Class 16

Martingales and Azuma-Hoeffding bound

Announcements

I lied!

- Question last time: “Is a coupling (X_t, Y_t) itself a Markov chain?”
- I think I said yes.
- That’s not true!! For example, it could be that the way in which X_t and Y_t transition together depends on X_s, Y_s for $s \neq t - 1$.

Recap

- $\{Z_t\}$ is a **martingale** with respect to $\{X_t\}$ if for all t:
  - Z_t is a function of $X_0, \dots, X_t$
  - $E[|Z_t|] < \infty$
  - $E[\, Z_t \mid X_0, X_1, \dots, X_{t-1}] = Z_{t-1}$

- **Doob Martingale**: $Z_t = E[\, A \mid X_0, \dots, X_t \,]$

Azuma-Hoeffding

- Let $\{Z_t\}$ be a martingale w.r.t. $\{X_t\}$ and suppose there are constants $c_1, \ldots, c_n$ so that for all $i \leq n$, $|Z_i - Z_{i-1}| \leq c_i$.

- For any $\lambda > 0$,

$$\Pr[\, |Z_n - Z_0| \geq \lambda] \leq 2 \exp\left(\frac{-\lambda^2}{2 \sum_i c_i^2}\right)$$

Questions?

Definition of Martingale? Azuma-Hoeffding? Quiz?

Quiz 1

- $X_0, X_1, \ldots$ are independent rolls of a six-sided die. Which are martingales w.r.t. $\{X_t\}$?

  - $Z_t = X_0 + \cdots + X_t$

  - $Z_t = \left(X_0 - \frac{7}{2}\right) + \left(X_1 - \frac{7}{2}\right) + \cdots + \left(X_t - \frac{7}{2}\right)$

  - $Z_t = E[\, A \mid X_0, \ldots, X_t \,]$ where $A = \prod_i X_i$

Quiz 2

Let $X_1, X_2, \ldots, X_n$ be independent random variables that lie in $[0, 1]$ with probability 1.
Let $A = \sum_{i=0}^{n} X_i^2$.

Apply the Azuma-Hoeffding inequality to the martingale $Z_t = \mathbb{E}[\sum_{j=1}^{n} X_j^2 | X_1, \ldots, X_t]$.
What does it say?

$\Pr[|A - \mathbb{E}A| > n^{2/3}] \leq$ ______

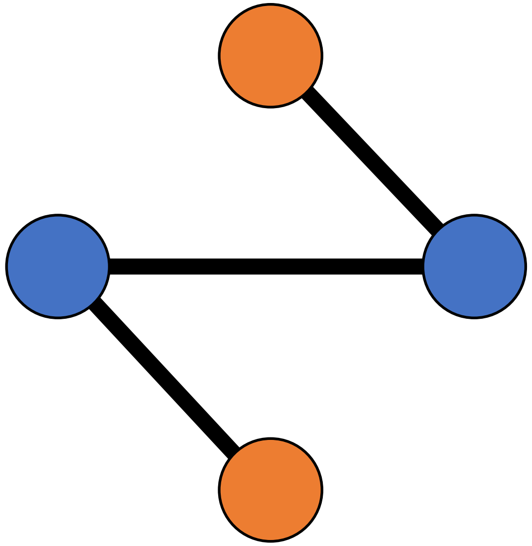
Today: Azuma-Hoeffding in Action

- Example 1: Chromatic number of random graphs
- Example 2: Gambling

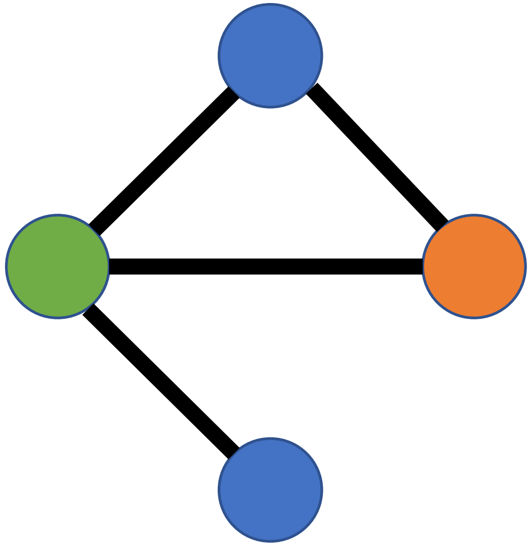
Chromatic number

(We saw this example in the mini-lectures)

- Let $G \sim G(n, p)$
- Let $A = \chi(G)$ be the minimum number of colors needed to properly color G.



$\chi(G) = 2$



$\chi(G) = 3$

Vertex exposure martingale

- Let $G \sim G(n, p)$
- Let $A = \chi(G)$
- $X_t =$ status of edges between vertex t and vertices $1,2, \ldots, t-1$
- $Z_t = \mathbb{E}[A \mid X_1, X_2, \ldots, X_t]$

Note: this is slightly different than the lecture notes! Both work fine for this example, this one is maybe more standard.

Edge exposure martingale

- Let $G \sim G(n, p)$
- Let $A = \chi(G)$
- X_t = status of edge t, for $t = 1, \ldots, \binom{n}{2}$
- $Z_t = \mathbb{E}[A \mid X_1, X_2, \ldots, X_t]$

Group work:

Apply Azuma-Hoeffding both ways!

1. Use Azuma-Hoeffding with the vertex exposure martingale to bound

$$\Pr[\ |A\ - E[A]| > c\sqrt{n}] \le 2\exp\left(-\frac{c^2}{2}\right)$$

2. What happens with the edge exposure martingale?
3. (bonus) what can you say about E[A]?

1. Vertex exposure martingale

- Need to bound $|Z_t - Z_{t-1}| \leq$ ______________

1. Vertex exposure martingale

- Applying Azuma-Hoeffding:

$$\Pr[\,|Z_n - Z_0| > c\sqrt{n}\,] \leq 2\exp\left(-\frac{c^2 n}{2\sum_i c_i}\right)$$

$$\leq 2\exp(-c^2)$$

2. Edge exposure martingale

- Need to bound $|Z_t - Z_{t-1}| \leq$ ______________

Technically…

See lecture notes

- For any a_t,

$$
\begin{aligned}
&\mathbb{E}[A|X_1, \ldots, X_{t-1}, X_t = a_t] \\
&\quad = \sum_{a_{t+1}, \ldots, a_n} \mathbb{E}[A|X_{\leq t-1}, X_t = a_t, X_{\geq t+1} = a_{\geq t+1}] \Pr[X_{\geq t+1} = a_{\geq t+1}|X_{\leq t-1}, X_t = a_t] \\
&\quad = \sum_{a_{t+1}, \ldots, a_n} \mathbb{E}[A|X_{\leq t-1}, X_t = a_t, X_{\geq t+1} = a_{\geq t+1}] \Pr[X_{\geq t+1} = a_{\geq t+1}]
\end{aligned}
$$

Using independence since in our case the X_t happen to be independent.

Technically…

See lecture notes

$$
\begin{aligned}
&\mathbb{E}[A|X_1,\ldots,X_{t-1},X_t=a_t] \\
&\quad= \sum_{a_{t+1},\ldots,a_n} \mathbb{E}[A|X_{\le t-1},X_t=a_t,X_{\ge t+1}=a_{\ge t+1}]\Pr[X_{\ge t+1}=a_{\ge t+1}|X_{\le t-1},X_t=a_t] \\
&\quad= \sum_{a_{t+1},\ldots,a_n} \mathbb{E}[A|X_{\le t-1},X_t=a_t,X_{\ge t+1}=a_{\ge t+1}]\Pr[X_{\ge t+1}=a_{\ge t+1}]
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}[A|X_1,\ldots,X_{t-1},X_t=a_t] - \mathbb{E}[X_1,\ldots,X_{t-1},X_t] \\
&\quad= \sum_{a_{t+1},\ldots,a_n} \left(\mathbb{E}[A|X_{\le t-1},X_t=a_t,X_{\ge t+1}=a_{\ge t+1}] - \mathbb{E}[A|X_{\le t},X_{\ge t+1}=a_{\ge t+1}]\right)\Pr[X_{\ge t+1}=a_{\ge t+1}]
\end{aligned}
$$

2. Edge exposure martingale

- Applying Azuma-Hoeffding:

$$\Pr[\,|Z_n - Z_0| > c\sqrt{n}\,] \le 2\exp\left(-\frac{c^2 n}{2\sum_i c_i}\right)$$

Next example: gambling

- A fun game:
  - At each step t, you can bet $b_t \in [0, B]$ and guess “heads or tails”
  - Flip a fair coin. If you were right, you win b_t. Otherwise you lose b_t.

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- Notice that b_t can depend on everything that's happened so far.
- Let Z_t be the amount of money you have at time t.
- It's okay for $Z_t < 0$ (the casino knows you're good for it…)

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- Quick question:
  - Is Z_t a sum of independent random variables?

Group Work

- At each step t, you can bet $b_t \in [0, B]$ and guess "heads or tails"
- Flip a fair coin. If you were right, you win b_t. Otherwise you lose b_t.
- $Z_t =$ amount of money at time t

1. Define some random variables X_t so that $\{Z_t\}$ is a martingale w.r.t. $\{X_t\}$
2. Use Azuma-Hoeffding to bound $\Pr[\, |Z_n| \geq cB\sqrt{n}\,]$
3. Does the proof above work if your betting strategy is randomized? If not, make it work.

1. Setting up a martingale

- Z_t = amount of $$ at time t
- X_t =

- This is a martingale because…

  1. Z_t is a function of $X_0, \ldots, X_t$?

  2. $E[|Z_t|] < \infty$?

  3. $E[Z_t | X_0, \ldots, X_{t-1}] = Z_{t-1}$?

2. Using Azuma-Hoeffding

- $|Z_t - Z_{t-1}| \leq$ __________

- Azuma-Hoeffding:
- $\Pr[\, |Z_n - Z_0| \geq c\sqrt{n}\,] \leq 2\exp\left(-\frac{c^2 n}{\sum_i c_i}\right)$

With a randomized betting strategy?

- Z_t = amount of $$ at time t
- $X_t =$

- This is a martingale because…

  1. Z_t is a function of $X_0, \dots, X_t$?
  2. $E[|Z_t|] < \infty$?
  3. $E[\, Z_t \mid X_0, \dots, X_{t-1} \,] = Z_{t-1}$?

Recap

- We got some practice applying Azuma-Hoeffding.
- Moral of the story I:
  - Azuma's inequality can bound sums of random variables that aren't necessarily independent!
- Moral of the story II:
  - One useful case is when you want to establish concentration for a random variable A that doesn't depend too much on any of the underlying variables X_i.
- Moral of the story II:
  - Sometimes there's more than one relevant martingale, and one might be better than the other.