

# CS265/CME309: Randomized Algorithms and Probabilistic Analysis

## Lecture #2: Linearity of Expectation, Karger's Min-Cut Algorithm, and Quicksort with Random Pivot.

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### 1 Linearity of Expectation

I expect that you've seen linearity of expectation before, but it can be so useful that it's worth writing it out explicitly. Linearity of expectation says that if you have random variables  $X_1, \dots, X_n$ , and scalars  $a_1, \dots, a_n$ , then

$$\mathbf{E}\left[\sum_i a_i X_i\right] = \sum_i a_i \mathbf{E}[X_i].$$

Let's do some quick examples. The first one is a bit contrived, and the second is quite useful!

**Example 1.** Suppose that  $G$  is a  $d$ -regular bipartite graph, with  $n$  vertices on each side. Suppose that the left hand vertices are colored half blue and half red. Let  $X$  be a random variable that is defined as the number of red neighbors of a randomly chosen vertex on the right. What is  $\mathbf{E}[X]$ ? Let  $v_i$  be the  $i$ 'th vertex on the left, and let  $X_i = 1$  if the randomly chosen vertex on the right is attached to  $v_i$ . We can use linearity of expectation to compute:

$$\begin{aligned} \mathbf{E}[X] &= \mathbf{E}\left[\sum_{i=1}^n \mathbf{1}[v_i \text{ is red}] X_i\right] \\ &= \sum_i \mathbf{1}[v_i \text{ is red}] \mathbf{E}[X_i] \\ &= \sum_i \mathbf{1}[v_i \text{ is red}] \frac{d}{n}, \end{aligned}$$

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using the fact that since  $v_i$  has  $d$  neighbors, the probability that a random vertex on the right is one of them is  $d/n$ . But this simplifies to

$$\frac{d}{n} \cdot (\text{number of red vertices}) = \frac{d}{n} \cdot \frac{n}{2} = \frac{d}{2}.$$

Thus, the expected number of red neighbors of a random vertex is  $d/2$ . This makes intuitive sense, but without linearity of expectation (or some switching-of-summations that would end up being equivalent), it might be a bit tricky to prove.

**Example 2.** Suppose we are flipping a  $p$ -biased coin repeatedly. Let  $X$  be the number of flips necessary until you get heads. You may recognize  $X$  as a geometric random variable, and perhaps you know that  $\mathbf{E}[X] = 1/p$ . However, there are a few different ways to prove this, and the most straightforward one of writing out the definition of the expectation will result in a pretty messy computation. Here's a slick way to do it, which (sort of) uses linearity of expectation.

Suppose we flip our first coin. Either it comes up heads (probability  $p$ ), or it comes up tails (probability  $1-p$ ). If it comes up heads, the number of coins we need to flip to get heads is 1. If it comes up tails, the number of coins we need to flip to get heads is 1, plus...however many more times we need to flip a  $p$ -biased coin to get heads. This logic allows us to write the equation:

$$\mathbf{E}[X] = \mathbf{E}[p \cdot 1 + (1-p) \cdot (1 + X)] = p + (1-p)(1 + \mathbf{E}[X]).$$

Solving for  $\mathbf{E}[X]$ , we see that  $\mathbf{E}[X] = 1/p$ , as desired.

To see a third (optional) example that you might have already seen, go to the end of the lecture notes and check out the discussion of QuickSort.

## 2 Karger's Min-Cut Algorithm

First, we'll discuss an incredibly elegant randomized algorithm for finding the minimum cut in a graph, due to David Karger from 1993 [1]. Given an undirected, unweighted graph, the min-cut problem asks us to partition the vertices into two sets, so as to minimize the number of edges that cross from one side of the partition to the other. Formally, a *cut* of a graph  $G = (V, E)$  is a partition of  $V$  into two disjoint, nonempty sets,  $S_1 \subset V$  and  $S_2 \subset V$  so that  $S_1 \cup S_2 = V$ . We say an edge  $(u, v)$  *crosses* a cut  $S_1, S_2$  if  $u \in S_1$  and  $v \in S_2$  (or the other way around). A *minimum cut* of  $G$  is one that minimizes the number of edges in  $G$  that cross the cut.

This fundamental problem (and its many variants, including partitioning into  $k > 2$  sets), have many applications, including clustering webpages and social graphs, as well as documents (say two documents have an edge between them if one references the other, or we could have a weighted edge corresponding to the Jaccard similarity between the sets of words used in the documents, for example...if you haven't come across Jaccard similarity before, maybe do a quick wikipedia search.). Similar problems are also used to segment images, assign computation to processors, etc.

In contrast to polynomial identity testing (from Lecture 1), where we do not know of any efficient (polynomial time) deterministic algorithm, min-cut *does* have efficient deterministic algorithms. Still, the randomized algorithm we present below is so clean and elegant, that it is worth including in the first week of this course. Additionally, with several tweaks (one of which we'll work out together in our class activities this week), the runtime of this randomized algorithm becomes competitive with the best deterministic algorithms.

## 2.1 High-Level Intuition

Karger's min-cut algorithm starts with the original graph, and iteratively reduces the number of vertices via a series of *edge contractions*. In each step, an edge is chosen uniformly at random, and then the two endpoints of that edge are merged into a single vertex, and all edges are preserved (we will allow multiple parallel edges between two vertices) except for the 'self-loop' that is created by the merge. The algorithm proceeds iteratively until there are only two vertices left—call them  $u_1$  and  $u_2$ —at which point the algorithm returns the partition (i.e. the cut) where all the vertices that were merged into  $u_1$  are in one set, and all the vertices that were merged into  $u_2$  are in the other set.

To see one intuition for why this algorithm might be expected to perform well, imagine that the original graph consists of two disconnected components. In this case, the above algorithm will (with probability 1) return the correct min-cut corresponding to these two disconnected components. Now imagine that the graph consists of two components (each of which has quite a lot of internal edges), where the two components are connected via a single edge (or, more generally, relatively few edges). The algorithm will be successful provided, at *every* step of the algorithm, it avoids selecting one of the edges that cross between the components. (Why? Well if the algorithm contract an edge that crosses between the components, then at least one of the vertices of the graph will have been merged into the “wrong” side of the cut.) If there is just a single edge crossing between these components, then we certainly don't expect it to be picked early in the algorithm (since there are so many other edges to choose from), and we will be able to argue that there will be a reasonable chance that we never contract that edge.

If you work out a few examples to try to understand this intuition, you might realize that, while there's a “reasonable” chance that we find the correct min-cut, this reasonable chance is not high. Fortunately, we can repeat the algorithm described above many times, independently, to boost the success probability. We'll show that “reasonable” is big enough that we don't have to repeat too many times.

## 2.2 The Algorithm

We begin by formally defining an *edge contraction*.

**Definition 2.1** Given a graph  $G = (V, E)$  with  $n$  vertices  $V = \{v_1, \dots, v_n\}$ , and an edge  $e \in E$  that connects vertices  $v_i, v_j$ , the graph resulting from *contracting* edge  $e$  will have  $n - 1$  vertices, namely  $V \setminus \{v_i\}$ , and edge set defined as follows: for every edge  $e' \in E$  that

does not have  $v_i$  as an endpoint,  $e'$  is an edge of the new graph. For every edge  $e' \in E$  that connects  $v_i, v_k$  for  $k \neq j$ , we add the edge  $(v_j, v_k)$  to the edge set.

The following figure illustrates what's going on: we merge the two endpoints of the edge, keeping all edges except self-loops.



Figure 1: Example of a graph before and after contracting the orange edge.

We are now ready to formally state the algorithm.

**Algorithm 1**

MIN-CUT ALGORITHM

Given a graph  $G = (V, E)$  with  $|V| = n$  vertices:

1. For  $i = 1$  to  $n - 2$ :
  - Choose one of the remaining edges of  $G$  uniformly at random, and contract that edge, so that  $G$  now has  $n - i$  vertices.
2. Return the partition corresponding to the final 2-vertex graph (each of these two vertices represents a subset of  $V$  corresponding to all the vertices that eventually became merged).

At this point, you might be wondering how efficiently one can actually perform an edge contraction. Naively, this might require a fair bit of book-keeping, as much as the degree of the vertices being contracted (which could be as large as  $n$ ). Still, there are some clever data structures can be used to help with this; it turns out that one can complete one run of Karger's algorithm in time  $O(|E|)$ . For the remainder of these notes, however, we'll ignore how we actually implement a contraction, and instead focus on understanding the probability that the algorithm returns the minimum cut.

**Theorem 2.2** *The probability that the above algorithm returns a minimum cut is at least  $\frac{2}{n(n-1)} \geq 2/n^2$ .*

The proof of the above theorem relies on the following lemma, which argues that, at some intermediate stage of the algorithm, as long as we haven't yet contracted an edge that crosses the minimum cut of the original graph, then that minimum cut will also be a minimum cut of the graph we currently have.

**Lemma 2.3** *Let  $S_1, S_2 \subset V$  be a minimum cut. Suppose we contract an edge  $(u, v)$  that does not cross the cut. Then  $S_1, S_2$  will be a minimum cut of the new graph resulting from the contraction.*

*Proof:* For any cut in the new graph (after the contraction) that cuts  $s$  edges, that cut corresponds to a partition in the previous graph that also cuts  $s$  edges. Hence the number of edges cut by the smallest cut cannot decrease when we contract an edge. Finally, if we contract an edge that does not cross from  $S_1$  to  $S_2$ , then partition  $S_1, S_2$  still exists in the new graph, and cuts the same number of edges as before, and hence must still be a min-cut in the new graph. ■

*Proof of Theorem 2.2.* Although there might be more than one minimum cut, we will actually prove that, for any minimum cut,  $C$ , the probability the algorithm returns that particular minimum cut  $C$  is at least  $\frac{2}{n(n-1)}$ . So let  $C$  be some minimum cut.

By Lemma 2.3, the algorithm will return  $C$  if, and only if, each of the  $n - 2$  edge contractions contract an edge that does not cross the cut  $C$ . Let  $E_i$  denote the event that we do not contract an edge crossing  $C$  in the  $i$ th step of the algorithm. We have the following:

$$\begin{aligned} Pr[\text{output } C] &= Pr[E_1 \wedge E_2 \wedge \dots \wedge E_{n-2}] \\ &= Pr[E_1] \cdot Pr[E_2|E_1] \cdot Pr[E_3|E_1, E_2] \cdot \dots \cdot Pr[E_{n-2}|E_1, E_2, \dots, E_{n-3}]. \end{aligned}$$

Letting  $k$  denote the number of edges crossing  $C$  in the original graph, we trivially have that  $Pr[E_1] = 1 - \frac{k}{\text{total number edges}}$ . Since  $C$  is, by assumption, a minimum cut, the degree of every vertex must be at least  $k$ , which implies that the total number of edges must be at least  $nk/2$ . This is because each of the  $n$  vertices has degree at least  $k$ , so the sum of degrees of the vertices will be at least  $nk$ , but this double-counts all the edges, hence the factor of 2. Hence we have that

$$Pr[E_1] \geq 1 - \frac{k}{nk/2} = 1 - \frac{2}{n} = \frac{n-2}{n}.$$

From Lemma 2.3, conditioned on  $E_1, \dots, E_{i-1}$ , we now that  $C$  is a minimum cut of the graph before the  $i$ th contraction, and hence

$$Pr[E_i|E_1, E_2, \dots, E_{i-1}] \geq 1 - \frac{2}{n-i+1} = \frac{n-i-1}{n-i+1}.$$

Combining these terms we conclude

$$Pr[\text{output } C] \geq \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdot \dots \cdot \frac{2}{4} \cdot \frac{1}{3} = \frac{2}{n(n-1)},$$

where the last equality follows from observing that all numerators and denominators cancel except the denominators of the first two terms, and numerators of the last two terms. ■

Should we be happy with a probability of success of  $\approx 2/n^2$ ? If we simply repeat the above algorithm  $t = cn^2/2$  times, and return the smallest cut that was found in any of the  $t$

runs, then the probability of failure becomes at most  $(1 - \frac{2}{n^2})^{cn^2/2} \leq e^{-\frac{2}{n^2} \cdot \frac{cn^2}{2}} = e^{-c}$ , where we used the trick from Lecture 1 where we noted that  $1 - x \leq e^{-x}$ . So, this implies that if we want a probability of success of, say, 0.9, we would need to perform  $O(n^2 \cdot n)$  edge contractions— $n$  for each of the  $O(n^2)$  runs of the algorithm.<sup>1</sup>

How can we do better? One approach to improving this is based on the following intuition: suppose we were told that a given run of the algorithm was not successful. If we needed to guess which iteration destroyed the minimum cut, we would probably guess that it was one of the later iterations. After all, the probability that our first contraction preserves the minimum cut is at least  $\frac{n-2}{n} \approx 1$ , whereas, even if everything has gone perfectly, the probability (according to our pessimistic calculations) that the very last contraction is successful might only be  $1/3$ . This motivates the following idea: rather than repeating the entire algorithm, including all the work we have done during the first few iterations (which probably did not destroy the minimum cut), why not just re-do the last few edge contractions (with freshly chosen random choices for the edges to contract)? We'll explore this idea in more detail during our in-class exercises.

For even more improvements, check out [2], which gives a randomized algorithm for min-cut that runs in time  $\tilde{O}(m)$ .

**Note: The material below on QuickSort is not covered in the lecture videos or in class, and it's optional reading.**

### 3 Analysis of Quicksort with a Random Pivot

Many of you may have seen QuickSort and its analysis in CS161. We're not going to spend any time on it in class, but we've included some notes here since it's a nice reminder of the facts that:

1. Linearity of expectation is extremely powerful!
2. When analyzing the expectation of a random variable, it is often a good idea to represent that random variable as a sum of other random variables that are easy to analyze, and apply linearity of expectation.

If you haven't seen this analysis before, be sure to check it out; and even if you have, it's a great time for a refresher!

Recall the recursive Quicksort algorithm for sorting a list of  $n$  numbers. For clarity, we describe the algorithm and its analysis assuming that the  $n$  numbers are all distinct (no repetitions), though the algorithm and analysis naturally extend to the general case.

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<sup>1</sup>In terms of running time, since we can execute one run of Karger's algorithm in time  $O(|E|)$  with appropriate data structures, the running time is  $O(n^2|E|)$ .

**Algorithm 2**

QUICKSORT (WITH RANDOM PIVOT)

Given set/list of  $n$  distinct numbers,  $S = (x_1, \dots, x_n)$ :

1. If  $|S| = 0$  return the empty list.
2. Otherwise, select  $i$  uniformly at random from  $\{1, \dots, n\}$ .
3. Compare every element of  $S$  to  $x_i$ , forming two sets,  $S_<$  and  $S_>$  consisting respectively of the number less than  $x_i$ , and the numbers that are greater than  $x_i$ .
4. Return the list corresponding to the concatenation of  $Quicksort(S_<), (x_i), Quicksort(S_>)$ .

The above algorithm, in the worst case, might require almost  $n^2/2$  comparisons if the pivot that is chosen at every step of the recursion happens to be the minimum element of the set. Nevertheless, as we argue below, in expectation, the number of comparisons is at most  $2n \log n + O(n)$ , which is extremely good.

**Theorem 3.1** *The expected runtime of the above algorithm is at most  $2n \log n + O(n)$ .*

*Proof:* To compute the expected number of comparisons, we will simply apply linearity of expectation, and then analyze the expected number of times that every pair of inputs is compared. For convenience, assume that the input set contains the numbers  $z_1 < z_2 < \dots < z_n$ . First note that for any pair  $z_i, z_j$ , we either compare them 0 times, or once, since if we do compare them at some iteration of the recursion, that means that  $z_i$  or  $z_j$  was a pivot, and we will never compare that pair again, because the pivot is not included in the sets  $S_<$  or  $S_>$ . Given this, by linearity of expectation, we have

$$E[\# \text{ comparisons}] = \sum_{i=1}^{n-1} \sum_{j=i+1}^n Pr[z_i \text{ is compared to } z_j \text{ at any point during the algorithm}].$$

To analyze this, we just need to think about the probability that  $z_i$  and  $z_j$  ever get compared during the algorithm.

One slick way of analyzing this is as follows: at every recursive execution of the algorithm,  $z_i$  and  $z_j$  will both end up in the same set (either  $S_<$  or  $S_>$ ) until the first time that an element of the set  $R_{i,j} = z_i, z_{i+1}, \dots, z_{j-1}, z_j$  is chosen as the pivot. After the first time that happens,  $z_i$  and  $z_j$  will be split up, and will never be able to be compared to each other. Hence, the probability they are compared is exactly equal to the probability that, the first time a number in  $R_{i,j}$  is chosen, it happens to be either  $z_i$  or  $z_j$ . Hence this probability is

exactly  $\frac{2}{|R_{i,j}|} = \frac{2}{j-i+1}$ . The rest is just some tedious calculations:

$$\begin{aligned}
 E[\# \text{ comparisons}] &= \sum_{i=1}^{n-1} \sum_{j=i+1}^n \frac{2}{j-i+1} \\
 &= \sum_{i=1}^{n-1} \sum_{k=2}^{n-i+1} \frac{2}{k} = \sum_{k=2}^n \frac{2}{k} (n+1-k) \\
 &= -2(n-1) + (n+1) \sum_{k=2}^n \frac{2}{k} \\
 &< 2(n+1)(1 + \log n) = 2n \log n + O(n).
 \end{aligned}$$

To get from the second-to-last line to the last line, we used the fact that the harmonic sum  $\sum_{i=1}^n \frac{1}{i}$  is between  $\log n$  and  $1 + \log n$ , since  $\int_{x=1}^n \frac{1}{x} = \log n$ , and  $\int_{x=1}^n \frac{1}{x} < \sum_{i=1}^n \frac{1}{i} < 1 + \int_{x=1}^{n+1} \frac{1}{x}$ . ■

## References

- [1] David R Karger. Global Min-cuts in RNC, and Other Ramifications of a Simple Min-Cut Algorithm. In *Symposium on Discrete Algorithms (SODA)*, volume 93, pages 21–30, 1993.
- [2] David R Karger. Minimum cuts in near-linear time. *Journal of the ACM (JACM)*, 47(1):46–76, 2000.