Class 16

Practice with Azuma-Hoeffding bound

Group Work I

Vertex exposure martingale

Let $A = \chi(G_{n,p})$

Consider the Doob martingale $\mathbb{E}[A \mid X_1, X_2, \dots, X_i] = Z_i$

vertex exposure

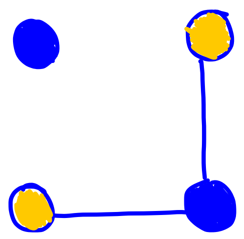
X_i = neighborhood of vertex i

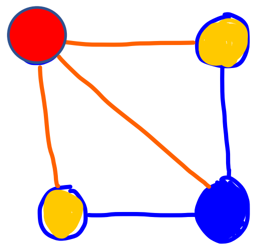
Applying Azuma-Hoeffding

<u>CLAIM</u> $\left| \mathbb{E}[A \mid X_1, \ldots, X_{i-1}] - \mathbb{E}[A \mid X_1, \ldots, X_i] \right| \leq 1$

Intuitively, changing one vertex can't change $\chi(G)$ by more than 1,

since you'd only change the color of that vertex.





CLAIM $\left| \mathbb{E}[A \mid X_1, \dots, X_{i-1}] - \mathbb{E}[A \mid X_1, \dots, X_i] \right| \leq 1$

More formally, for any choice $x_1, \dots, x_n$ for $X_1, \dots, X_n$:

$$\mathbb{E}\left[A \mid X^{(\leq i-1)} = x^{(\leq i-1)}\right] - \mathbb{E}\left[A \mid X^{(\leq i)} = x^{(\leq i)}\right]$$

$$= \mathop{\mathbb{E}}_{\substack{y_i \sim X_i \\ x^{(>i)} \sim X^{(>i)}}} \left[\mathbb{E}\left[A \mid x^{(\leq i-1)}, y_i, x^{(>i)}\right] - \mathbb{E}\left[A \mid x^{(\leq i-1)}, x_i, x^{(>i)}\right] \right]$$

$$\leq \sup_{y_i, x^{(>i)}} \left[\mathbb{E}\left[A \mid x^{(\leq i-1)}, y_i, x^{(>i)}\right] - \mathbb{E}\left[A \mid x^{(\leq i-1)}, x_i, x^{(>i)}\right] \right]$$

CLAIM $\left| \mathbb{E}[A \mid X_1,\dots,X_{i-1}] - \mathbb{E}[A \mid X_1,\dots,X_i] \right| \leq 1$

More formally, for any choice $x_1,\dots,x_n$ for $X_1,\dots,X_n$:

$$\mathbb{E}[A \mid X^{(\leq i-1)} = x^{(\leq i-1)}] - \mathbb{E}[A \mid X^{(\leq i)} = x^{(\leq i)}]$$

$$\leq \sup_{y_i,\, x^{(>i)}} \Big[\mathbb{E}[A \mid \underbrace{x^{(\leq i-1)}, y_i, x^{(>i)}}_{\text{this completely determines a graph}}] - \mathbb{E}[A \mid \underbrace{x^{(\leq i-1)}, x_i, x^{(>i)}}_{\text{this determines a graph that only differs at one vertex.}}] \Big]$$

these $\mathbb{E}$'s are not doing anything

$$\leq \max_{\substack{G,G' \text{ differ} \\ \text{on only one} \\ \text{vertex nbhd}}} \left| \chi(G) - \chi(G') \right| \leq 1$$

Applying Azuma-Hoeffding

CLAIM $\left| \mathbb{E}[A \mid X_1, \dots, X_{i-1}] - \mathbb{E}[A \mid X_1, \dots, X_i] \right| \leq 1$

$$\mathbb{P}\left[|Z_n - Z_0| \geq \lambda \right] \leq 2e^{-\lambda^2 / 2\sum_i c_i^2}$$

$Z_n = \mathbb{E}[A \mid \text{whole graph}] = A$

$Z_0 = \mathbb{E}[A]$

$c_i = 1 \;\forall i$

$$\mathbb{P}\left[|A - \mathbb{E}A| \geq \lambda \right] \leq 2e^{-\lambda^2/2n}$$

$$\mathbb{P}\left[|A - \mathbb{E}A| \geq c\sqrt{n} \right] \leq 2\exp(-c^2/2)$$

Notice that we don't know what $\mathbb{E}[A]$ is!!

But whatever it is, A is pretty concentrated around it.

Edge exposure martingale

If we used the EDGE EXPOSURE martingale instead, we still naively would have $|Z_i - Z_{i-1}| \leq 1$

(one edge can change $\chi(G)$ by 1)

but now we get $\mathbb{P}\left[|A - \mathbb{E}A| > c\sqrt{n}\right] \leq 2\exp\left(-\frac{(c\sqrt{n})^2}{2\cdot\binom{n}{2}\cdot 1}\right)$

$= 2\exp(-\Theta(c^2/n))$

Much worse!

Group Work II

Gambling Game

- At time t, you can bet <u>any</u> amount in $[0, B]$
- Flip a fair coin:
  - heads, you win your bet
  - tails, you lose your bet

You are allowed to be in debt.

<u>NOTE</u> This is <u>NOT</u> a Markov chain, since how much you bet can depend on EVERYTHING so far.

Setting up a martingale

Z_t = amt of money at time t.

X_i = coin flip at time t

This is legit if your betting strategy is deterministic, since

- Z_t is a fn of $X_1, \dots, X_t$
- $\mathbb{E}[Z_t \mid X_1, \dots, X_{t-1}] = \frac{1}{2}(+\mathrm{bet}(X_1, \dots, X_{t-1})) + \frac{1}{2}(-\mathrm{bet}(X_1, \dots, X_{t-1})) + Z_{t-1}$

$= 0 + Z_{t-1}$

Azuma-Hoeffding

$$\mathbb{P}\left[\,|Z_n - \underbrace{Z_0}_{\text{zero}}| > \lambda\,\right] \le 2\exp\left(\frac{-\lambda^2}{8\,n\,B^2}\right)$$

since $|Z_i - Z_{i-1}| \le 2B$ always.

$$\mathbb{P}\left[\,|Z_n| > c \cdot B \cdot \sqrt{n}\,\right] \le 2\exp\left(-\frac{c^2}{8}\right)$$

So if B is small, you are VERY unlikely to win more than $O(\sqrt{n})$ \$ in B rounds.

What if the strategy is randomized?

$\{Z_t\}$ is no longer a Martingale w/r/t $\{X_t\}$,

since Z_t is NOT a function of $X_1, \ldots, X_t$.

Define $Y_t = \left(X_t, \text{ how much you bet at time } t+1 \right)$

Now $\{Z_t\}$ is a martingale w/r/t $\{Y_t\}$ and the same analysis holds.

Recap

- We got some practice applying Azuma-Hoeffding.
- General trend:
  - If you are looking at the Doob martingale for some r.v. A w/r/t $X_1, X_2, \ldots$
  - and A doesn't depend too much on any one of the $X_i's$
  - then Azuma-Hoeffding does well