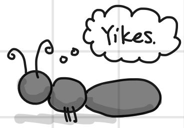
CS250/EE387 - LECTURE 14 - LOCALITY!

AGENDA

1. LOCALLY CORRECTABLE CODES
2. RM CODES as LCCs
3. HIGH-RATE LCCs [sketch]

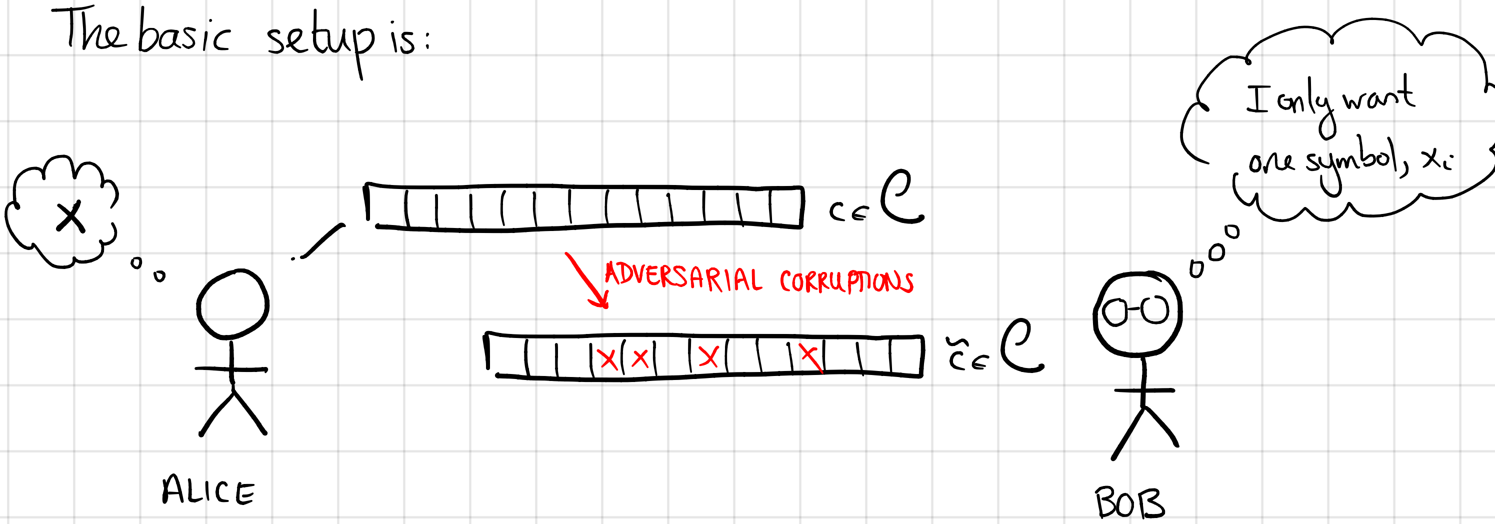
TODAY'S ANT FACT

A species of Malaysian ant has a defense mechanism where workers can perform an act of suicidal altruism by essentially causing their own head to explode in a shower of venom.



Today we will talk about LOCALLY DECODABLE CODES.

The basic setup is:



If Bob only wants one symbol of Alice's message (or her codeword), then he COULD decode the whole thing and figure out x_i.

But that seems wasteful...

The idea of LOCAL DECODING is to allow Bob to figure out x_i in SUBLINEAR TIME. In particular, he won't even have enough time to look at all of $\tilde{c}$!

① LOCALLY CORRECTABLE CODES.

Let us try to formalize this goal:

NOT THE CORRECT DEF.

$\mathcal{C} \subseteq \mathbb{F}_q^n$ is a (δ, Q)- LOCALLY CORRECTABLE CODE (LCC) if there is an algorithm A so that the following holds:

for all $w \in \mathbb{F}_q^n$ so that $\exists c \in \mathcal{C}$ s.t. $\Delta(c,w) \le \delta n$, and $\forall i \in [n]$,

$A^{(w)}(i)$ makes at most Q queries to w and returns c_i.

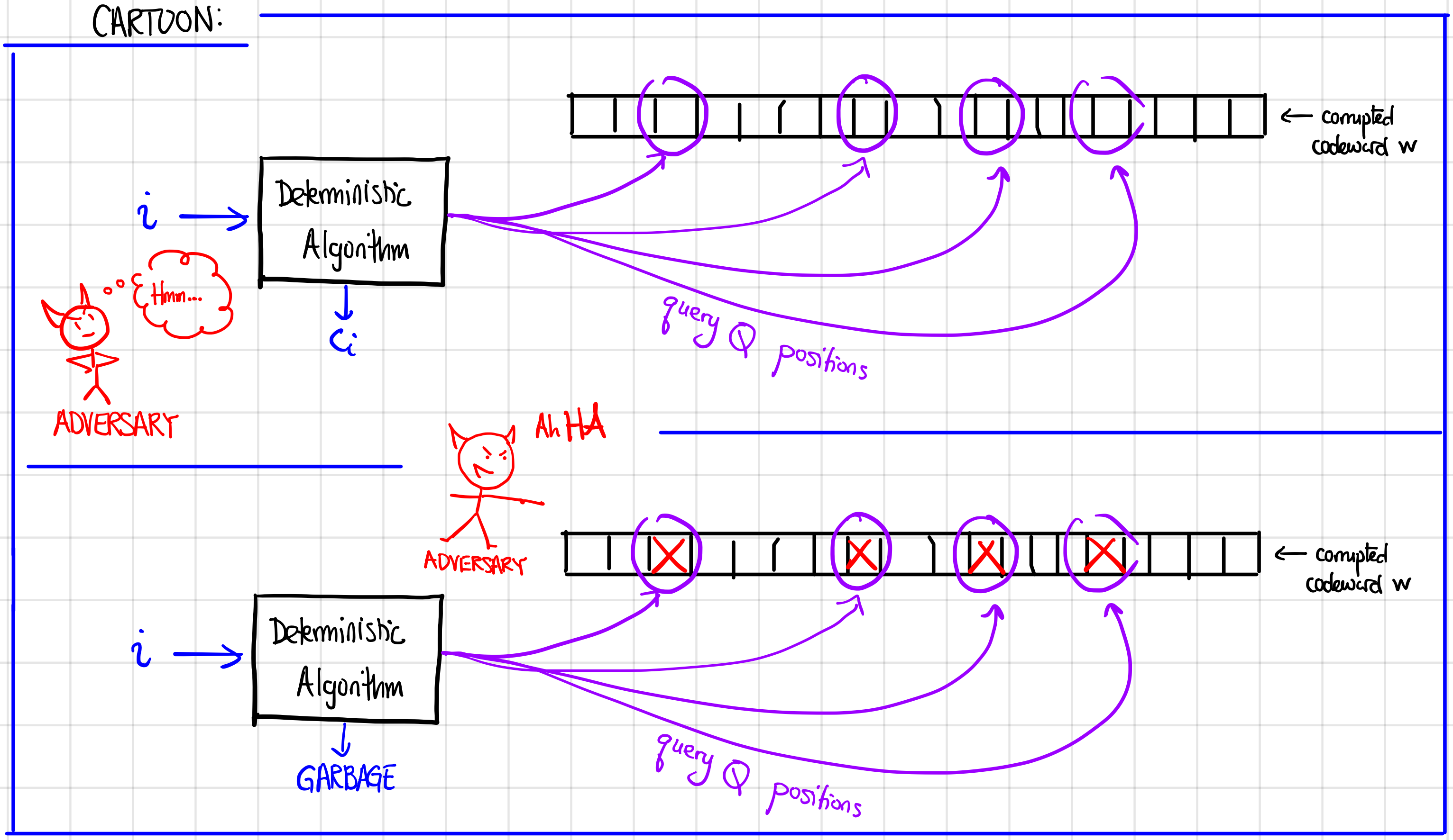
input is i

A has oracle access to w

Does this make sense? NO.

Sure, it parses, but if $Q = o(n)$ then this is a vacuous definition.

CARTOON:



That is, if the queries are deterministic, and $Q < \delta n$, then the adversary can COMPLETELY mess up the algorithm's view.

Instead, we will need to RANDOMIZE the queries if we want to deal with an adversary.

DEF. $\mathcal{C} \subseteq \mathbb{F}_q^n$ is a (δ, Q, γ)- LOCALLY CORRECTABLE CODE (LCC) if there is a randomized algorithm A so that the following holds:

for all $w \in \mathbb{F}_q^n$ so that $\exists c \in \mathcal{C}$ s.t. $\Delta(c, w) \leq \delta n$, and $\forall i \in [n]$,

- $A^{(w)}(i)$ makes at most Q queries to w

  input is i

  A has oracle access to w

- $A^{(w)}(i) = c_i$ with probability at least $1-\gamma$.

OTHER NOTIONS of LOCALITY:

- If you only want to recover a MESSAGE SYMBOL x_i instead of a CODEWORD SYMBOL c_i, it's called a LOCALLY DECODABLE CODE.

- If there's no adversary and you just want to be able to recover any symbol in SOME local way (not including that symbol) it's a LOCALLY REPAIRABLE CODE. (or LOCALLY RECOVERABLE CODE)

- Also: REGENERATING CODES, LOCALLY TESTABLE CODES, RELAXED LCCs, MAXIMALLY RECOVERABLE CODES, ...

BRIEF LIT. REVIEW on LCC's.

Q	n (as a function of k)	Comments
2	$n = \Theta(2^k)$	Matching upper + lower bounds here.
3	$\exp\left(\left(k/q^2\right)^{1/8}\right) \overset{(*)}{\leq} n \overset{(**)}{\leq} \exp\left(\exp\left(O\left(\sqrt{\log k}\,\log\log k\right)\right)\right)$	The upper bound is actually an LDC [weaker than LCC]. The lower bound holds only for LCCs [not LDCs].
$O(\log(n))$	$k \leq n \overset{(***)}{\leq} \mathrm{poly}(k)$	
$O(n^\varepsilon)$	$k \leq n \overset{(****)}{\leq} (1+\alpha)k$ for any $\alpha > 0$	

(*) [Kothari, Manohar, 2023] (**) [Efremenko '09], [Yekhanin '08]

(***) [RM codes, below] (****) We'll discuss one such construction below.

Today we'll see how RM codes fit in, starting at $Q=2$ and ending at $Q=n^\varepsilon$.

② RM codes as LCCs. First let's recall the def. of REED-MULLER CODES:

Recall that $\mathbb{F}_q[X_1, \dots, X_m]$ is the space of m-variate polynomials over $\mathbb{F}_q$.
The (total) DEGREE of a monomial $X_1^{i_1} X_2^{i_2} \cdots X_m^{i_m}$ is $\sum_{j=1}^m i_j$.
The DEGREE of $f \in \mathbb{F}_q[X_1, \dots, X_m]$ is the largest degree of any monomial in f.

DEF. The m-VARIATE REED-MULLER CODE of DEGREE r over $\mathbb{F}_q$ is

$$RM_q(m,r) = \left\{ \left(f(\vec{\alpha}_1), \dots, f(\vec{\alpha}_{q^m})\right) : f \in \mathbb{F}_q[X_1, \dots, X_m],\ \deg(f) \leq r \right\}$$

REMARK. Note that we may assume that each X_i has degree $< q$, since $\alpha = \alpha^q$ for all $\alpha \in \mathbb{F}_q$.

We saw BINARY RM CODES back in Lecture 6 when we were trying to figure out how to get good binary codes.

Let's start with $RM_2(m, 1)$: that is, codewords are just the evaluations of LINEAR polynomials

$$f(X_1, X_2, \dots, X_m) = \sum_i a_i X_i$$

This is also called the HADAMARD CODE. You saw it on your HW.

NOTE: Technically for $RM_2(m,1)$ we should also have a constant term here, but it will be convenient for us to ignore it...

Consider the following algorithm for locally decoding the Hadamard code:

ALG.

Input: Query access to $g : \mathbb{F}_2^m \to \mathbb{F}_2$ s.t. $\Delta(g, f) < 2^{m-2}$ for some $f \in RM_2(m,1)$, and an index $\alpha \in \mathbb{F}_2^m$

Output: A guess for $f(\alpha)$

Choose $\beta \in \mathbb{F}_2^m$ at random.

RETURN $g(\beta) + g(\beta + \alpha)$

CLAIM: $RM_2(m, 1)$ is a $(\delta, 2, 1-2\delta)$-LCC for any $\delta < 1/4$.

proof. If $g(\beta) = f(\beta)$ and $g(\beta+\alpha) = f(\beta+\alpha)$ (*)

then $g(\beta) + g(\beta+\alpha) = f(\beta) + f(\beta+\alpha) = f(\alpha)$ since $\deg(f) = 1$.

Here I'm using the assm that f has no constant term.

(*) happens with probability $\geq 1 - 2\delta$, since

$$\mathbb{P}\{ g(\beta) \neq f(\beta) \} = \mathbb{P}\{ g(\beta+\alpha) \neq f(\beta+\alpha) \} \leq \delta,$$ since β and $\alpha+\beta$ are both uniformly random.

(Notice that they are NOT jointly uniform, but each marginal is uniform).

GREAT! Now we have a 2-query LCC. But the rate is not great: $m/2^m$.

QUESTIONS:

1. Can we do better for $Q=2$?

   NO. See [Kerenedis+Wolf]

2. What if $Q=\omega(1)$?

   YES! Coming up next.

(2B) $Q = \log(n)$

We'd like to use the same idea, but there's a problem.

If our strategy is "hope that our $\log(n)$ queries completely avoid the errors," we'll be in trouble. Indeed, whp there will be about $\delta \cdot \log(n)$ errors in our $\log(n)$ queries.

The idea will be to make our queries themselves somewhat robust to error.

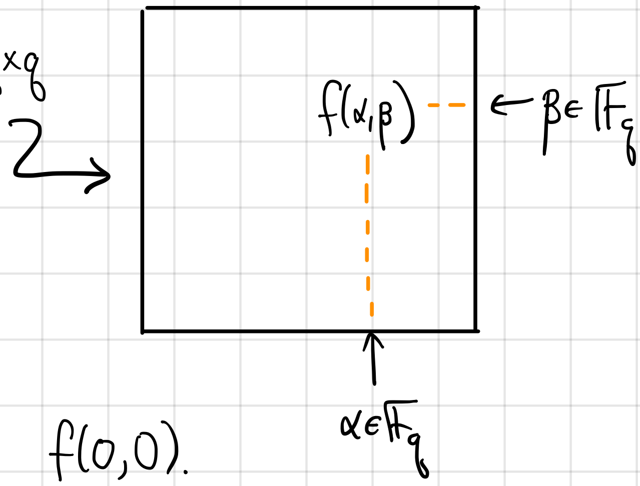
For motivation, consider $RM_q(2,r)$.

That is, the codewords of $RM_q(2,r)$ are evaluations of bivariate polynomials

$$f(X,Y) = \sum_{i+j \le r} c_{i,j} X^i Y^j .$$

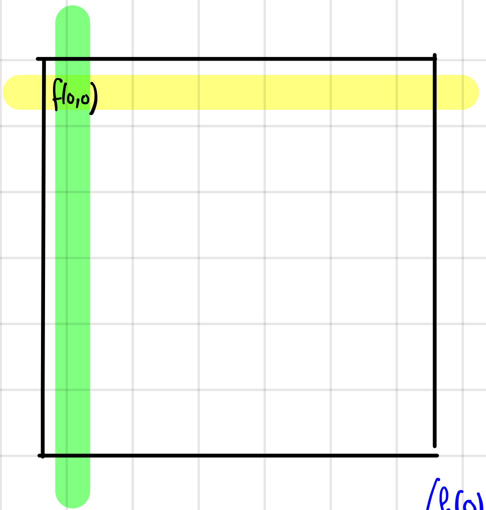
GOAL: Recover a single symbol (say, $f(\alpha,\beta)$) given query access to $g: \mathbb{F}_q^2 \to \mathbb{F}_q$ with $\Delta(g,f) \le \delta$.

We can think of codewords as $q \times q$ grids of evaluation points.



Suppose I want to recover $f(0,0)$.

As before, we want to find a bunch of LOCAL, LINEAR relationships involving $f(0,0)$.



← This row is $\left(f(0,0), f(0,\gamma), \ldots, f(0,\gamma^{q-1}) \right)$

$=: \left(g(0), g(\gamma), \ldots, g(\gamma^{q-1}) \right)$

where $g(Y) := f(0,Y) = \sum_{i+j \le r} c_{ij} 0^i \cdot Y^j$

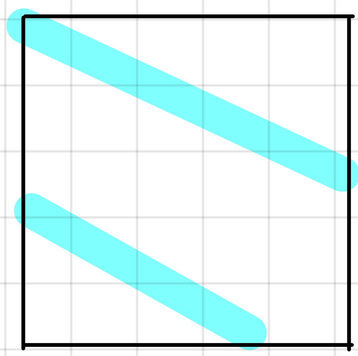
$= \sum_{j \le r} c_{0j} Y^j$

Similarly, this column is $\begin{pmatrix} h(0) \\ h(\gamma) \\ \vdots \\ h(\gamma^{q-1}) \end{pmatrix}$ where $h(X) = \sum_{j \le r} c_{j0} X^j$

Hey, those are univariate polynomials! (aka, RS codewords).

Moreover, the restriction of f to ANY line is an RS codeword!

Consider the line $L(Z) = (a_1 Z + b_1, a_2 Z + b_2)$, $a_i, b_i \in \mathbb{F}_q$.



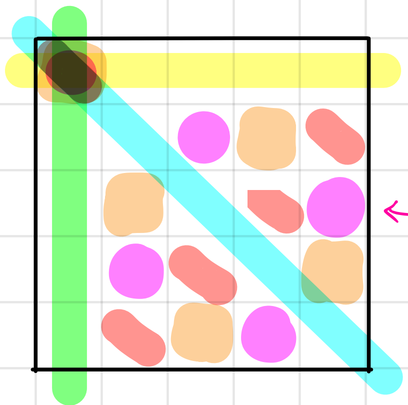
Then $f(L(Z)) = \sum_{i+j \le r} (a_1 Z + b_1)^i (a_2 Z + b_2)^j$

$=$ some degree $\le r$ polynomial in Z.

The lines through $f(0,0)$ have the properties we want:

- There are not too many (only q) points per line.
- Any two lines through $f(0,0)$ don't intersect anywhere else.

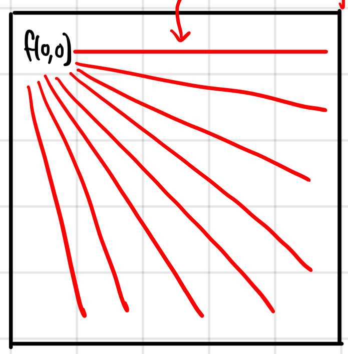
PICTURE(S):



These pink dots form a line in $\mathbb{F}_5$

Confusing but more accurate

The lines through $(0,0)$ are disjoint and cover everything.



Less accurate but hopefully more clear.

This inspires an algorithm:

ALG. (Let $r<q$ and suppose $\delta < \frac{1}{2}(1-r/q)$.)

Input: Query access to $g: \mathbb{F}_q^2 \to \mathbb{F}_q$ s.t. $\Delta(g,f) < \delta \cdot q^2$ for some $f \in RM_q(2,r)$.
and an index $(\alpha,\beta) \in \mathbb{F}_q^2$

Output: A guess for $f(\alpha,\beta)$.

Choose $(\sigma,\tau) \in \mathbb{F}_q^2 \setminus \{(0,0)\}$ at random, let $L(Z) = (\sigma \cdot Z + \alpha, \tau Z + \beta)$.

Query $g(L(\lambda))$ for all $\lambda \in \mathbb{F}_q$ and let $\tilde{h}(Z) := g(L(Z))$.

Use RS decoding to find an $h \in \mathbb{F}_q[Z]$, $\deg(h) \le r$, so that $\Delta(h,\tilde{h}) < \frac{q-r}{2}$

half the distance of the RS code.

RETURN $h(0)$.

CLAIM. For any $\delta > 0$, ALG is correct with prob. $\geq 1 - \left(\frac{2\delta q}{q-r-1}\right)$

Proof. The RS decoder will successfully find $h(Z) = f(L(Z))$ as long as the number of errors on $\{L(\lambda) : \lambda \in \mathbb{F}_q\}$ is $< \lfloor \frac{q-r}{2} \rfloor$, since $f(L(Z)) \in RS_q(q, r+1)$.

$\mathbb{E}\{\#\text{errors on a line}\} = \delta q$, so by Markov's inequality,

$$\mathbb{P}\left\{\#\text{errors on a line} \geq \left\lfloor \frac{q-r}{2} \right\rfloor\right\} \leq \frac{\delta q}{\lfloor \frac{q-r}{2} \rfloor} \leq \frac{2\delta q}{q-r-1}$$

NOTE: If $\delta < \frac{1}{2}(1 - r/q) = \frac{1}{2}\,\mathrm{dist}(RM_q(2,r))$, then the failure probability above is interesting, otherwise it reads "with prob. ≥ 0."

For example:

$N = q^2$ here

COR. $RM_q(2, r = q/2) \subseteq \mathbb{F}_q^N$, is a $(Q = \sqrt{N}, \delta, 4\delta)$-LCC for any $\delta < \frac{1}{4}$. The rate is $\approx 1/8$ and the distance is $1/2$.

We can do EXACTLY the same thing with $m > 2$.

LARGE-but-CONSTANT m:

Then we get $Q = q = N^{1/m}$, since $N = q^m$.

However, as $m \uparrow$ then the rate $\downarrow$. Recall $R = \binom{q+m}{m} / q^m \leq \left(\frac{c}{m}\right)^m \to 0$ as $m \to \infty$.

But this does give us a constant-rate code w/ $Q = N^{1/100}$ (say).

EVEN LARGER m:

Choose $m = q/\log(q)$.

Then $N = q^{q/\log(q)} = 2^q$ so $Q = q = \log(N)$

But the rate is even worse, and in fact goes to 0 like $1/\mathrm{poly}(n)$.

This simple construction is the state-of-the-art for $\log(n)$ queries. Can you do better???

So far, we have seen how to use RM codes to get:

Q	n (as a function of k)	Code
2	$n = \Theta(2^k)$	$RM_2(m, 1)$
$\log(n)$	$n = \mathrm{poly}(k)$	$RM_q(m, r)$ for $m \approx q/\log(q)$, $q > r$
n^{ε}	$n = \Theta_{\varepsilon}(k)$	$RM_q(m, r)$ for $m = 1/\varepsilon$, $q > r$
$\sqrt{n}$	$n = 8k$	$RM_q(2, q/2)$

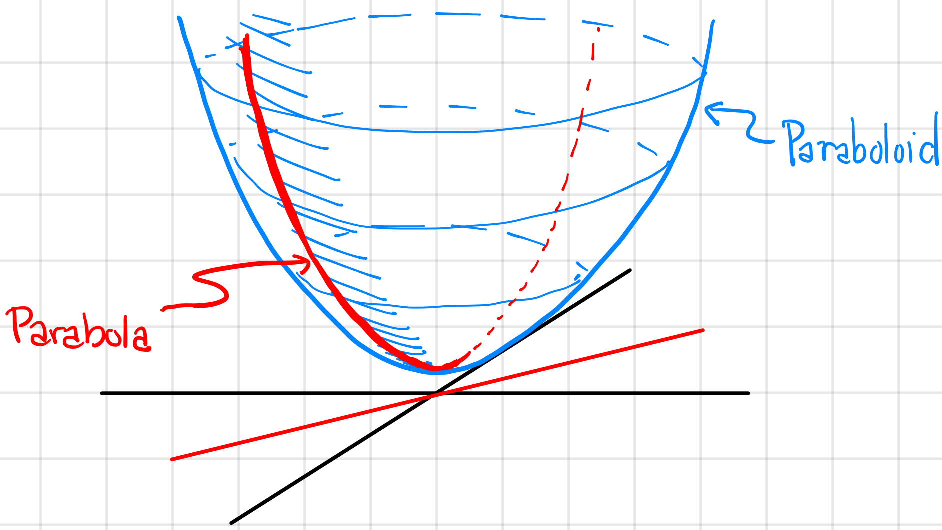
All of these have pretty low rate. Could we get an LCC with rate $\to 1$?

For $Q = n^{\varepsilon}$ (and even a bit smaller), the answer is YES.

There are several constructions. Here's a sketch of one based on RM codes.

(2) HIGH-RATE LCCs.

The thing we needed from RM codes are that restrictions to lines are low-deg polys.



To make the rate better, we might try

$$\mathcal{C} = \left\{ (f(\alpha_1), \ldots, f(\alpha_{q^m})) : f \in \mathbb{F}_q[X], \text{ AND } \deg(f(L(z))) \le r \ \forall \text{ lines } L \right\}$$

This would be a win as long as $|C| \geq |RM_q(m,r)|$, aka, as long as there are high-degree polynomials whose restrictions to lines are low-degree.

QUESTION. Does there exist a polynomial $f: \mathbb{F}_q^m \to \mathbb{F}_q$ of degree $> r$ so that, $\forall$ lines $L: \mathbb{F}_q \to \mathbb{F}_q^m$, $\deg(f(L(z))) \leq r$? (for $r < q-1$)

This means $\exists\, g(z)$ w/ $\deg(z) \leq r$ s.t. $g(\lambda) = f(L(\lambda))$ $\forall \lambda \in \mathbb{F}_q$

ANSWER.

Over $\mathbb{R}$ or $\mathbb{C}$: NO. (fun exercise!)

Over $\mathbb{F}_p$, for prime p: NO. (See [Rubinfeld-Sudan '96])

Over $\mathbb{F}_q$, and $q > 2 \cdot r$: NO. (" " ")

Over $\mathbb{F}_q$, and $q \approx r(1+\epsilon)$: YES, and there are LOTS of them.
[Guo, Kopparty, Sudan '13]

EXAMPLE. Consider $f(X,Y) = X^2 Y^2$ over $\mathbb{F}_4$.

The degree of f is 4.
Any restriction of f to a line is equivalent to a polynomial of deg $\leq 3 = q-1$. (Not too helpful).

CLAIM $\forall$ lines $L: \mathbb{F}_4 \to \mathbb{F}_4^2$, $\deg(f(L(z))) \leq 2$.

pf. Say $L(z) = (\sigma Z + \alpha, \tau Z + \beta)$.

$$
\begin{aligned}
f(L(Z)) &= (\sigma Z + \alpha)^2 (\tau Z + \beta)^2 \\
&= (\sigma^2 Z^2 + \alpha^2)(\tau^2 Z^2 + \beta^2) && [(a+b)^2 = a^2 + b^2 \text{ in } \mathbb{F}_{2^t}] \\
&= \sigma^2\tau^2 Z^4 + (\alpha^2\tau^2 + \sigma^2\beta^2) Z^2 + \alpha^2\beta^2 && [\text{algebra}] \\
&\equiv (\alpha^2\tau^2 + \sigma^2\beta^2) Z^2 + \sigma^2\tau^2 Z + \alpha^2\beta^2. && [\alpha^4 = \alpha \text{ in } \mathbb{F}_4]
\end{aligned}
$$

That's just one example, but it turns out there are actually LOTS, enough so that

$$C = \{ (f(\alpha_1), \dots, f(\alpha_{q^m})) : f \in \mathbb{F}_q[X], \text{ AND } \deg(f(L(z))) \le r \ \forall \text{ lines } L \}$$

has $|C| \geq q^{(1-\epsilon)\cdot(q^m)}$, aka $\mathrm{RATE}(C) \geq 1-\epsilon$.

C is called a "LIFTED CODE."

Thm (Guo, Kopparty, Sudan)

$\forall m > 0$, $q = 2^t$, $\forall \varepsilon > 0$, $\exists \varepsilon' > 0$ s.t. the set

$$S = \{ f : \mathbb{F}_q^m \to \mathbb{F}_q \mid f \text{ has degree} \le (1-\varepsilon')\cdot q \text{ restrictions to ALL lines} \}$$

has $\dim(S) \geq (1-\varepsilon)\cdot q^m$

COR. $\forall \ \varepsilon, \alpha > 0$, $\exists \ \delta > 0$ and $\gamma > 0$ s.t. there exists a family of codes $C \subseteq \mathbb{F}_q^n$ so that C is a $(n^\alpha, \delta, \gamma)$-LCC of rate $1-\varepsilon$.

(One can do a bit better than this: see [Kopparty, Meir, Ron-Zewi, Saraf, 2015].)

RECAP: • RM codes have nice local structure
• They are LCCs with $Q = 2, \log(n), n^{1/100}$, although the rate gets bad.
• To get rate $1-\varepsilon$ with $Q = n^{1/100}$, we can "lift" RM codes.

• In general, there are TONS of open questions about LCCs!

QUESTIONS TO PONDER

1. Can you show that k must be at least $n^{2+\epsilon}$ for 3-query LCC's?
2. Can you beat RM codes for $Q = \log(n)$?
3. Can you do anything with the Hadamard code when $\delta > 1/4$?