Class 13

Markov Chains I

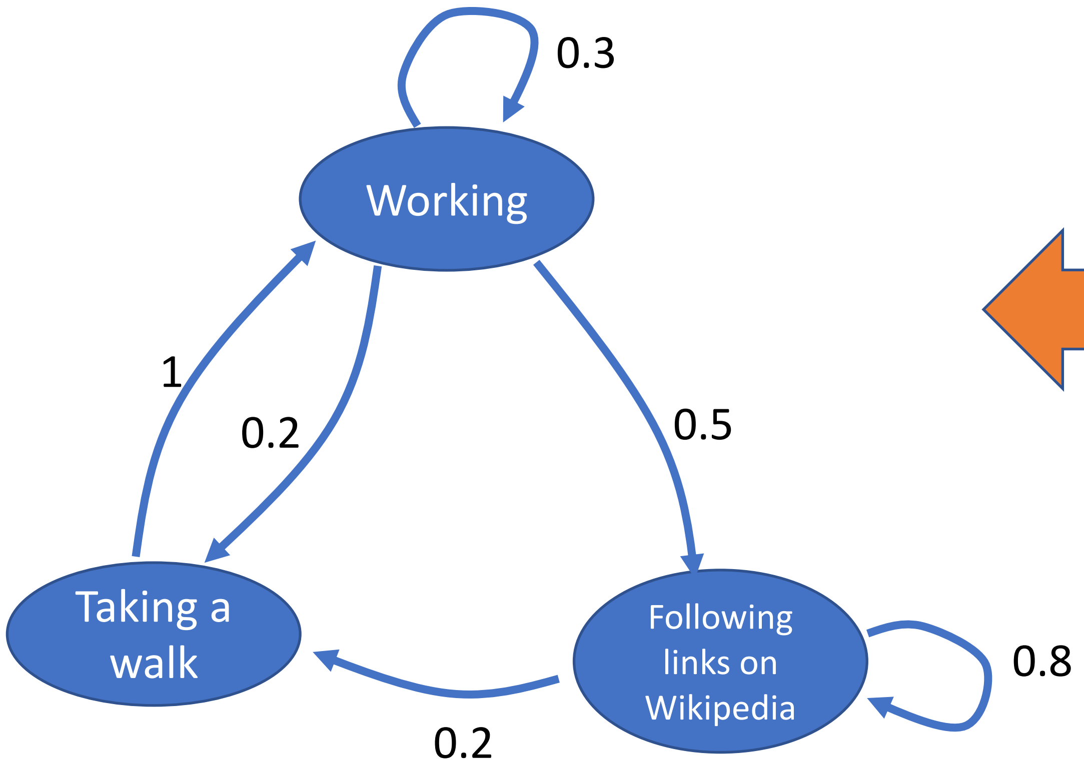
Announcements

- HW6 due tomorrow!
- HW7 will be out next week, it's not due until the week **after** Thanksgiving.
- Did anyone leave a pair of headphones in my office?

Recap

(Time homogeneous, finite) Markov Chains!

Memorylessness: $\Pr[\,X_t = a \mid X_0, \ldots, X_{t-1}] = \Pr[\,X_t = a \mid X_{t-1}]$



	Working	Walking	Wikipedia
Working	0.3	0.2	0.5
Walking	1	0	0
Wikipedia	0	0.2	0.8

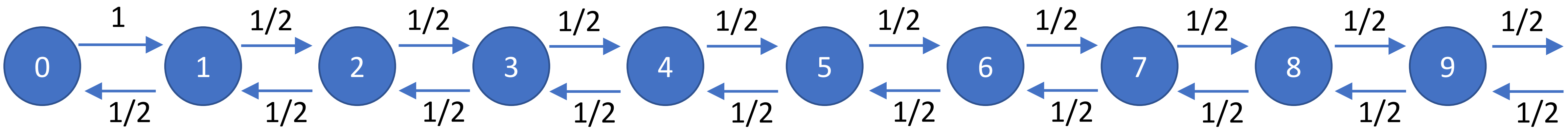
e.g., $\varphi = (x_1 \vee x_2) \wedge (\overline{x_1} \vee x_3) \wedge \cdots \wedge (x_5 \vee \overline{x_7})$

Recap

Randomized algorithm for 2SAT!

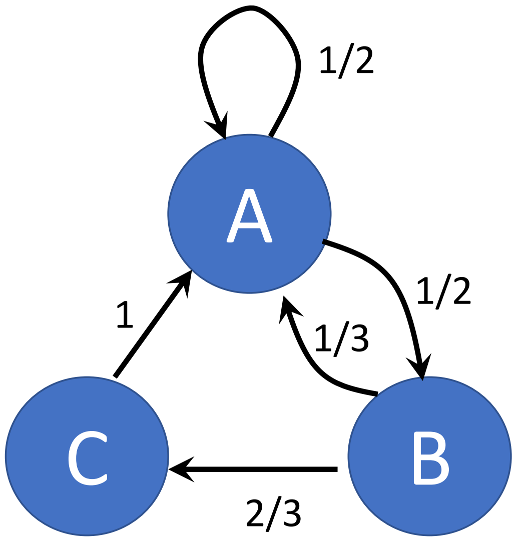
- Algorithm:
  - While not done:
    - Find an unsatisfied clause, flip one of the variables at random.

- Analysis: This Markov chain hits n in no more than $100n^2$ steps whp:



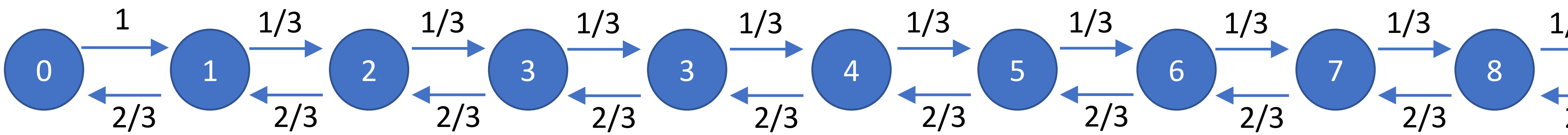
Questions?

Q1



- $X_0 = A$ with probability 1.
- Q1.1: Pr[$X_1 = B$]?
- Q1.2: Pr[$X_2 = B$]?
- Q1.3: Pr[$X_{10} = B$]?

Q2



- $r_i = E[\min_t\{t : Z_n = n\} \mid Z_0 = i]$

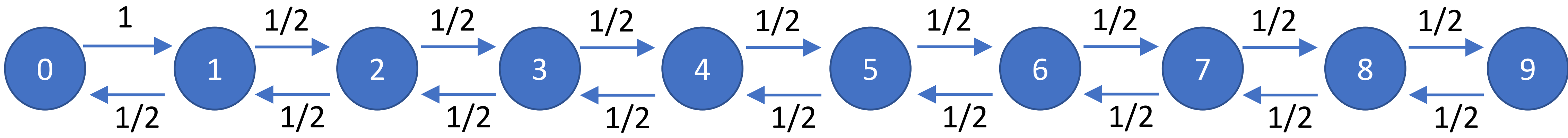
- Q2.1. What expression do the r_i satisfy?

- Q2.2. Sanity-check: prove that $r_i = f_i + r_{i+1}$ where $f_i = 2f_{i-1} + 3$ and $f_0 = 1$.

- Q2.3. $r_0 = 2^{\Omega(n)}$

In the mini-lectures

- We saw one way to analyze a chain that looked like this:

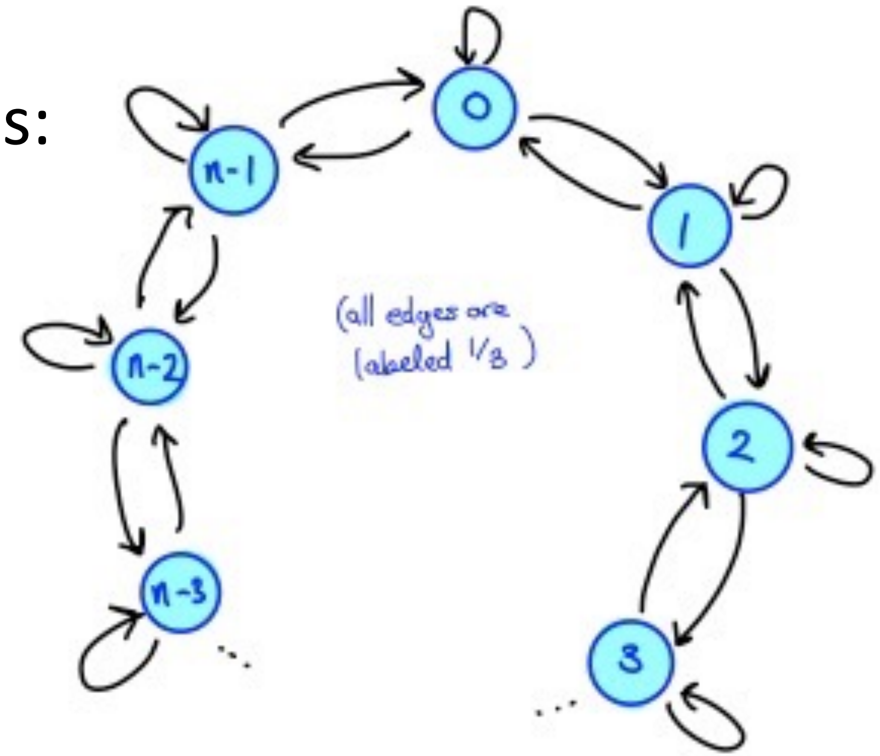


- Whp, it took $\Theta(n^2)$ steps to reach n.

Today

- We will see another way to use the transition matrix to analyze (certain nice) Markov chains.

- We'll analyze a Markov chain that looks like this:

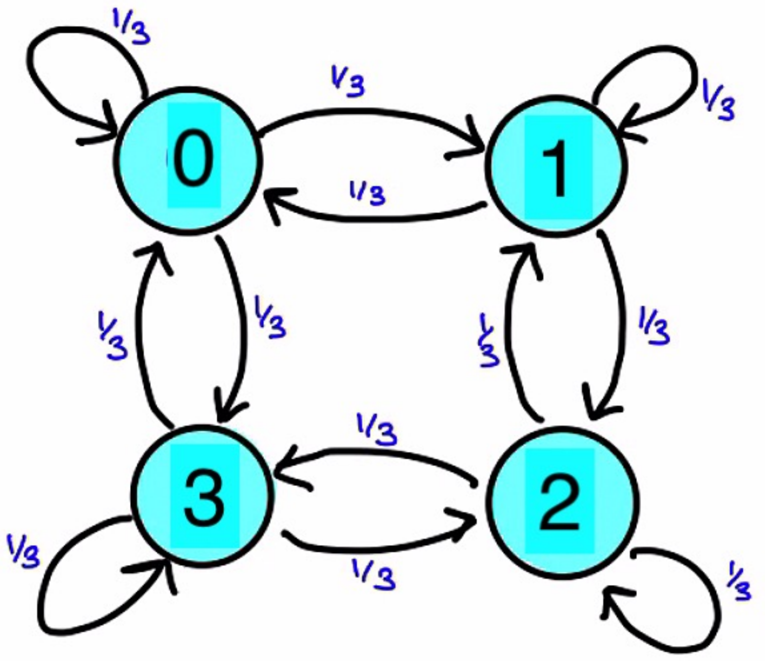


- What should happen if the Markov chain runs for long enough?

- What's your intuition about how long "long enough" is?

Let's do this first group work together.

- Transition matrix?



- $\Pr[\, X_t = 2 \mid X_0 = 0 \,]$ for t=1, 2, …, 100?

- As $t \to \infty$, what do you think this prob. should go to?

Next group work!

$$F = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

- Show that:

$$P = F \cdot \begin{pmatrix} 1 & & & \\ & 1/3 & & \\ & & -1/3 & \\ & & & 1/3 \end{pmatrix} \cdot F^*.$$

- Use this to find a nicer way of computing $\Pr[\, X_t = 2 \mid X_0 = 0 \,]$

Show $P = F \cdot \begin{pmatrix} 1 & & & \\ & 1/3 & & \\ & & -1/3 & \\ & & & 1/3 \end{pmatrix} \cdot F^*.$

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Whoops, everything on this slide should be divided by 3.

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Eigenvector with eigenvalue 3

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

Eigenvector with eigenvalue −1

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix}$$

Eigenvector with eigenvalue 1

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix} = \begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix}$$

Eigenvector with eigenvalue 1

$$\Rightarrow P \cdot F = F \cdot \begin{pmatrix} 3 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \quad \Rightarrow P = F \cdot \underbrace{\begin{pmatrix} 3 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}}_{\text{call this } \Lambda} F^*$$

A nicer way of computing $\Pr[\, X_t = 2 \mid X_0 = 0 \,]$

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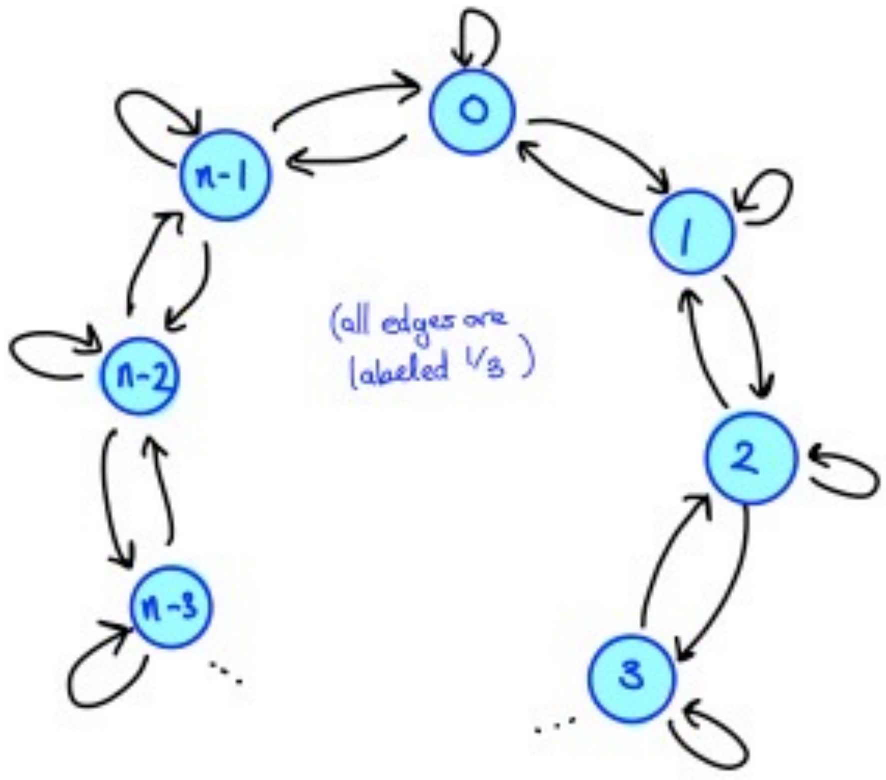
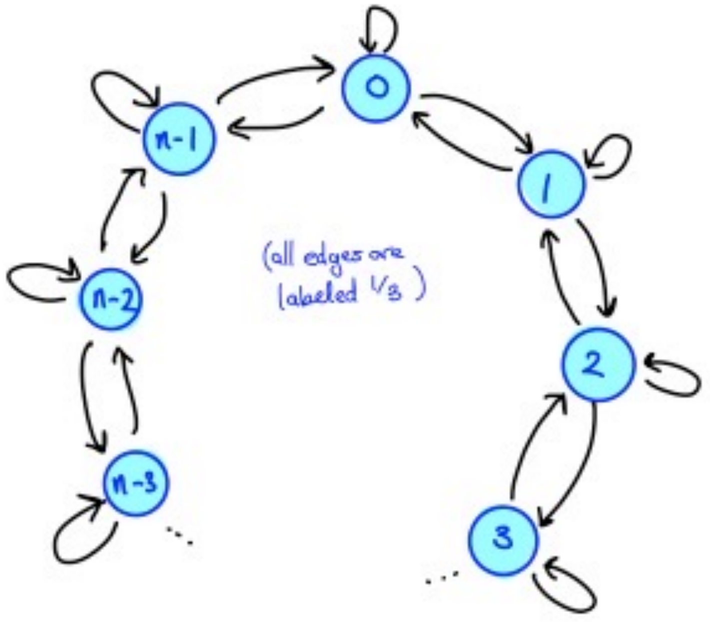
Back to group work!

Do the same thing for a large cycle.

- We'll do this somewhat synchronously since the math is a bit gross.



Part 1: Decomposing the adjacency matrix



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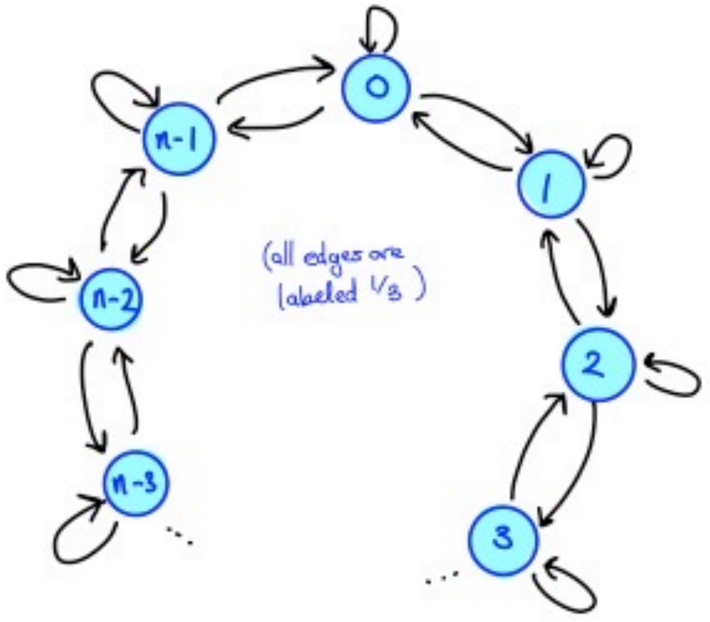

Part 2: $\Pr[X_t = 0 \,|X_0 = 0]$

Part 2: $\Pr[X_t = 0 \mid X_0 = 0]$

Part 3: What happens as t gets big?

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$$\mathbb{P}[X_t = 0 \mid X_0 = 0] = \frac{1}{n} + \frac{1}{n}\sum_{j=1}^{n-1} \underbrace{\left(\frac{1 + 2\cos(2\pi j/n)}{3}\right)^t}$$

this $\to 0$ as $t \to \infty$

since $1 + 2\cos\left(\frac{2\pi j}{n}\right) < 3$

Part 4: How big does t need to be?

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Take-aways

- If we can diagonalize the transition matrix, it can make analyzing a Markov chain easier.
  - This is called “spectral analysis.”
- If the second eigenvalue of a (symmetric) transition matrix is bounded away from 1, then the Markov chain “mixes” quickly.
  - We’ll see what it means to “mix quickly” more formally next week.