Class 14

Markov Chains II

Announcements

- No HW at the moment!
- Next week is fall break!
- This Friday is the last day to change to/from CR/NC, and also the last day to withdraw.

Recap: More Markov Chains!

- Definitions!
  - A chain is **irreducible** if you can get to anywhere from anywhere else.

  - A state is **recurrent** if you'll return to it eventually with probability 1.
  - Otherwise it is **transient**.

  - A chain is **periodic** if there's a state that you can only reach on multiples of c, for some integer c > 1.
  - Otherwise it is **aperiodic.**
    - Useful fact: any irreducible chain with a self-loop is aperiodic!

Recap: Fundamental Theorem of Markov Chains

- Any irreducible and aperiodic Markov chain over a finite state space has a unique **stationary distribution π.**
- As t gets big, $X_t \rightarrow \pi$
- $\pi P = \pi$, aka if $X_t \sim \pi$, then $X_{t+1} \sim \pi$
- If you start in state i, the expected amount of time to return is $\frac{1}{\pi_i}$

Tie-in to last time…

- Proposition: If a Markov chain has symmetric transitions, and is aperiodic and irreducible, then the stationary distribution is uniform.

Metropolis algorithm and MCMC

- **Markov Chain Monte Carlo:**
  - Set up a Markov Chain with a particular desired stationary distribution.
  - To sample from that distribution, run the chain for a while!

- **Metropolis Algorithm:**
  - A particular way to set up such a chain.

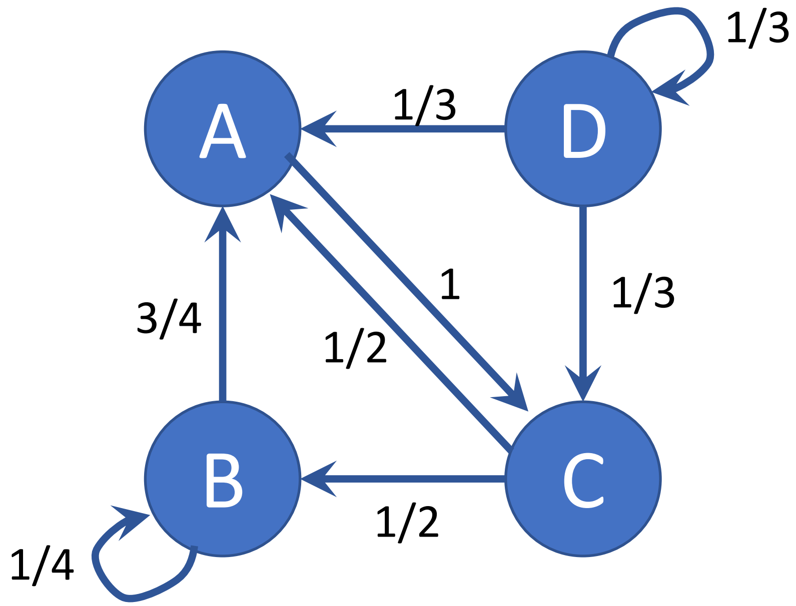
$$P_{i,j} = \begin{cases} 0 & \text{if } i, j \text{ not neighbors} \\ \frac{1}{d}\min(1, \frac{\pi(j)}{\pi(i)}) & \text{if } i \neq j \text{ and they are neighbors} \\ 1 - \sum_{\ell \neq i} P_{i,\ell} & \text{if } i = j. \end{cases}$$

Questions?

Fundamental Theorem, Metropolis Alg? Quiz?

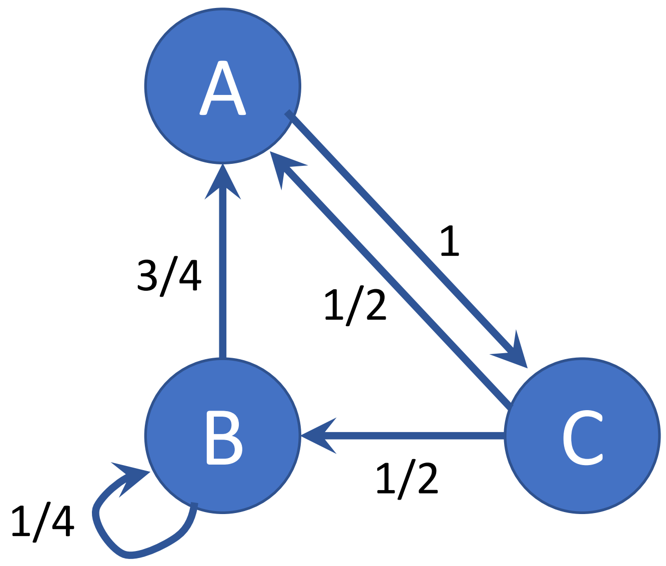
Question 1

- Which states are recurrent?
  - A,B,C
- Which states are transient?
  - D
- Is this chain irreducible?
  - No, it's reducible into {D}, {A,B,C}.
- Is this chain periodic?
  - No, it's aperiodic.



Question 2

- What is the stationary distribution?
  - (3/8, 1/4, 3/8)



Question 3

- Choose a card uniformly at random.
- Move it to the top of the deck.

- Is the MC symmetric? (No)
- Is it irreducible? (Yes)
- Is it periodic? (No)
- Is the stationary distribution uniform? (Yes)

Today: Gibbs Sampling!

- An MCMC algorithm for multivariate distributions.

- Set-up:
  - π is a joint distribution on random variables X, Y
    - More generally $X_1, X_2, \dots, X_m$
  - It's hard to sample from π
  - But it's easy to sample from $\pi(X \mid Y = y)$ or $\pi(Y|X = x)$ for any fixed x, y.

Gibbs Sampling
(for two variables)

- Say $(X_t, Y_t) = (x, y)$
- Draw $x' \sim \pi(X|Y = y)$
- Draw $y' \sim \pi(Y|X = x')$
- Set $(X_{t+1}, Y_{t+1}) = (x', y')$

Group Work!

1. Show that the uniform distribution is a stationary distribution.
2. Under what conditions on π does FToMC hold?
3. What is the take-away in the context of MCMC?
4. How would you use Gibbs sampling to sample random colorings?
5. How would you use Gibbs sampling to sample a uniformly random 7-word sentence from the distribution of all reasonable such sentences?
6. Any other applications of Gibbs sampling/MCMC that you've encountered?

- Say $(X_t, Y_t) = (x, y)$
- Draw $x' \sim \pi(X|Y = y)$
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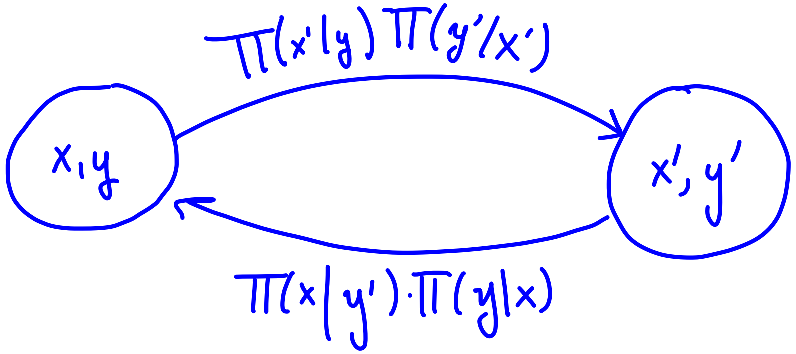


1. Stationary dist is π

- Show $\pi(x,y) = \sum_{x',y'} \pi(x',y')\Pr[(x',y') \to (x,y)]$

2. Do the conditions hold?

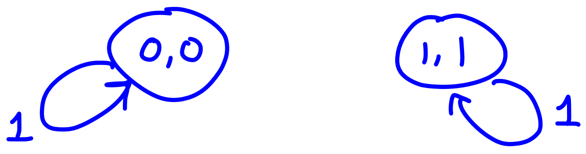
- Aperiodic:
- Irreducible:
- Finite:

2. Do the conditions hold?

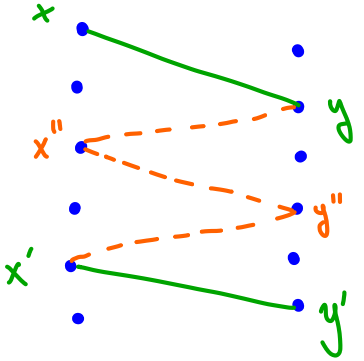
- Not necessarily irreducible:

eg, if $x, y \in \{0,1\}$, π is:

X \ Y	0	1
0	1/2	0
1	0	1/2



- It is irreducible if this bipartite graph is connected:



One way to get from (x,y) to (x',y'):

$(x,y) \mapsto (x'',y) \mapsto (x'',y'') \mapsto (x',y'') \mapsto (x',y')$

3. Why is this useful?

- If we can easily sample from $\pi(X \mid Y = y)$ or $\pi(Y|X = x)$, then we can sample (X_t, Y_t).
- As $t \to \infty$, this will converge to π, so eventually we can sample from π!
  - How long does it take to converge????

- Say $(X_t, Y_t) = (x, y)$
- Draw $x' \sim \pi(X|Y = y)$
- Draw $y' \sim \pi(Y|X = x')$
- Set $(X_{t+1}, Y_{t+1}) = (x', y')$

4. Graph Coloring

- What is the right multivariate generalization?

- How to apply to sampling a random proper coloring?

5. Sampling 7-word sentences

- What is the algorithm? What task do you need to be able to do?

6. Other examples?

- Y'all come from many different areas – have any of you used Gibbs sampling or any other MCMC method before? For what applications?

Another Example: Image Denoising

- Say you get a noisy (black and white, say) image $X = (x_1, \ldots, x_N)$.
  - Each pixel x_p is ± 1
- Sample an "un-noisy" version $Y = (y_1, \ldots, y_N)$, so that the probability of Y is proportional to:

$$\exp(\eta \sum_p x_p y_p + \beta \sum_{p \sim p'} y_p y_{p'})$$

Recap

- The fundamental theorem of Markov chains can be useful!
- But it sure would be more useful if we knew how fast we approached the stationary distribution…
  - Next time!