma \boldsymbol{x}(s')^T \boldsymbol{w}}_{G(s)} - \boldsymbol{x}(s)^T \boldsymbol{w}) \boldsymbol{x}(s)
\end{aligned}
$$

MC

TD(0) Linear Value Function Approximation for Policy Evaluation

Initialize $\mathbf{w} = \mathbf{0}$, $k = 1$
loop
    Sample tuple (s_k, a_k, r_k, s_{k+1}) given π
    Update weights:

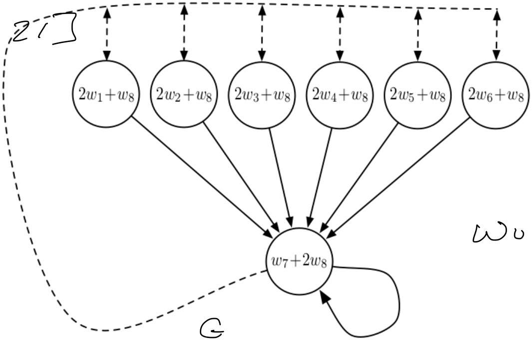
$$\boldsymbol{w} = \boldsymbol{w} + \alpha(r + \gamma \boldsymbol{x}(s')^T \boldsymbol{w} - \boldsymbol{x}(s)^T \boldsymbol{w})\boldsymbol{x}(s)$$

    $k = k + 1$
end loop

Baird Example with TD(0) On Policy Evaluation [1]

$s_1 = [2\ 0\ 0\ 0\ 0\ 0\ 0\ 1]$

$s_7 = [0\ 0\ 0\ 0\ 0\ 0\ 2\ 1]$



rewards 0

$w_0 = [1\ 1\ 1\ 1\ 1\ 1\ 1\ 1]$

- TD update: $\Delta \boldsymbol{w} = \alpha(r + \overbrace{\gamma \boldsymbol{x}(s')^T \boldsymbol{w}} - \boldsymbol{x}(s)^T \boldsymbol{w})\boldsymbol{x}(s)$

$s_1,\ a_1,\ 0\ s_7 \qquad 0 + \gamma 3$

$\Delta w = \alpha(\gamma 3 - 3)\underbrace{x(s_1)}_{[2\ 0\ 0\ 0\ 0\ 0\ 0\ 1]}$

[1] Figure from Sutton and Barto 2018

Convergence Guarantees for Linear Value Function Approximation for Policy Evaluation

- Define the mean squared error of a linear value function approximation for a particular policy π relative to the true value as

$$MSVE(\boldsymbol{w}) = \sum_{s \in S} d(s)(V^{\pi}(s) - \hat{V}^{\pi}(s;\boldsymbol{w}))^2$$

- where
  - $d(s)$: stationary distribution of π in the true decision process
  - $\hat{V}^{\pi}(s;\boldsymbol{w}) = \boldsymbol{x}(s)^T\boldsymbol{w}$, a linear value function approximation
- TD(0) policy evaluation with VFA converges to weights $\boldsymbol{w}_{TD}$ which is within a constant factor of the minimum mean squared error possible:

$$MSVE(\boldsymbol{w}_{TD}) \le \frac{1}{1-\gamma} \min_{\boldsymbol{w}} \sum_{s \in S} d(s)(V^{\pi}(s) - \hat{V}^{\pi}(s;\boldsymbol{w}))^2$$

$\gamma = .9$ 10.

Check Your Understanding: Poll

- Monte Carlo policy evaluation with VFA converges to the weights $\mathbf{w}_{MC}$ which has the minimum mean squared error possible:

$$MSVE(\mathbf{w}_{MC}) = \min_{\mathbf{w}} \sum_{s \in S} d(s)(V^{\pi}(s) - \hat{V}^{\pi}(s; \mathbf{w}))^2$$

- TD(0) policy evaluation with VFA converges to weights $\mathbf{w}_{TD}$ which is within a constant factor of the minimum mean squared error possible:

$$MSVE(\mathbf{w}_{TD}) \leq \frac{1}{1-\gamma} \min_{\mathbf{w}} \sum_{s \in S} d(s)(V^{\pi}(s) - \hat{V}^{\pi}(s; \mathbf{w}))^2$$

- If the VFA is a tabular representation (one feature for each state), what is the MSVE for MC and TD? [select all]

1. MSVE=0 for MC
2. MSVE $>$ 0 for MC
3. MSVE $=$ 0 for TD
4. MSVE $>$ 0 for TD
5. Not sure

Convergence Rates for Linear Value Function Approximation for Policy Evaluation

- Does TD or MC converge faster to a fixed point?
- Not (to my knowledge) definitively understood
- Practically TD learning often converges faster to its fixed value function approximation point

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Control using Value Function Approximation

- Use value function approximation to represent state-action values $\hat{Q}^\pi(s, a; \boldsymbol{w}) \approx Q^\pi$
- Interleave
  - Approximate policy evaluation using value function approximation
  - Perform ϵ-greedy policy improvement
- Can be unstable. Generally involves intersection of the following:
  - Function approximation
  - Bootstrapping
  - **Off-policy learning**

Action-Value Function Approximation with an Oracle

- $\hat{Q}^\pi(s, a; \boldsymbol{w}) \approx Q^\pi$
- Minimize the mean-squared error between the true action-value function $Q^\pi(s, a)$ and the approximate action-value function:

$$J(\boldsymbol{w}) = \mathbb{E}_\pi[(Q^\pi(s, a) - \hat{Q}^\pi(s, a; \boldsymbol{w}))^2]$$

- Use stochastic gradient descent to find a local minimum

$$\begin{aligned} -\frac{1}{2}\nabla_{\boldsymbol{W}} J(\boldsymbol{w}) &= \mathbb{E}\left[(Q^\pi(s, a) - \hat{Q}^\pi(s, a; \boldsymbol{w}))\nabla_{\boldsymbol{w}}\hat{Q}^\pi(s, a; \boldsymbol{w})\right] \\ \Delta(\boldsymbol{w}) &= -\frac{1}{2}\alpha\nabla_{\boldsymbol{w}} J(\boldsymbol{w}) \end{aligned}$$

- Stochastic gradient descent (SGD) samples the gradient

Linear State Action Value Function Approximation with an Oracle

- Use features to represent both the state and action

$$\mathbf{x}(s,a) = \begin{pmatrix} x_1(s,a) \\ x_2(s,a) \\ \dots \\ x_n(s,a) \end{pmatrix}$$

- Represent state-action value function with a weighted linear combination of features

$$\hat{Q}(s,a;\mathbf{w}) = \mathbf{x}(s,a)^T\mathbf{w} = \sum_{j=1}^{n} x_j(s,a)w_j$$

- Stochastic gradient descent update:

$$\nabla_{\mathbf{w}} J(\mathbf{w}) = \nabla_{\mathbf{w}} \mathbb{E}_{\pi}[(Q^{\pi}(s,a) - \hat{Q}^{\pi}(s,a;\mathbf{w}))^2]$$

Incremental Model-Free Control Approaches

- Similar to policy evaluation, true state-action value function for a state is unknown and so substitute a target value
- In Monte Carlo methods, use a return G_t as a substitute target

$$\Delta \boldsymbol{w} = \alpha(G_t - \hat{Q}(s_t, a_t; \boldsymbol{w}))\nabla_{\boldsymbol{w}} \hat{Q}(s_t, a_t; \boldsymbol{w})$$

- For SARSA instead use a TD target $r + \gamma \hat{Q}(s', a'; \boldsymbol{w})$ which leverages the current function approximation value

$$\Delta \boldsymbol{w} = \alpha(r + \gamma \hat{Q}(s', a'; \boldsymbol{w}) - \hat{Q}(s, a; \boldsymbol{w}))\nabla_{\boldsymbol{w}} \hat{Q}(s, a; \boldsymbol{w})$$

Incremental Model-Free Control Approaches

- Similar to policy evaluation, true state-action value function for a state is unknown and so substitute a target value
- In Monte Carlo methods, use a return G_t as a substitute target

$$\Delta \boldsymbol{w} = \alpha(G_t - \hat{Q}(s_t, a_t; \boldsymbol{w}))\nabla_{\boldsymbol{w}} \hat{Q}(s_t, a_t; \boldsymbol{w})$$

- For SARSA instead use a TD target $r + \gamma \hat{Q}(s', a'; \boldsymbol{w})$ which leverages the current function approximation value

$$\Delta \boldsymbol{w} = \alpha(r + \gamma \hat{Q}(s', a'; \boldsymbol{w}) - \hat{Q}(s, a; \boldsymbol{w}))\nabla_{\boldsymbol{w}} \hat{Q}(s, a; \boldsymbol{w})$$

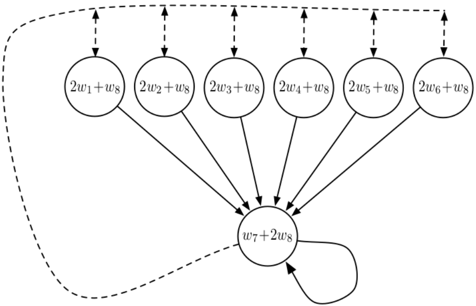
- For Q-learning instead use a TD target $r + \gamma \max_{a'} \hat{Q}(s', a'; \boldsymbol{w})$ which leverages the max of the current function approximation value

$$\Delta \boldsymbol{w} = \alpha(r + \gamma \max_{a'} \hat{Q}(s', a'; \boldsymbol{w}) - \hat{Q}(s, a; \boldsymbol{w}))\nabla_{\boldsymbol{w}} \hat{Q}(s, a; \boldsymbol{w})$$

Convergence of TD Methods with VFA

- Informally, updates involve doing an (approximate) Bellman backup followed by best trying to fit underlying value function to a particular feature representation
- Bellman operators are contractions, but value function approximation fitting can be an expansion

Challenges of Off Policy Control: Baird Example [1]



$\pi(\text{solid}|\cdot) = 1$

$\mu(\text{dashed}|\cdot) = 6/7$

$\mu(\text{solid}|\cdot) = 1/7$

$\gamma = 0.99$

- Behavior policy and target policy are not identical
- Value can diverge

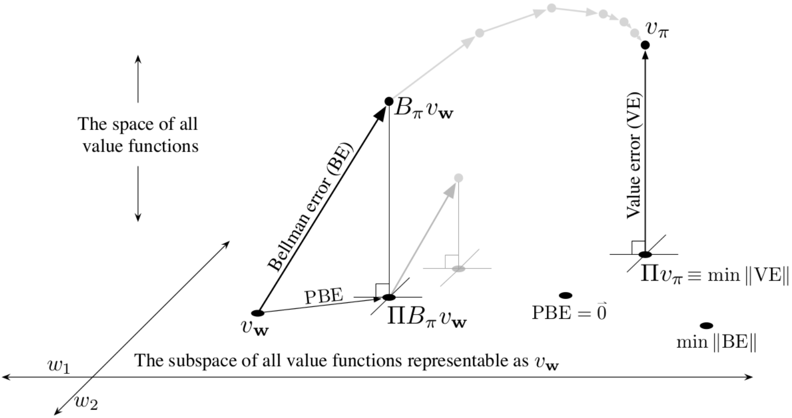
Convergence of Control Methods with VFA

Algorithm	Tabular	Linear VFA	Nonlinear VFA
Monte-Carlo Control			
Sarsa			
Q-learning			

Hot Topic: Off Policy Function Approximation Convergence

- Extensive work in better TD-style algorithms with value function approximation, some with convergence guarantees: see Chp 11 SB
- Exciting recent work on batch RL that can converge with nonlinear VFA (Dai et al. ICML 2018): uses primal dual optimization
- An important issue is not just whether the algorithm converges, but **what** solution it converges too
- Critical choices: **objective function and feature representation**

Linear Value Function Approximation[5]



[5] Figure from Sutton and Barto 2018

What You Should Understand

- Be able to implement TD(0) and MC on policy evaluation with linear value function approximation
- Be able to define what TD(0) and MC on policy evaluation with linear VFA are converging to and when this solution has 0 error and non-zero error.
- Be able to implement Q-learning and SARSA and MC control algorithms
- List the 3 issues that can cause instability and describe the problems qualitatively: function approximation, bootstrapping and off policy learning

Class Structure

- Last time: Control (making decisions) without a model of how the world works
- This time: Value function approximation
- **Next time**: Deep reinforcement learning

Batch Monte Carlo Value Function Approximation

- May have a set of episodes from a policy π
- Can analytically solve for the best linear approximation that minimizes mean squared error on this data set
- Let $G(s_i)$ be an unbiased sample of the true expected return $V^\pi(s_i)$

$$\arg\min_{\boldsymbol{w}} \sum_{i=1}^{N} (G(s_i) - \boldsymbol{x}(s_i)^T \boldsymbol{w})^2$$

- Take the derivative and set to 0

$$\boldsymbol{w} = (X^T X)^{-1} X^T \boldsymbol{G}$$

- where $\boldsymbol{G}$ is a vector of all N returns, and X is a matrix of the features of each of the N states $\boldsymbol{x}(s_i)$
- Note: not making any Markov assumptions