Sampling-based Median

Finding the median of n things

- You may have seen an $O(n)$ time algorithm in CS161.
  - It was pretty complicated.
- Today: a simpler randomized algorithm (with optimal constant in the big-Oh runtime)!

Array S of n distinct numbers:

9	5	34	1	2	33	12	4	15	3	6	8	10	18	0

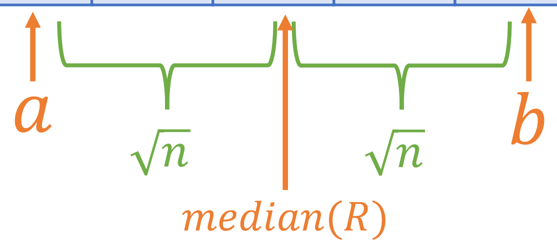
$n = 15$ here.

Choose a set R of size $n^{3/4}$ by drawing that many things uniformly at random, independently.

5	12	15	5	10	3	33

Sort R:

3	5	5	10	12	15	33



Find all the things in S between a and b (time $O(n)$), to form a list T:

9	5	12	15	6	8	10

If $|T| < 4n^{3/4}$, sort T:
(otherwise output FAIL)

5	6	8	9	10	12	15

- We can see in time $O(n)$ that there are 5 things in S less than a, and 3 things in S larger than b.
- The median is the 8'th smallest thing in S, which is the $8 - 5 = 3$'rd smallest thing in T.
- Return 8

If this calculation shows that the median is not in T, output FAIL.

Group work…

Solutions to group work

2. Suppose that:
   - With probability at least 0.9, the median of S is in T.
   - With probability at least 0.9, $|T| < 4t$.

- Then the algorithm returns the correct answer with probability at least 0.8.

Array S of n distinct numbers:

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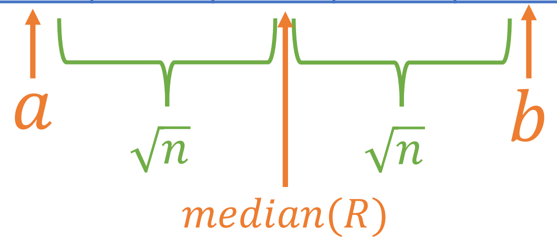

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If this calculation shows that the median is not in T, output FAIL.

Solutions to group work

2. Suppose that:
   - With probability at least 0.9, the median of S is in T.
   - With probability at least 0.9, $|T| < 4t$.

- Then the algorithm returns the correct answer with probability 0.8.

- If both events happen, then the algorithm never returns FAIL.
- If it doesn’t return FAIL, then it returns the right answer by construction.

Solutions to group work

3. The running time is $O(n)$ operations.....actually (3/2)n + o(n)

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$n = 15$ here.

Choose a set R of size $n^{3/4}$ by drawing that many things uniformly at random, independently.

5	12	15	5	10	3	33

$O(n^{3/4}) = o(n)$ operations.

Sort R:

3	5	5	10	12	15	33

$O\left(n^{\frac{3}{4}} \log\left(n^{\frac{3}{4}}\right)\right) = o(n)$ operations.

a $\sqrt{n}$ $median(R)$ $\sqrt{n}$ b

Find all the things in S between a and b (time $(3/2)n$), to form a list T:

5	9	12	15	6	8	10

$O(n)$

If $|T| < 4n^{3/4}$, sort T:
(otherwise output FAIL)

5	6	8	9	10	12	15

$O\left(n^{\frac{3}{4}} \log\left(n^{\frac{3}{4}}\right)\right) = o(n)$ operations.

- We can see in time $O(n)$ that there are 5 things in S less than a, and 3 things in S larger than b. $O(n)$
- The median is the 8'th smallest thing in S, which is the $8 - 5 = 3$'rd smallest thing in T. $O(1)$
- Return 8

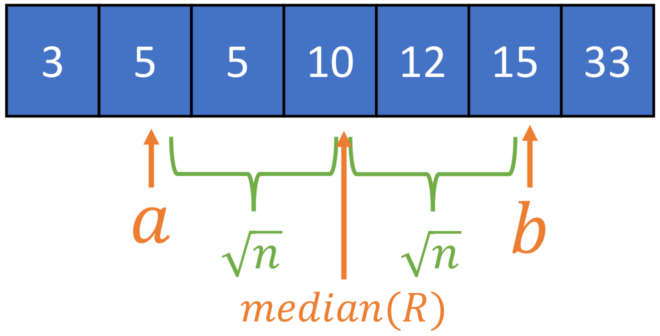
If this calculation shows that the median is not in T, output FAIL.

Solutions to group work

- Question 4: want to show that $median(S) \in T$ w.h.p.

Solutions to group work

Sorted version of R:

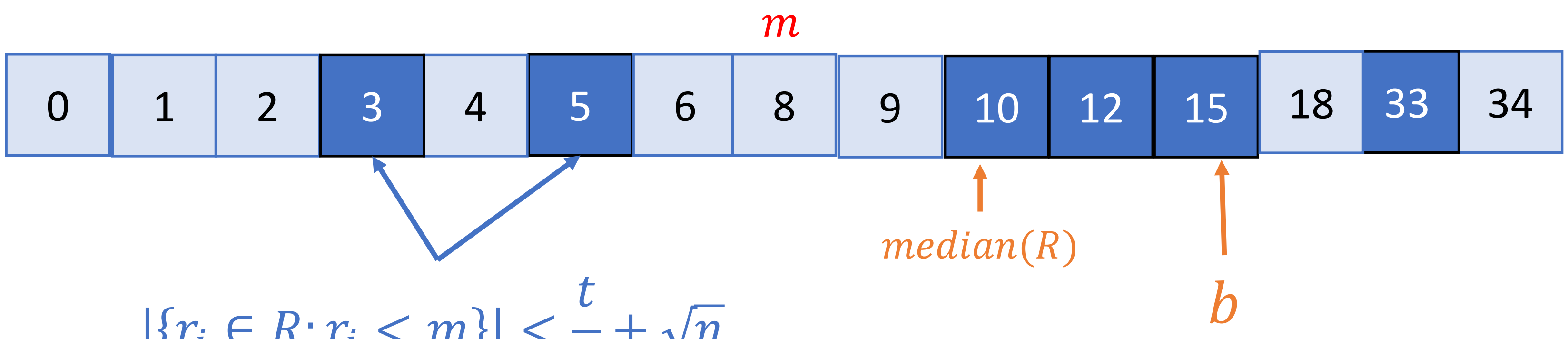


4a. Consider two events:

$$|\{r_i \in R: r_i < m\}| < \frac{t}{2} + \sqrt{n}$$

$$|\{r_i \in R: r_i > m\}| < \frac{t}{2} + \sqrt{n}$$

Sorted version of S:



$$|\{r_i \in R: r_i < m\}| < \frac{t}{2} + \sqrt{n}$$

$$\Rightarrow b \geq m$$

$\frac{t}{2} + \sqrt{n}$'th smallest thing in R.

Solutions to group work

4a. Consider two events:

$$|\{r_i \in R: r_i < m\}| < \frac{t}{2} + \sqrt{n}$$

$$\Rightarrow b \geq m$$

$$|\{r_i \in R: r_i > m\}| < \frac{t}{2} + \sqrt{n}$$

$$\Rightarrow a \leq m$$

- Then $a \leq m \leq b$, aka $m \in T$

Solutions to group work

4b. Let $X = |\{r_i \in R : r_i < m\}|$

- Then $X = \sum_i X_i$ where $X_i = 1$ iff $r_i < m$ and 0 otherwise, for $i = 1, \dots, t$
- $\mathbf{E}[X_i] = \Pr[r_i < m] \le \frac{1}{2}$,
- $\mathrm{Var}[X_i] \le \frac{1}{4}$

- $\Pr\left[\sum_i X_i \ge \frac{t}{2} + \sqrt{n}\right] \le \Pr[\sum_i (X_i - \mathbf{E}X_i) \ge \sqrt{n}] \le \frac{t/4}{n} = \frac{1}{4n^{1/4}} = o(1)$

Solutions to group work

4c. Consider two events:

$$|\{r_i \in R : r_i < m\}| < \frac{t}{2} + \sqrt{n} \quad\quad |\{r_i \in R : r_i > m\}| < \frac{t}{2} + \sqrt{n}$$

$$\Rightarrow b \geq m \quad\quad \Rightarrow a \leq m$$

- Then $a \leq m \leq b$, aka $m \in T$

Both have probability at least $1 - O\left(n^{-1/4}\right)$

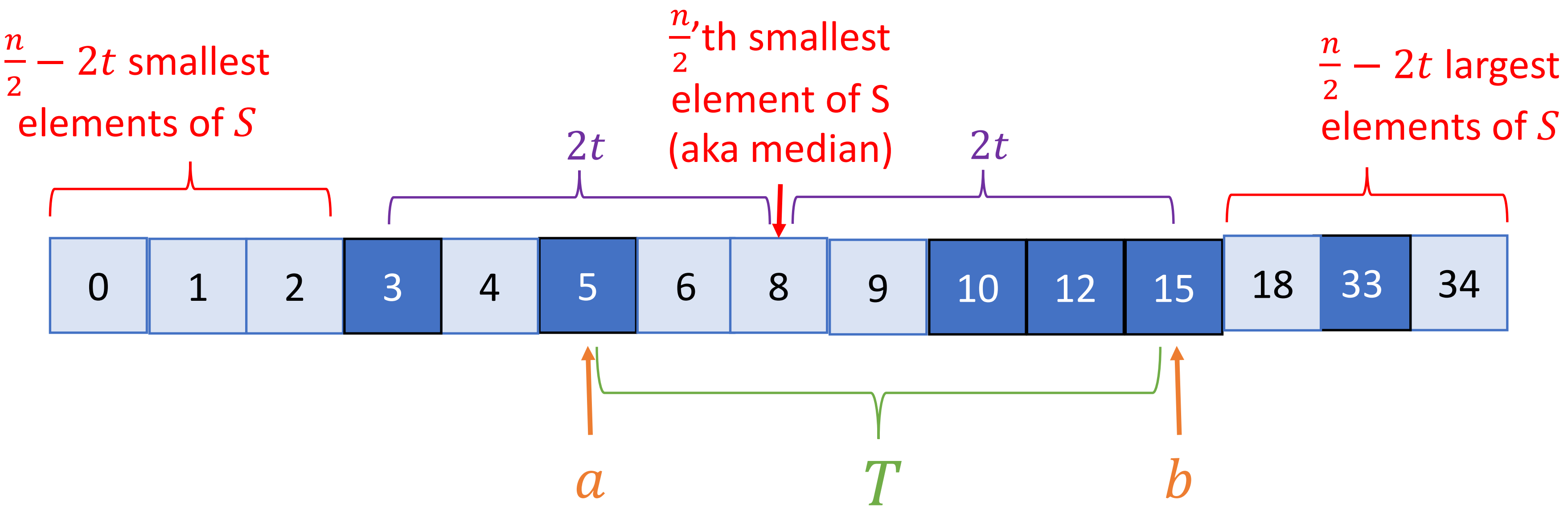
$$\Pr[m \in T] \geq 1 - O(n^{-1/4})$$

Solutions to group work

- Question 5: want to show that $|T| < 4t$ w.h.p.

Solutions to group work

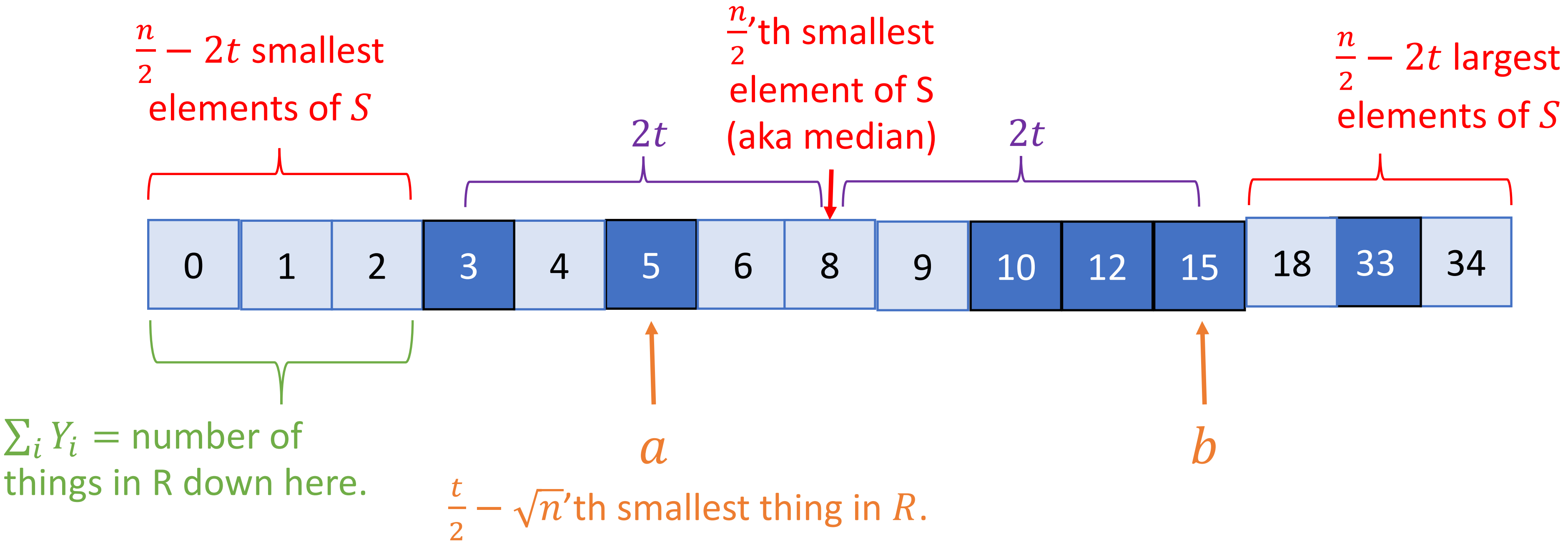
- 5(a). Say that a is not one of the $\frac{n}{2} - 2t$ smallest elements of S
- Say that b is not one of the $\frac{n}{2} - 2t$ largest elements of S



- Then $|T| < 4t$

Solutions to group work

- 5(b) Let $Y_i = 1$ iff r_i is in the $\frac{n}{2} - 2t$ smallest elements of S, 0 else



- $\sum_i Y_i \geq \frac{t}{2} - \sqrt{n} \iff a$ is among the $\frac{n}{2} - 2t$ smallest elements of S

- $\sum_i Y_i \geq \frac{t}{2} - \sqrt{n} \iff a$ is among the $\frac{n}{2} - 2t$ smallest elements of S

Solutions to group work

- 5(b) Let $Y_i = 1$ iff r_i is in the $\frac{n}{2} - 2t$ smallest elements of S, 0 else.
  - $\mathbf{E}Y_i = \frac{1}{2} - \frac{2t}{n} = \frac{1}{2} - \frac{2}{n^{1/4}}$

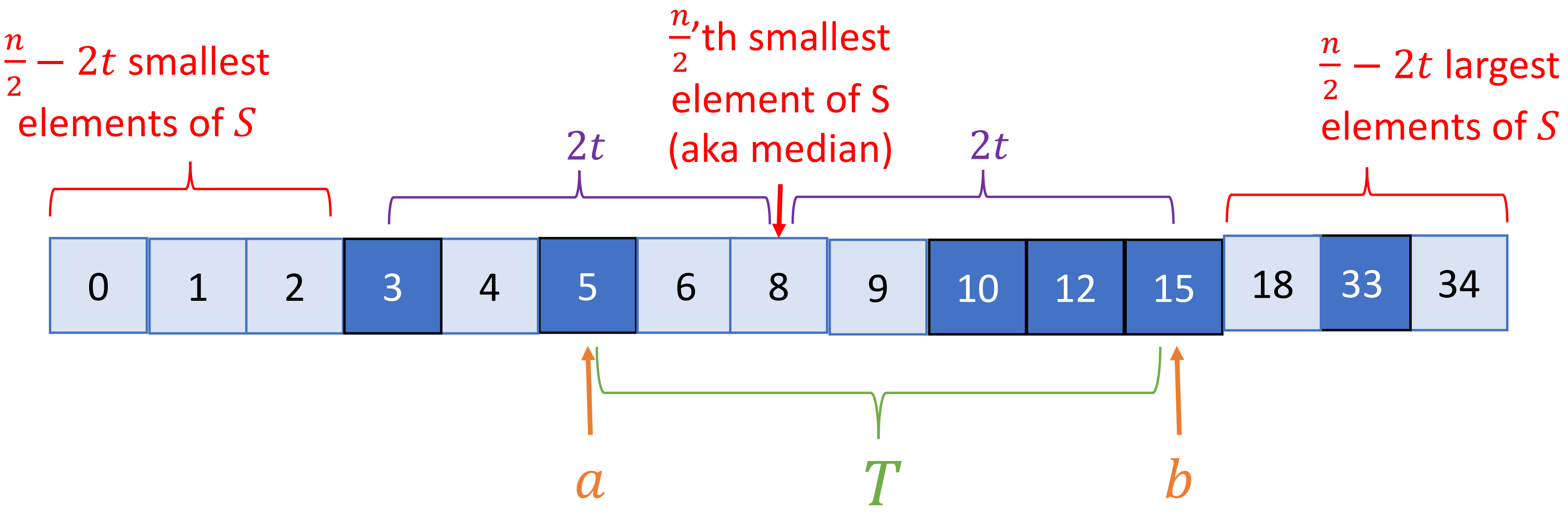
- $\Pr\left[\sum_i Y_i \geq \frac{t}{2} - \sqrt{n}\right] \leq \Pr\left[\sum_i (Y_i - \mathbf{E}Y_i) \geq \frac{2t}{n^{1/4}} - \sqrt{n}\right]$
- $= \Pr[\sum_i (Y_i - \mathbf{E}Y_i) \geq \sqrt{n}]$
- $\leq \frac{\mathrm{Var}[\sum_i Y_i]}{n}$
- $\leq \frac{t}{4n} = \frac{1}{4n^{1/4}} = o(1)$

$\mathrm{Var}[\sum_i Y_i] = \sum_i \mathrm{Var}[Y_i] \leq \frac{t}{4}.$

Solutions to group work

Both have probability at least $1 - O(n^{-1/4})$

- 5(c). Say that a is not one of the $\frac{n}{2} - 2t$ smallest elements of S
- Say that b is not one of the $\frac{n}{2} - 2t$ largest elements of S



- Then $|T| < 4t$

$\Rightarrow \Pr[|T| < 4t] \geq 1 - O(n^{-1/4})$

All together:

- Question 2: To show that this algorithm works whp, it's enough to show that :
  - whp, $median(S) \in T$
  - whp, $|T| < 4t$

- Question 4: whp, $median(S) \in T$
- Question 5: whp, $|T| < 4t$

- (And Question 2: it runs in time $O(n)$).