# Class 17

Martingale Stopping Theorem; Wald’s equation

# Announcements

- HW7 due tomorrow!
- HW8 out now!
- ***You are now all done with quizzes!!!***
- FINAL EXAM will be Thursday Dec 15, 12:15-3:15 in 420-040.
  - We'll release a practice exam with our cheat sheet soon.
- Check out Ed for an addendum to my flailing at the end of Tuesday's class.

- Plan for week 10:
  - Tuesday: Fun "bonus" material on pseudorandomness. (No quiz, will not be on HW or exam).
  - Thursday: The research frontier! (At least 2 short research talks).

# Recap!

- Stopping times
- Martingale Stopping theorem
- Applications

# Stopping times

- $T$ is a **stopping time** for $\{X_t\}$ if the event that $T = i$ is mutually independent of all the random variables $X_j |\, X_0, \dots, X_i$, for all $j > i$.
  - Informally, you should be able to tell that $T$ has occurred at time $T$, without looking into the future.
  - Example: The first time $X_t$ hits 100.
  - Non-example: The last time $X_t$ hits 100.

# Martingale Stopping Theorem

- Let $\{Z_t\}$ be a martingale w.r.t. $\{X_t\}$.
- Let $T$ be a stopping time for $\{X_t\}$.
- Then $\mathbf{E}[Z_T] = \mathbf{E}[Z_0]$ if at least one of the following hold:
  - There is a c so that $|Z_i| < c$ for all i
  - There is a c so that $T < c$ with probability 1
  - There is a c so that $\mathbf{E}[|Z_{i+1} - Z_i| \mid X_0, \dots, X_i] < c$ for all i, and $\mathbf{E}[T] < \infty$

# Why do we care?

- Hitting time of random walks!

- Set up a martingale $Z_t$ so that $\mathbf{E}[Z_T]$ has something to do with something you care about.
  - E.g., $\mathbf{E}[Z_T] = \Pr[\, Z_T = b] \cdot b - \Pr[Z_T = a] \cdot a$
- You know what $\mathbf{E}[Z_0]$ is.
  - E.g., $\mathbf{E}[Z_0] = 0.$
- Solve FTW.
  - E.g., $\Pr[Z_T = b] = \frac{a}{a+b}$

# Questions?

Stopping times, martingale stopping thm, quiz?

# Q1. Is it a stopping time?

- $Z_t$ is the sum of t rolls of a fair six-sided die.
- $T_1 = \min\{t: Z_t \geq 12\}$
- $T_2 = \min\{t: Z_{t+2} \geq 12\}$
- $T_3 = \min\left\{t: Z_t - \frac{7t}{2} \geq 12\right\}$

# Q2.

- $T_1 = \min\{t: Z_t \geq 12\}$.
- $T_2 = \min\{t: Z_{t+2} \geq 12\}$
- $T_3 = \min\left\{t: Z_t - \frac{7t}{2} \geq 12\right\}$

- Same $Z_t$ as before.
- $Y_t = Z_t - \frac{7t}{2}$

- To which of the stopping times does the MST apply?
  - $T_1$ and the $Y_t$'s: Yes, since $T_1 < 13$ with probability 1.
  - $T_1$ and the $Z_t$'s: No, since the $Z_t$'s are not a martingale.
  - $T_3$ and the $Y_t$'s: No, since the conditions of the MST aren't met.
    - Intuitively, $Y_t$ can wander off to $-\infty$ without ever hitting 12.

# Q3. Hitting times

- $Q_0 = 0$
- $Q_t = \begin{cases} Q_{t-1} + 3 & \Pr \frac{1}{2} \\ Q_{t-1} - 3 & \Pr \frac{1}{2} \end{cases}$
- $\mathbf{E}[\min\{t : |Q_t| = 30\}] = 100$

# Plan for today

- Wald's Equation
- (If time) Ballot counting thm

# Wald's Equation

**Theorem.**

- Suppose that $X_1, X_2, \ldots$ are non-negative i.i.d. random variables, $X_i \sim X$.
- Let $T$ be a stopping time for $\{X_i\}$.
- Suppose that $\mathbf{E}[T], \mathbf{E}[X] < \infty$.
- Then

$$\mathbf{E}\left[\sum_{i=1}^{T} X_i\right] = \mathbf{E}[T] \cdot \mathbf{E}[X]$$

# Group Work

1. Find an example where Thm 1 fails if the hypotheses aren't met.
   - Try to violate as few of the hypothesis as you can!
2. Let $Z_i = \sum_{j=1}^{i}\left(X_j - E[X]\right)$. Prove that $\{Z_t\}$ is a martingale wrt $\{X_t\}$
3. Show that the martingale stopping thm applies to $\{Z_t\}$ and $T$.
4. Use the martingale stopping thm to prove Wald's equation.
5. Consider rolling a fair six-sided die repeatedly. Let X be the the sum of all of the rolls up until the first "6" is rolled. (Not including that first "6"). What is E[X]?

# 1. Examples where Wald’s eqn doesn’t hold

# 1. An example where the conclusion does not hold.

- Let $X_1, X_2, \dots,$ be i.i.d. $\{0,1\}$ random variables, mean $1/2$.
  - Actually we only need $X_1$
- Let $T$ be $1 - X_1$. (Note that T is not a stopping time!)

- Then $\mathbf{E}\left[\sum_{i=1}^{T} X_i\right] = 0$.
  - If $X_1 = 1$ then we sum zero things
  - If $X_1 = 0$ then we sum one thing which is equal to 0
- But $\mathbf{E}[T] \cdot \mathbf{E}[X] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

# 2. $Z_i$ is a martingale

Let $Z_i = \sum_{j=1}^{i} (X_j - \mathbb{E}[X])$

- $\mathbf{E}[Z_i | X_1, \ldots, X_{i-1}] = Z_{i-1}$

- $\mathbf{E}[|Z_i|] < \infty$

# 3. The Martingale Stopping Thm applies

**Theorem 1** (Martingale Stopping Theorem). *Letting $\{Z_t\}$ denote a martingale with respect to $\{X_t\}$, and $T$ a stopping time for $\{X_t\}$, then $\mathbf{E}[Z_T] = \mathbf{E}[Z_0]$ if at least one of the following conditions hold:*

1. *If there exists a constant $c$ such that for all $i$, $|Z_i| < c$.*
2. *If there exists a constant $c$ such that with probability 1, $T < c$.*
3. *If $\mathbf{E}[T] < \infty$, and there exists a constant $c$ such that for all $i$, $\mathbf{E}[|Z_{i+1} - Z_i| \| X_0, \ldots, X_i] < c$.*

# 3. The Martingale Stopping Thm applies

- Apply the third condition:

$\mathbb{E}[T] < \infty$ (by assm)

$\mathbb{E}\big[|Z_{i+1} - Z_i| \cdot | X_1, \dots, X_i\big] = \mathbb{E}\big|X_{i+1} - \mathbb{E}[X_{i+1}]\big|$

$\leq 2\mathbb{E}[X]$ is bounded. (by assm)

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# 4. Apply the martingale stopping theorem

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$$\mathbb{E}[Z_T] = \mathbb{E}[Z_1] = 0.$$

$$0 = \mathbb{E}[Z_T] = \mathbb{E}\left[\sum_{j=1}^{T} (X_j - \mathbb{E}X)\right]$$

$$= \mathbb{E}\left[\left(\sum_{j=1}^{T} X_j\right) - T \cdot \mathbb{E}[X]\right]$$

$$= \mathbb{E}\left[\sum_{j=1}^{T} X_j\right] - \mathbb{E}[T] \cdot \mathbb{E}[X]$$

$$\Rightarrow \mathbb{E}\left[\sum_{j=1}^{T} X_j\right] = \mathbb{E}[T] \cdot \mathbb{E}[X].$$

# 5. Application of Wald's equation

- X = sum of die-rolls up until you get a six.  (Not including that six).
- $E[X] =$

# Ballot counting

- Election with two candidates, A and B, and n voters.
- A will win, receiving $N_A > N_B$ votes. (so $N_A + N_B = n$).
- Ballots are counted in a random order.
- What is the probability that A remains ahead the whole time?

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- Ballots are counted in a random order.
- What is the probability that A remains ahead the whole time?

- Let $A_t$ be number of votes for A at time t
- Let $B_t$ be number of votes for B at time t
- Let $Z_t = \frac{A_{n-t} - B_{n-t}}{n-t}$

# Group Work

$$Z_t = \frac{A_{n-t} - B_{n-t}}{n-t}$$

1. Let $T$ be smallest t so that $Z_t = 0$; if this never occurs, $T = n - 1$. Explain why MST applies to $\{Z_t\}$ and T.
   - Assume for now that $\{Z_t\}$ is a martingale.

2. Apply MST to $\{Z_t\}$ and T, and use it to compute the probability that A was ahead the whole time.

3. Show that $\{Z_t\}$ is indeed a martingale.

1. Let $T$ be smallest t so that $Z_t = 0$; if this never occurs, $T = n - 1$. Explain why MST applies to $\{Z_t\}$ and T.

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$$Z_t = \frac{A_{n-t} - B_{n-t}}{n-t}$$

## 2. Apply MST to compute the prob. A was ahead the whole time.

- $E[Z_T] = E[Z_0] =$

- OTOH, $E[Z_T] =$

$$Z_t = \frac{A_{n-t} - B_{n-t}}{n-t}$$

# 3. Show $Z_t$ is indeed a martingale

- Start two piles of ballots.  Take one from a random pile at each step.

# Recap

- The Martingale Stopping Theorem is useful!