CS250/EE387 – LECTURE 11 – GURUSWAMI-SUDAN ALG.

AGENDA

(0) Recap
(1) Aside: How hard is decoding RS codes?
(2) Sudan Algorithm
(3) Guruswami-Sudan Algorithm.

TODAY'S ANT FACT

There is a type of fungus, found in tropical forests, which infects ants and turns them into "zombies." The ants are compelled to climb to an appropriate height, bite onto a leaf, and wait to die. When they do, the fungus erupts from their corpse and sends its spores down to infect more ants. Pretty gruesome!

(0) RECAP.

Last time, we saw LIST-DECODING:

DEF. A code $\mathcal{C} \subseteq \Sigma^n$ is (p, L)-**LIST-DECODABLE** if $\forall y \in \Sigma^n$,

$$\left| \{ c \in \mathcal{C} : \delta(c, y) \leq p \} \right| \leq L.$$

The LIST-DECODING CAPACITY THM says that there $\exists$ codes that are $(p, 1/\epsilon)$-list-decodable with rate $R = 1 - H_2(p) - \epsilon$, for $p \leq 1 - 1/q$.

- That's the same trade-off as for random errors!
- Moreover, notice that p can get as big as $1 - 1/q$. If we demand unique decoding, the Plotkin bound says we can't hope to do better than $\frac{1 - 1/q}{2}$ with $R > 0$.

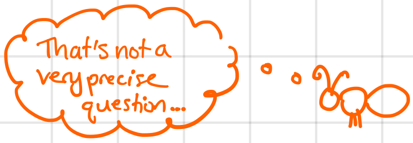
OUR NEXT QUESTION: How list-decodable are codes we know and love? For example, Reed-Solomon Codes?

Last time we saw the JOHNSON BOUND which says that codes with good distance are decently list-decodable.

- For RS codes, the Johnson bound says that an RS code of rate R is list-decodable up to distance $\rho \approx 1-\sqrt{R}$.

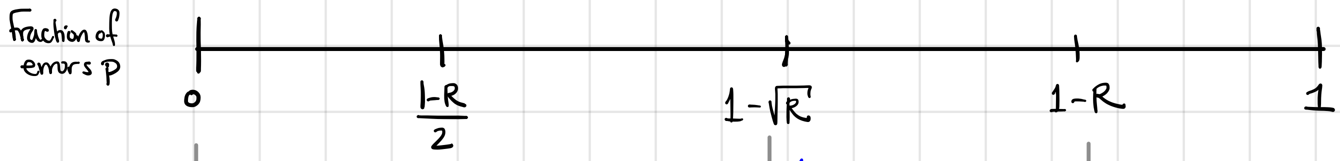
- Notice that list-decoding capacity is $\rho = H_q^{-1}(1-R) \approx 1-R$ for large q. So $\rho = 1-\sqrt{R}$ is less good than it could be.

① ASIDE: How hard is it to decode RS codes?



More precisely, given $w \in \mathbb{F}_q^n$, find $c \in RS_q(n,k)$ so that $\delta(w,c)$ is minimized.

How hard this depends a lot on the assumptions we can make about $\min_{c \in RS} \delta(w,c)$. For example, if $\exists c$ s.t. $\delta(c,w) < \frac{1-R}{2}$, then Welch-Berlekamp will do this in polynomial time. How about if $\delta(c,w)$ is larger?



List-decoding status	At most one codeword c with $\Delta(c,w) \leq \rho$	By the Johnson Bound, there are $\leq$ poly(n) codewords c with $\Delta(c,w) \leq \rho$	$\left(\begin{array}{l}\#c \text{ s.t.}\\ \delta(c,w) \leq \rho\end{array}\right) \leq ???$ For some choices of RS codes, it's exponential. For others, $O(1)$. In general, this is not well understood.	Definitely exponentially many codewords c s.t. $\delta(c,w) \leq \rho$ for any ρ in this range. (B/c of list.dec.cap. thm)
How hard is it to find the closest codeword?	We can find c efficiently w/ (eg) Welch-Berlekamp or else efficiently decide no such c exists.	TODAY! We will see how to find all these $\leq$ poly(n) codewords efficiently, and then we can search to find the closest.	Also ???. But there are some ρ's in here where doing MLD up to distance ρ is as hard as discrete log.	For large enough ρ, this is known to be NP-hard (A taste of this on HW!)

(2) SUDAN ALGORITHM

The Sudan Alg is a warmup to the GURUSWAMI-SUDAN alg, which will be able to efficiently list-decode RS codes up to the Johnson bound, $\rho = 1-\sqrt{R}$.

(2A) BIVARIATE POLYNOMIALS

A bivariate polynomial $Q(X,Y) \in \mathbb{F}_q[X,Y]$ is:

$$Q(X,Y) = \sum_{\substack{i=0,\dots,m_X \\ j=0,\dots,m_Y}} \alpha_{ij} X^i Y^j, \quad \text{where} \quad \begin{array}{l} m_X =: \deg_X(Q) \\ m_Y =: \deg_Y(Q) \end{array}$$

Notice that we can also think about Q as an element of $(\mathbb{F}_q[X])[Y]$:

$$Q(X,Y) = \sum_{j=0,\dots,m_Y} \underbrace{Q_j(X)}_{\text{The coefficients live in } \mathbb{F}_q[X]} \cdot Y^j$$

Polynomials in $(\mathbb{F}_q[X])[Y]$ behave a lot like a "normal" polynomial in Y.

FOR EXAMPLE: Consider $Q(Y) = Y^2 - 1$.
Then $Q(1) = 0$, which implies that $(Y-1) \mid Y^2-1$

Similarly, consider $Q(X,Y) = Y^2 - f(X)^2$.
Then $Q(X, f(X)) = 0$, which implies that $(Y-f(X)) \mid Q(X,Y)$

FACT. Let $Q(X,Y) \in \mathbb{F}_q[X,Y]$, and let $f \in \mathbb{F}_q[X]$. Then

$$Q(X, f(X)) \equiv 0 \iff (Y - f(X)) \mid Q(X,Y)$$

"$\equiv$" means "is identically 0" aka, all the coefficients are 0.

"divides." aka, $Q(X,Y) = (Y-f(X)) \cdot h(X,Y)$ for some $h \in \mathbb{F}[X,Y]$.

Moreover, we can find such f's efficiently, and there are at most $\deg_Y(Q)$ such f's.

(2B) RECALL the BERLEKAMP-WELCH ALGORITHM:

Given $y = (y_1, \dots, y_n) \in \mathbb{F}_q^n$:

Recall, $E(X)$ was supposed to be the ERROR LOCATOR POLY, $E(X) = \prod_{i : y_i \neq c_i} (X - \alpha_i)$, so that $E(\alpha_i) \cdot f(\alpha_i) = E(\alpha_i) \cdot y_i \ \forall i$

1. Find low-degree polynomials $E(X)$, $B(X)$ s.t. $E(\alpha_i) \cdot y_i = B(\alpha_i) \ \forall i = 1, \dots, n$
2. Return $f(X) = B(X)/E(X)$

We can recast this in terms of bivariate polys:

1. Find $Q(X,Y)$ (meant to be $Q(X,Y) = E(X) \cdot Y - B(X)$) s.t. $Q(\alpha_i, y_j) = 0 \ \forall i = 1, \dots, n$
2. Find a poly $f(X)$ s.t. $Q(X, f(X)) \equiv 0$, and return f.

(Notice that $f(X) = B(X)/E(X)$ will work in the Q we were supposed to find).

We'll use the same framework for SUDAN'S ALGORITHM for list-decoding.

PROBLEM: Given $y = (y_1, \dots, y_n)$, k, and t, find all polynomials $f \in \mathbb{F}_q[X]$ s.t.:

- $\deg(f) < k$
- $f(\alpha_i) = y_i$ for at least t of the α_i's.

What t's can we handle? We'll see later!

(2C) Finally, SUDAN'S ALG.

In this context, Berlekamp-Welch is:

INTERPOLATION STEP 1. Find a LOW-DEGREE polynomial $Q(X,Y)$ so that $Q(\alpha_i, y_i) = 0 \ \forall i = 1, \dots, n$

What exactly should this mean?

ROOT-FINDING STEP 2. Factor $Q(X,Y)$ to find polynomials $f(X)$ s.t. $Q(X, f(X)) \equiv 0$.
Return all such f's.

- We can do STEP 1 as long as we have more variables (coeffs of Q) than constraints.
- To make sure that STEP 2 is correct, we'll have to argue that whenever $f(\alpha_i) = y_i$ for $\geq t$ values of i, then $Q(X, f(X)) \equiv 0$. The fact that the list is small will follow from the fact that Q is low-deg.

This algorithm basically works, and is called SUDAN'S ALGORITHM.

THM If $t > 2\sqrt{nk}$, then we can solve the list-decoding problem in polynomial time.

Before we prove the THM, we can ask how good this is.

$$(\#\text{agreements between } f \text{ and } y) = t > 2\sqrt{nk}$$

$$\Delta(f,y) = n - t < n - 2\sqrt{nk}$$

So this works up to radius $p \leq \frac{n-t}{n} = 1 - 2\sqrt{R}$.
Remember that we were shooting for $1-\sqrt{R}$, so this isn't quite right — but we'll get there!

Now we'll prove the THM, and finish specifying the alg. along the way.

pf/algorithm:

STEP 1 (INTERPOLATION). Choose $\ell = \sqrt{nk}$.

Find $Q(X,Y)$ s.t. $\deg_X(Q) \leq \ell$ and $\deg_Y(Q) \leq n/\ell$,
so that $Q(\alpha_i, y_i) = 0 \ \forall i = 1, \dots, n$.

To do this, we need:

$$\underbrace{(\#\text{coeffs in } Q)}_{(\ell+1)(\frac{n}{\ell}+1) \text{ of these}} > \underbrace{(\#\text{constraints})}_{n \text{ of these}}$$

and indeed we have $(\ell+1)(\frac{n}{\ell}+1) = n + \frac{n}{\ell} + \ell + 1 > n$ ↘

STEP 2 on NEXT PAGE

pf ctd.

STEP 2. (ROOT-FINDING STEP) Return all $f(X)$ s.t. $Q(X, f(X)) \equiv 0$.

Note that we can do this efficiently, and the size of our list will be at most $\deg_Y(Q) = n/\ell = n/\sqrt{kn} = 1/\sqrt{R}$, a constant.

Now we need to argue why this step is a good idea.

Suppose $\deg(f) < k$ and that $f(\alpha_i) = y_i$ for $\geq t$ vals of i.
We need to show that we will return f, so we need to show $Q(X, f(X)) \equiv 0$.

Let $R(X) := Q(X, f(X))$.
Then $\deg(R) \leq \deg_X(Q) + \deg(f) \cdot \deg_Y(Q) < \ell + k \cdot \frac{n}{\ell} = 2\sqrt{nk}$

↖ This is why we chose $\ell = \sqrt{nk}$, to balance these two terms.

But $R(\alpha_i) = Q(\alpha_i, f(\alpha_i)) = Q(\alpha_i, y_i) = 0$
for at least t values of i.

So R has degree $< 2\sqrt{nk}$, but $t > 2\sqrt{nk}$ roots, hence $R(X) \equiv 0$, as desired.

③ GURUSWAMI-SUDAN ALG.

Now we'll fix this up so that we can actually get up to $p = 1 - \sqrt{R}$, meeting the JOHNSON BOUND.

TWO CHANGES:

1. We will change how we measure "LOW-DEGREE"
2. We will require something a bit stronger than $Q(\alpha_i, y_i) = 0$; we'll ask for Q to vanish with high MULTIPLICITY.

CHANGE 1.

DEF. The $(1,k)$-degree of X^iY^j is $i+kj$

The $(1,k)$-degree of $Q(X,Y)$ is the max $(1,k)$-degree of any monomial in Q.

Just this change is enough to make SOME progress:

THM If $t > \sqrt{2nk}$, then we can solve the list-decoding problem in polynomial time.

pf. sketch Same alg, but now demand the $(1,k)$-degree of Q is $\leq \sqrt{2kn}$

STEP 1. INTERPOLATION.

Find $Q(X,Y)$ s.t.

$(1,k)$-deg is $\leq \sqrt{2kn}$.

Turns out (FUN EXERCISE!) there are $> D^2/2k$ coeffs in a poly w/ $(1,k)$-deg $\leq D$ So

$$(\#\text{variables}) > \frac{(2kn)}{2k} = n = (\#\text{constraints})$$

and we can find Q.

STEP 2. ROOT-FINDING.

(same as before)

Now we have $\deg(R) = \deg(Q(X,f(X))) < (1,k)\text{-deg of } Q \leq \sqrt{2nk}$

So the argument goes through as before with a slightly better bound.

But we want $1-\sqrt{R}$! Not $1-\sqrt{2R}$!

CHANGE 2.

DEF. $Q(X,Y)$ has a root of multiplicity r at (a,b) if $Q(X+a, Y+b)$ has no terms of total degree $< r$.

Example: $Q(X,Y) = (X-1)^2(Y-1)$ has a root of multiplicity 3 at $(1,1)$, because $Q(X+1,Y+1) = X^2 \cdot Y$ which has no terms of total degree < 3.

GURUSWAMI-SUDAN ALGORITHM.

Choose a parameter r

Suppose $t \geq \sqrt{kn(1+1/r)}$

1. INTERPOLATION STEP.

   Find a polynomial $Q(X,Y)$ with $(1,k)$-degree $D = \sqrt{kn \cdot r \cdot (r+1)}$ so that $Q(\alpha_i, y_i) = 0$ with multiplicity r for $i=1,\ldots,n$.

2. ROOT-FINDING STEP.

   Return all f so that $Q(X, f(X)) \equiv 0$.

   [Notice that there are $\leq \deg_Y(Q) \leq D/k \approx r/\sqrt{R}$ of these.]

ANALYSIS:

Again we need to show that 1. is possible and that 2. is a good idea.

1. FUN EXERCISE: The number of constraints in "$Q(\alpha_i, y_i) = 0$ w/ mult. r $\forall i$" is $n \cdot \binom{r+1}{2}$.

   So that's MORE constraints than before, which seems like a bad thing....

   we'll see later why it's actually good.

   The number of variables is still $> D^2/2k$, so we need

   $$D^2 \geq 2k \cdot n \cdot \binom{r+1}{2} = knr(r+1),$$

   which is TRUE by our choice of D.

2. Let $R(X) = Q(X, f(X))$ as before.

   Then not only does $R(X)$ have $\geq t$ roots [as before], it has $\geq t$ roots which EACH have multiplicity r.

Justification on next page...

CLAIM. If $f(\alpha_i)=y_i$, then $(X-\alpha_i)^r \mid R(X)$.

aka, $R(X)$ has a root of multiplicity r at each α_i.

Pf. Let's drop the i subscripts for notational sanity.
Recall that since Q has a root of multiplicity r at (α, y),

$Q(X+\alpha, Y+y)$ has no terms of total degree $< r$.

Now, consider $\bar{f}(X) := f(X+\alpha)-y$. We have

$$R(X+\alpha) = Q(X+\alpha, f(X+\alpha)) = \underbrace{Q(X+\alpha, \bar{f}(X)+y)}$$

This is a sum of monomials $X^c \cdot \bar{f}(X)^d$ where $c+d \geq r$.

Now, since $f(\alpha)=y$, $\bar{f}(0)=0$, so $\bar{f}$ has no constant term.
Thus, those monomials $X^c\,\bar{f}(X)^d$ are all divisible by X^{c+d}, and hence are all divisible by X^r.

Then $X^r \mid R(X+\alpha)$, which means $(X-\alpha)^r \mid R(X)$, as desired.

Now given this claim, the fact that $f(\alpha_i)=y_i$ for at least t different i's means that $R(X)$ has $t \cdot r$ roots, counting multiplicities.

This is why it was OK to take a hit in the number of constraints! Now we get $r \cdot t$ roots instead of r.

Since $\deg(R) \leq D$, if R is nonzero we must have

$$tr < D$$
$$\sqrt{kn(1+1/r)} \cdot r < \sqrt{kn\,r(r+1)}$$
$$\sqrt{kn\,r(r+1)} < \sqrt{kn\,r(r+1)} \quad ↯$$

That's not true, so $R(X) \equiv 0$, and the proof concludes as before.

This proves the following theorem:

THM If $t > \sqrt{nk(1+1/r)}$ then we can solve the list-decoding problem in poly(n) time, with list size $r \cdot \sqrt{\frac{n}{k}}$.

Once again, we calculate that this means we can take $p = \frac{n-t}{n} = 1 - \sqrt{R(1+1/r)}$, so we conclude

THM. For all $r > 0$, RS codes of rate R are $\left(1-\sqrt{R(1+1/r)}, r/\sqrt{R}\right)$ List-decodable, and the Guruswami-Sudan algorithm can do the list-decoding in time poly(n,r).

Thus, we can rachet up r as large as we like (say, $r = \text{poly}(n)$) and approach the Johnson bound with polynomial-time algorithms. HOORAY!

The moral of the story:

WE CAN EFFICIENTLY LIST-DECODE RS CODES up to the JOHNSON BOUND

NOTE.
As presented, the Guruswami-Sudan algorithm runs in time $O(n^3)$, but people have optimized the heck out of it and it can be made to run in time $O(n\log(n))$.

see, eg, [Aleknovich, 2005]

disclaimer: maybe there's another loglog(n) factor in there

QUESTIONS TO PONDER

1. What breaks in the GS algorithm beyond the Johnson bound?
2. Can you come up with a "bad" list of close-together RS codewords beyond the Johnson bound?
3. What if I modify the constraints so that instead of "$f(\alpha_i) = y_i$" they are "$f(\alpha_i) \in \{y_i, y_i', y_i''\}$"