CS265/CME309, Winter 2022

Class 1: Agenda and in-class Questions

1 Welcome!

Welcome to CS265/CME309!

1.1 Introductions and Course Logistics

- All logistics info can be found on http://web.stanford.edu/class/cs265

2 Polynomial Identity Testing

2.1 Group work

- Find a small group to work with in one of the Breakout Rooms
  - The breakout rooms are in the room list on the left on OhYay. You might have to click the little "map" button on the upper right if you don't see the room list.
  - Each breakout room has a few polls in it. You can ignore those for now, but fill them out when we ask you to :)
  - If you're not in a position to interact right now, go to the "Quiet Room" and work on these problems on your own.
- If you have questions during group work, please flag one of the teaching team down!
  - Ask in chat (either to everyone or directly to the course staff).
  - Come to the "Teaching Staff" room and ask us.
  - Say in the chat that you have a question and what your room number is, and one of us will stop by.
  - We may be circulating.

Group Work

1. First, introduce yourselves to each other. What year/program are you in? What class/activity/etc are you most excited about for this quarter?
2. Quietly read the following definition.

   A multivariate polynomial $f(x_1, \dots, x_m)$ is *identically zero* if all its coefficients are zero. For example, the polynomial $f(x_1, x_2) = (x_1 + 1)(x_2 + 1) -$

$1 - x_1x_2 - (x_1 + x_2)$ is identically zero because when you expand it out, all of the terms cancel.

Now, work on the following questions with your group.

3. Which of the following two polynomials are identically zero?

$$f(x, y) = (x - y)^2 + 2xy + (x + y)^3 - y(3x^2 + y(3x + y + 1)) - (x + 1)x^2$$

$$g(x, y) = (x - 2y)^2 + xy + (x + y)^3 - y(3x^2 + y(3x + y + 1)) - (x + 1)x^2$$

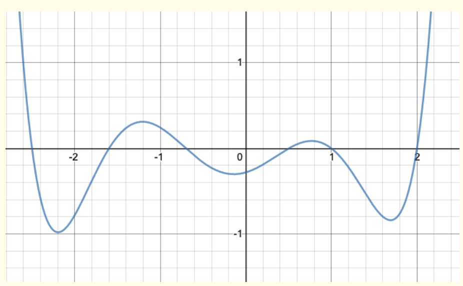
4. Suppose I were to give you a polynomial $f(x_1, x_2, \ldots, x_n)$ of total degree[a] n and with n variables and ask you if it is identically zero or not. How long, asymptotically, would it take you in the worst case if you were to do this in the straightforward way, by expanding out every term?
   (a) $O(n)$
   (b) polynomial in n
   (c) $2^{\Omega(n)}$

5. At the end of this group work, we are going to challenge you with two polynomials f and g, and ask you which is identically zero. You're going to have **one minute** to answer. Think now about an efficient way to answer this challenge.

   As part of your strategy, you may use a basic calculator (eg, `https://www.google.com/search?q=calculator`), but dumping the expression into WolframAlpha for it to plot or simplify (or something like that) is cheating.

   Hint: *Remember that this is a class on randomized algorithms. Can you think of a randomized strategy?*

   Hint: *You might take inspiration from the univariate case. Here's the graph of a polynomial $g(x)$ that is* not *identically zero. What is true about the values $g(x)$ for most choices of x?*



6. Would your strategy still work if Mary and Greg know it ahead of time? That is, if we know your strategy — but not necessarily the outcome of any randomness in the strategy — could we come up with polynomials f, g that would foil your strategy?

If your strategy would fail if we know it ahead of time, try to come up with another strategy that would (probably) succeed!

7. If you've finished all of the above, try to come up with an efficient *deterministic* strategy that would succeed at this task.

[a]the *total degree* of a monomial $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ is $\sum_{i=1}^{n} a_i$.

Group Work: Solutions

4. f is identically zero, and g is not.

5. Naively, this takes time $2^{\Omega(n)}$. That's because there are $\binom{2n}{n} = 2^{\Omega(n)}$ possible monomials of degree n, so I need to compute at least that many coefficients to check.

6. There are lots of strategies that would work! Some good ideas include:

   - Plug in 10000 (or some other big number) for all of the variables and see if you get zero. Since nonzero polynomials go to $\pm\infty$ as the inputs $x_i \to \infty$, and since Mary and Greg surely wouldn't choose a polynomial with super tiny coefficients, $f(10000, 10000, 10000, \ldots, 10000)$ will probably be non-zero if f is.
   - Compute just the constant term, or the coefficient on x_1^n. Both of those are pretty easy to compute just by looking at the function. If they're not zero, then say the polynomial is not zero. Again, if Mary and Greg don't know that you are going to be looking at the constant term (or whatever) if they choose a non-zero polynomial it probably won't have a zero constant term.
   - Choose a random evaluation point $(Z_1, Z_2, \ldots, Z_8)$ (say, according to a Gaussian). If $f(Z_1, \ldots, Z_8)$ is not zero, say that f is not identically zero. Otherwise, guess that f is identically zero.

     How likely is this to be correct? If f is identically zero, we will always be correct. If f is *not* identically zero, then we claim that we'll find a nonzero value of it with probability 1. Intuitively, the set of zeros of any degree-8 polynomial is a very small set—it has measure zero in $\mathbb{R}^8$—so no matter what nonzero polynomial is chosen, we will almost always hit a nonzero value.

     Notice that we can't *actually* choose a random real number to test our polynomial at — it would take an infinite amount of time to write that number down. Instead, we would *discretize*, and, say, choose a random element from some finite set that's fixed ahead of time. It turns out that this still works quite well, as we will see in the next part of the agenda!

   All of these are good ideas (and there are more good ideas!) However, the first two are not robust if I know them ahead of time—I could cook up some "bad" examples for them. The last one is robust!

8. This is actually an open question! (Sorry :))

2.2 Challenge and Discussion!

- Challenge time! Go to `http://PollEv.com/mkwoot` and get ready for your challenge!
- After that's done, refresh `http://PollEv.com/mkwoot` and describe your solution. If you see another group's solution that is similar to yours (or if you just like it), upvote it!
- A bit of lecture on Polynomial Identity Testing

3 What is a randomized algorithm?

- A bit of lecture about the basic framework for randomized algorithms, if time. (In case we don't get to this, there's also a YouTube video you can watch later – link on course website).

4 Wrap Up

Before next time:

- Watch the two short videos for Class 2 on YouTube (link on course website).
- Do the quiz on Gradescope.