Class 10

Derandomization Techniques

Solutions to Warm-up 1

Let $G=(V,E)$ be a graph.

- Choose a random $S \subseteq V$

- $$\mathbb{E}\left[\#\text{edges that cross } (S,\bar{S})\right] = \sum_{\{u,v\}\in E} \underbrace{\mathbb{P}\left\{\begin{matrix}\{u,v\}\text{ crosses}\\ S,\bar{S}\end{matrix}\right\}}_{1/2} = \frac{|E|}{2}$$

$$\Rightarrow \exists \text{ cut so that } \left(\begin{matrix}\#\text{edges}\\ \text{crossing cut}\end{matrix}\right) \geq \frac{|E|}{2}$$

Solutions to Warm-up 2

Let φ be a 3-CNF formula.

Choose a random assignment σ

$$\mathbb{E}\left[\begin{matrix}\#\text{satisfied clauses}\\ \text{in } \varphi(\sigma)\end{matrix}\right] = \sum_{\text{clauses } C} \underbrace{\mathbb{P}\left[C(\sigma) = \text{TRUE}\right]} = \frac{7}{8} \cdot \binom{\#\text{of}}{\text{clauses}}$$

e.g., $\mathbb{P}\{x_1 \vee x_2 \vee \bar{x}_3 = \text{TRUE}\} = 7/8$

since there's only one out of eight assignments so that it is false.

$\Rightarrow \exists\, \sigma$ s.t. $\varphi(\sigma)$ has $\geq 7/8$ of the clauses satisfied.

Questions about the minilectures/quiz?

Group Work!

- In those warm-ups (and in the minilectures) you saw ways to prove that something nice exists…but not how to find those nice things efficiently.

- Sometimes, it's possible to turn a probabilistic proof into a deterministic algorithm!

- Today, we'll see one technique called "Derandomization via conditional expectation."

Group Work

Problem 1

- $\mathbf{E}[X \mid v_1 \in S] = \mathbf{E}\left[X \middle| v_1 \in \bar{S}\right]$ by symmetry.
- $\frac{1}{2}\mathbf{E}[X|v_1 \in S] + \frac{1}{2}\mathbf{E}\left[X \middle| v_1 \in \bar{S}\right] = \mathbf{E}[X] = \frac{|E|}{2}$
- So both must be equal to $\frac{|E|}{2}$

Group Work

Problem 2

Let X be the #edges crossing $(S, \bar{S})$

$$\frac{|E|}{2} \le \mathbb{E}\left[X \,\middle|\, \begin{matrix}\text{choices for} \\ v_1, \ldots, v_{t-1}\end{matrix}\right] = \frac{1}{2} \cdot \underbrace{\mathbb{E}\left[X \,\middle|\, \begin{matrix}\text{choices for} \\ v_1, \ldots, v_{t-1}\end{matrix}, v_t \in S\right]} + \frac{1}{2} \cdot \underbrace{\mathbb{E}\left[X \,\middle|\, \begin{matrix}\text{choices for} \\ v_1, \ldots, v_{t-1}\end{matrix}, v_t \notin S\right]}$$

One of these must be $\ge |E|/2$

Group Work
Problem 3

$$\underbrace{\frac{1}{2}\cdot\mathbb{E}\left[X \,\middle|\, \begin{matrix}\text{choices for} \\ v_1,\dots,v_{t-1}\end{matrix}, v_t \in S\right]}_{} + \underbrace{\frac{1}{2}\cdot\mathbb{E}\left[X \,\middle|\, \begin{matrix}\text{choices for} \\ v_1,\dots,v_{t-1}\end{matrix}, v_t \notin S\right]}_{}$$

One of these must be $\geq |E|/2$

So we just want to see which of those is the case…

Consider: $\mathbb{E}\left[X \,\middle|\, \begin{matrix}\text{choices for} \\ v_1,\dots,v_{t-1}\end{matrix}, v_t \in S\right] - \mathbb{E}\left[X \,\middle|\, \begin{matrix}\text{choices for} \\ v_1,\dots,v_{t-1}\end{matrix}, v_t \notin S\right]$

- If this is positive, then we should put $v_t \in S$.
- Otherwise, put $v_t \in \overline{S}$

$$\mathbb{E}\left[X \,\middle|\, \begin{matrix}\text{choices for}\\ v_1,\dots,v_{t-1}\end{matrix},\ v_t \in S\right] - \mathbb{E}\left[X \,\middle|\, \begin{matrix}\text{choices for}\\ v_1,\dots,v_{t-1}\end{matrix},\ v_t \notin S\right]$$

$$= \sum_{\{u,v\}\in E} \underbrace{\left(\mathbb{P}\left\{ \begin{matrix}\{u,v\}\text{ crosses}\\ (S,\bar{S})\end{matrix} \,\middle|\, \begin{matrix}\text{choices for}\\ v_1,\dots,v_{t-1}\end{matrix},\ v_t \in S \right\} - \mathbb{P}\left\{ \begin{matrix}\{u,v\}\text{ crosses}\\ (S,\bar{S})\end{matrix} \,\middle|\, \begin{matrix}\text{choices for}\\ v_1,\dots,v_{t-1}\end{matrix},\ v_t \notin S \right\} \right)}$$

- If $\{u,v\}$ doesn't include v_t, this is 0
- If $\{u,v\} = \{v_i, v_t\}$ for $i>t$, this is $\frac{1}{2}-\frac{1}{2}=0$
- If $\{u,v\} = \{v_i, v_t\}$ for $i<t$ this is $\begin{cases} +1 & v_i \notin S \\ -1 & v_i \in S \end{cases}$

$$= \begin{pmatrix}\#i\le t \text{ s.t.}\\ v_i \notin S\end{pmatrix} - \begin{pmatrix}\#i\le t \text{ s.t.}\\ v_i \in S\end{pmatrix}$$

Group Work

Problem 3

$$\underbrace{\frac{1}{2}\cdot \mathbb{E}\left[X \,\middle|\, \begin{matrix}\text{choices for}\\ v_1,\dots,v_{t-1}\end{matrix},\ v_t \in S\right]}_{} + \underbrace{\frac{1}{2}\cdot \mathbb{E}\left[X \,\middle|\, \begin{matrix}\text{choices for}\\ v_1,\dots,v_{t-1}\end{matrix},\ v_t \notin S\right]}_{}$$

One of these must be $\geq |E|/2$

So we just want to see which of those is the case…

Consider: $\mathbb{E}\left[X \,\middle|\, \begin{matrix}\text{choices for}\\ v_1,\dots,v_{t-1}\end{matrix},\ v_t \in S\right] - \mathbb{E}\left[X \,\middle|\, \begin{matrix}\text{choices for}\\ v_1,\dots,v_{t-1}\end{matrix},\ v_t \notin S\right]$

$$= \left(\begin{matrix}\#i \leq t \text{ s.t.}\\ v_i \notin S\end{matrix}\right) - \left(\begin{matrix}\#i \leq t \text{ s.t.}\\ v_i \in S\end{matrix}\right)$$

- If this is positive, then we should put $v_t \in S$.
- Otherwise, put $v_t \in \overline{S}$

Aka, there are more edges from v_t to $\overline{S}$ than there are to S among the choices we've already made.

Group Work

Problem 4

ALGORITHM

$S = \emptyset$

$T = \emptyset$ // This will become $\bar{S}$

For $t = 1, 2, 3, \ldots, n$:

If v_t has more edges to S than T:

add v_t to T

Else:

add v_t to S

Return S

It's the greedy algorithm!

General Paradigm

Derandomization via conditional expectation

- Suppose you know that $\mathbf{E}[\text{something}]$ is good
- Suppose you can build [something] one choice at a time
- Then assuming that

$$\mathbf{E}[\text{something} \mid \text{choices } 1,2 \ \ldots, t-1\,] \text{ is good,}$$

there is a way to make t^th choice so that

$$\mathbf{E}[\text{something} \mid \text{choices } 1,2 \ \ldots, t\,] \text{ is good.}$$

- If you can find that way to make the t^th choice efficiently, you have an algorithm!

Let’s try another example!

(More groupwork)

Solutions to Group Work

Choose values (TRUE/FALSE) for $x_1, x_2, x_3, \dots, x_n$ one at a time.

At each step, make sure that $\mathbb{E}\left[\begin{matrix}\#\text{Sat.}\\ \text{clauses}\end{matrix} \;\middle|\; \begin{matrix}\text{choices for}\\ x_1,\dots,x_t\end{matrix}\right] \geq \frac{7m}{8}$

The choice exists by induction:

- base case = warmup exercise
- $\frac{7m}{8} \leq \mathbb{E}\left[\begin{matrix}\#\text{sat}\\ \text{clauses}\end{matrix} \;\middle|\; \begin{matrix}\text{choices for}\\ x_1,\dots,x_{t-1}\end{matrix}\right]$

one of these is $\geq 7m/8$

$$= \frac{1}{2}\,\mathbb{E}\left[\begin{matrix}\#\text{sat}\\ \text{Clauses}\end{matrix} \;\middle|\; \begin{matrix}\text{choices for}\\ x_1,\dots,x_{t-1},\end{matrix}\; x_t = \text{TRUE}\right] + \frac{1}{2}\,\mathbb{E}\left[\begin{matrix}\#\text{sat}\\ \text{Clauses}\end{matrix} \;\middle|\; \begin{matrix}\text{choices for}\\ x_1,\dots,x_{t-1},\end{matrix}\; x_t = \text{FALSE}\right]$$

How to make the choice efficiently?

Want to know when this is larger than $\frac{7m}{8}$

$$\mathbb{E}\left[\begin{matrix}\#\text{sat} \\ \text{clauses}\end{matrix} \,\middle|\, \begin{matrix}\text{choices for} \\ x_1, \ldots, x_{t-1}\end{matrix}, x_t = \text{TRUE}\right] = \sum_{\text{clauses } C} \underbrace{\mathbb{P}\left\{C = \text{TRUE} \,\middle|\, \begin{matrix}\text{choices for} \\ x_1, \ldots, x_{t-1}\end{matrix}, x_t = \text{TRUE}\right\}}$$

This is 1 if the choices have already made C true.

Otherwise it's $1 - 1/2^k$, where $k \in \{0,1,2,3\}$ is the # of free variables left in C.

In particular, we can compute this efficiently.

Time $O(m)$!

Algorithm

For $t = 1, \dots, n$:

Compute $\mathbb{E}\left[\begin{array}{c}\#\text{sat.} \\ \text{clauses}\end{array} \middle| \begin{array}{c}\text{choices for} \\ x_1, \dots, x_{t-1}\end{array}, x_t = \text{TRUE}\right]$ Time $O(m)$

If it is at least $7m/8$, set $x_t = \text{TRUE}$

Otherwise, set $x_t = \text{FALSE}$

Time $O(nm)$ total!