CS265, Winter 2022

Class 11: Agenda, Questions, and Links

1 Announcements

- HW5 was due today, HW6 due next Wednesday!

2 Recap/Questions?

Any questions from the minilectures and/or the quiz (second moment method and LLL)?

3 Practice with the LLL

Recall the k-*SAT* problem. There are n variables $x_1, \ldots, x_n$. We consider clauses that looks like $(x_{i_1} \vee x_{i_2} \vee \overline{x_{i_3}} \vee \cdots \vee x_{i_k})$; that is, a clause is the OR of k literals. **For today, assume that each clause has k distinct variables that appear in it.** We have a formula φ that is the AND of m clauses. We would like to know: is φ satisfiable? That is, is there a way to assign values to the variables $x_1, x_2, \ldots$ so that φ evaluates to TRUE?

Group Work

Suppose that each variable x_i is in at most t clauses, for some parameter t that will depend on k and that you'll work out in this problem. Apply the LLL to get a statement like the following:

> Suppose that each variable is in at most t clauses of φ. Then φ is satisfiable.

(You should try to get t to be as large as possible. It's not hard to see that the statement above is true if, say, $t = 1$, but you should get a value of t that grows with k.)

Hint: *Recall that to apply the LLL, you need to define a probability distribution and a set of "bad" events. We set up this example in the minilecture video, we just didn't work out the conclusion. In the set-up of the video, we considered the probability distribution to correspond to assigning TRUE/FALSE to each variable $x_1, \ldots, x_n$ independently with probability $1/2$ each, and we defined the bad event A_i to be the event that clause i is not satisfied.*

3.1 More Practice with LLL and Mutual Independence

Consider a set of equations over variables $x_1, \ldots, x_n$, where each equation has the form $a_1 x_{i_1} + a_2 x_{i_2} + \ldots + a_r x_{i_r} \equiv a_{r+1} \mod 17$, for some r (that might vary from equation to equation) and set of coefficients $a_1, \ldots, a_r \in \{1, 2, \ldots, 16\}$, and $a_{r+1} \in \{0, \ldots, 16\}$. Additionally, suppose that each variable, x_i, occurs in at most 4 equations.

Group Work

Prove that there exists an assignment to the variables such that *none* of the equations are satisfied.

Hint*: Recall that because 17 is prime, for any $a \in \{1, \ldots, 16\}$ and any $b \in \{0, \ldots, 16\}$, the equation $ax \equiv b$ has a unique solution for $x \in \{0, \ldots, 16\}$.*

Hint*: It might be helpful to go back to the definition of mutual independence when arguing about the value of d when applying the LLL.*

4 Practice with the second moment method

In a graph $G = (V, E)$, say that a vertex v is **isolated** if it has no neighboring vertices.

Group Work

Let $G \sim G_{n,p}$ be a random graph where each edge is present independently with probability p, where $p = \frac{c \ln n}{n}$ for some constant $0 < c < 1$.

1. Use the Second Moment Method to show that, with probability at least $1 - o(1)$, there is some isolated vertex in G.

   For this exercise, feel free to use the approximation $e^{-x} \approx 1 - x$ when x is small as an equality without worrying about it.

   Hint*: Consider the random variable X that is the number of isolated vertices in G, and recall that the second moment method says that $\Pr[X = 0] \leq \frac{\mathrm{Var}[X]}{(\mathbb{E}X)^2}$.*

   Hint*: When computing the variance of X, you may want to consider the following question: given two distinct vertices u, v of G, what is the probability that* both *u and v are isolated?*

2. If you finish the previous part, what statement can you make about the case that $c > 1$?