Class 19

Extractors and Expanders

Warm-up

- Say that X is a k-source on $\{0,1\}^n$. Let $N = 2^n$.
- Let $\sigma \in \mathbb{R}^N$ correspond to the pmf for X.

1. Why is $\|\sigma\|_\infty \leq 2^{-k}$?
2. Argue that $\|\sigma\|_2 \leq 2^{-k/2}$.
   - Hint: $\|x\|_2^2 \leq \|x\|_\infty \|x\|_1$

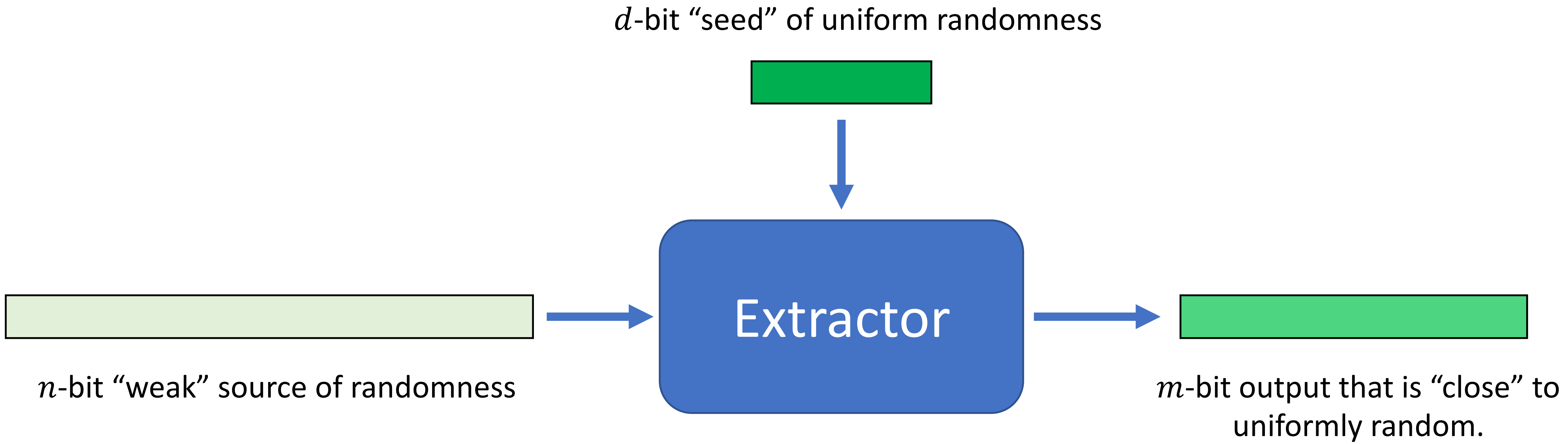
Announcements

- Welcome to week 10!!!
- HW8 due Friday.
- Practice exam is out now. (With solutions).
  - We hope it's about the same difficulty as the real final, although TBH I think it's not as "good" an exam as the real final... (good exams are hard to write).
- Today:
  - Pseudorandomness! Not on the exam.
- Thursday:
  - Research talks! Also not on the exam.
- EXAM: Thursday 12/15, 12:15-3:15pm, Room 420-040.

Pseudorandomness

- Deterministic (or not-so-random) objects that behave like random ones.
- Useful for derandomization.

Extractors



d-bit "seed" of uniform randomness
Extractor
n-bit "weak" source of randomness
m-bit output that is "close" to uniformly random.

Expanders

- Let $G = (V, E)$ be an unweighted, undirected, regular graph with degree D and with N vertices.
- Let A be the normalized adjacency matrix of G.
- Say that the eigenvalues of A are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$

- The **expansion** of A is $\lambda(G) = \max\{\lambda_2, |\lambda_n|\}$

Theorem:

- Let $\{X_t\}$ be a random walk on $G = (V, E)$.
- The stationary distribution of $\{X_t\}$ is π = uniform on V.
- If $\lambda(G) < 0.99$, then $\tau_{mix} = O(\log n)$

Questions?

Minilectures, Warm-up?

Warm-up:

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    - Hint: $\|x\|_2^2 \leq \|x\|_\infty \|x\|_1$

$\sigma_i = \mathbb{P}[X = i], \quad \forall i \in \{0, \dots, N-1\}$

or, the binary expansion of i.

Warm-Up

- Say that X is a k-source on $\{0,1\}^n$
- Let $N = 2^n$, and let σ be the "vectorized" version of the distribution of X

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Warm-Up

- Say that X is a k-source on $\{0,1\}^n$
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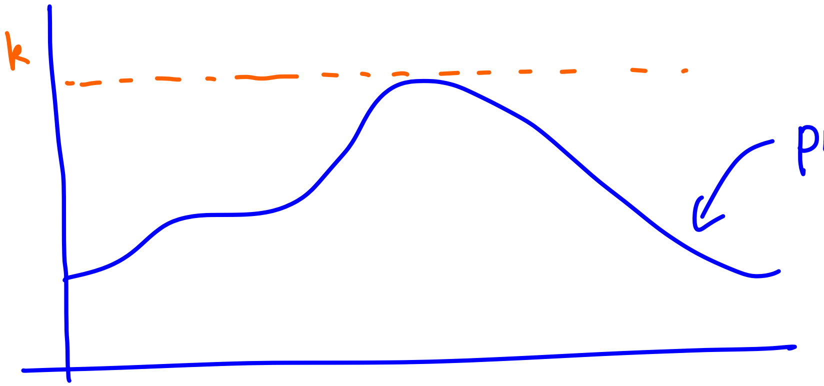
Then $\|\sigma\|_\infty \leq 2^{-k}$, by def of k-source:

$H_\infty(X) \leq 2^{-k}$



pmf of $X = \sigma$

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Warm-Up

- Say that X is a k-source on $\{0,1\}^n$
- Let $N = 2^n$, and let σ be the "vectorized" version of the distribution of X

$$\|\sigma\|_2 = \left(\sum_{i\in[N]} \sigma_i^2 \right)^{1/2} \leq \|\sigma\|_\infty^{1/2} \Big(\underbrace{\sum_{i\in[N]} \sigma_i}_{=1} \Big)^{1/2} = \|\sigma\|_\infty^{1/2} \leq 2^{-k/2}$$

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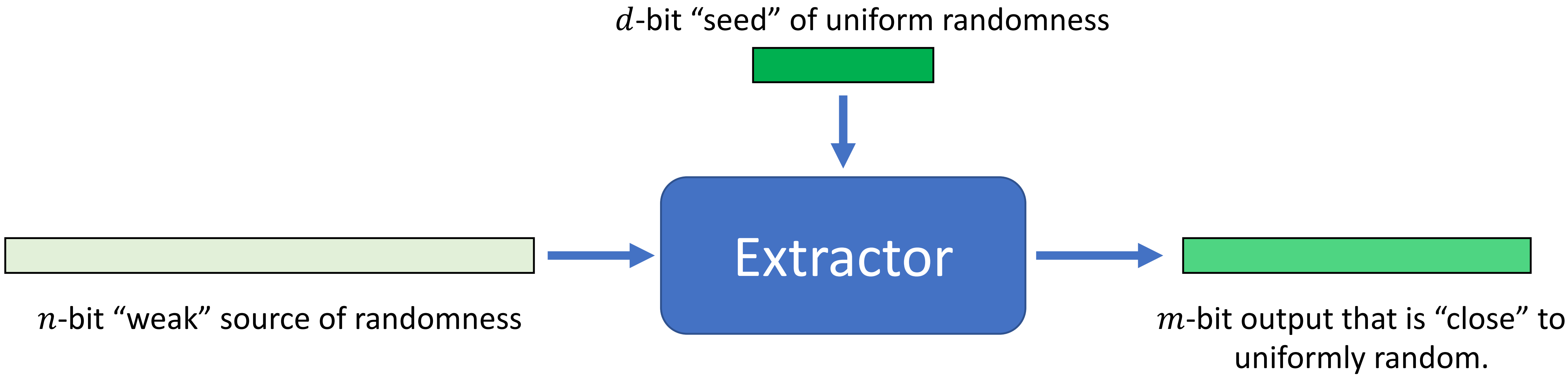

Today

- We will consider a way to make an extractor out of an expander graph.

- Recall: An expander graph looks like this:

This is $\frac{1}{D}$ times the standard adjacency matrix.

Degree D graph with N vertices

Normalized adjacency matrix
$A \in \mathbb{R}^{N \times N}$

- The eigenvalues of A are
  $$1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$$
- The expansion is
  $$\lambda(G) = \max\{\lambda_2, |\lambda_n|\}$$
- For an expander, $\lambda(G)$ is decently less than 1.

Our extractor

$N = 2^n$, and choose $k \leq n$ and $\epsilon > 0$

Let $d = \ell \cdot \log(D)$, where $\ell = \frac{n-k}{2} + \log\left(\frac{1}{\epsilon}\right)$

G =Degree D graph with N vertices

- Associate each vertex of G with a string in $\{0,1\}^n$

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- Associate each vertex of G with a string in $\{0,1\}^n$
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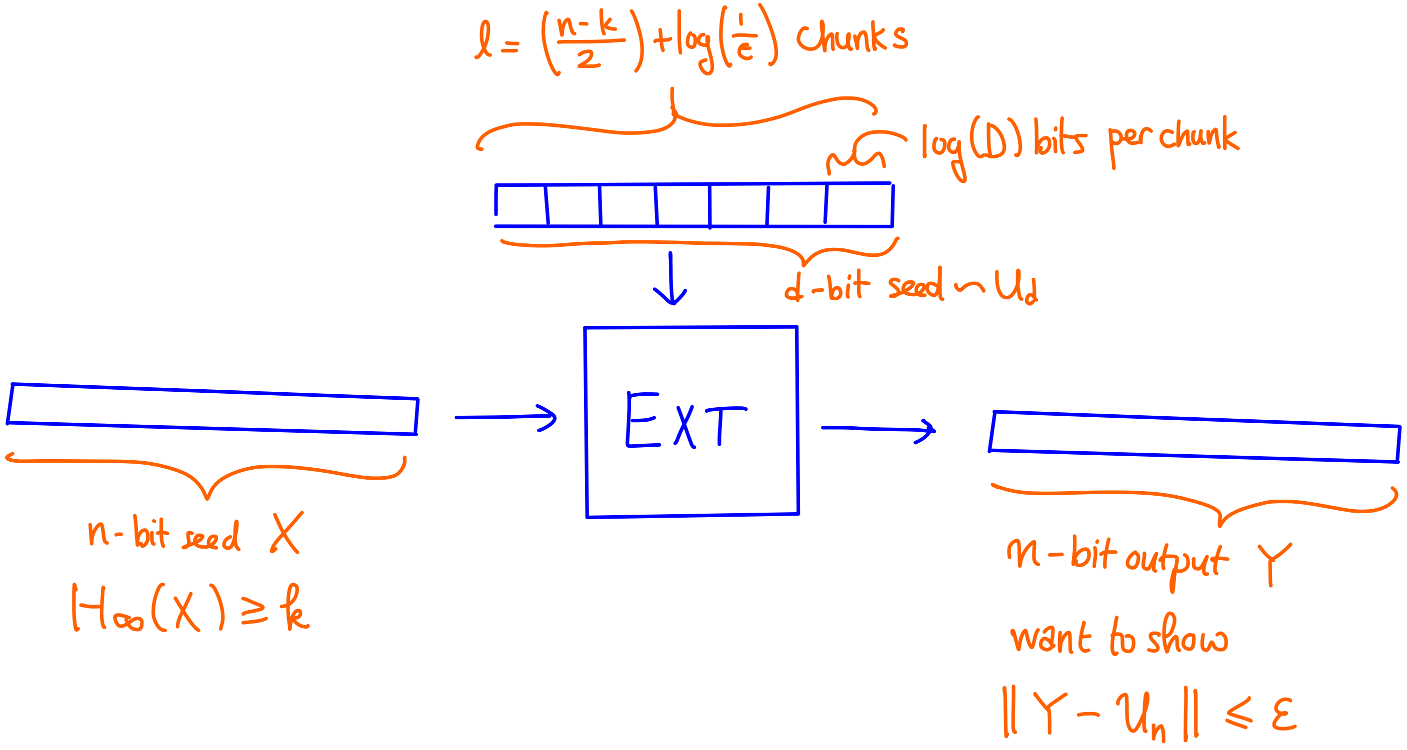

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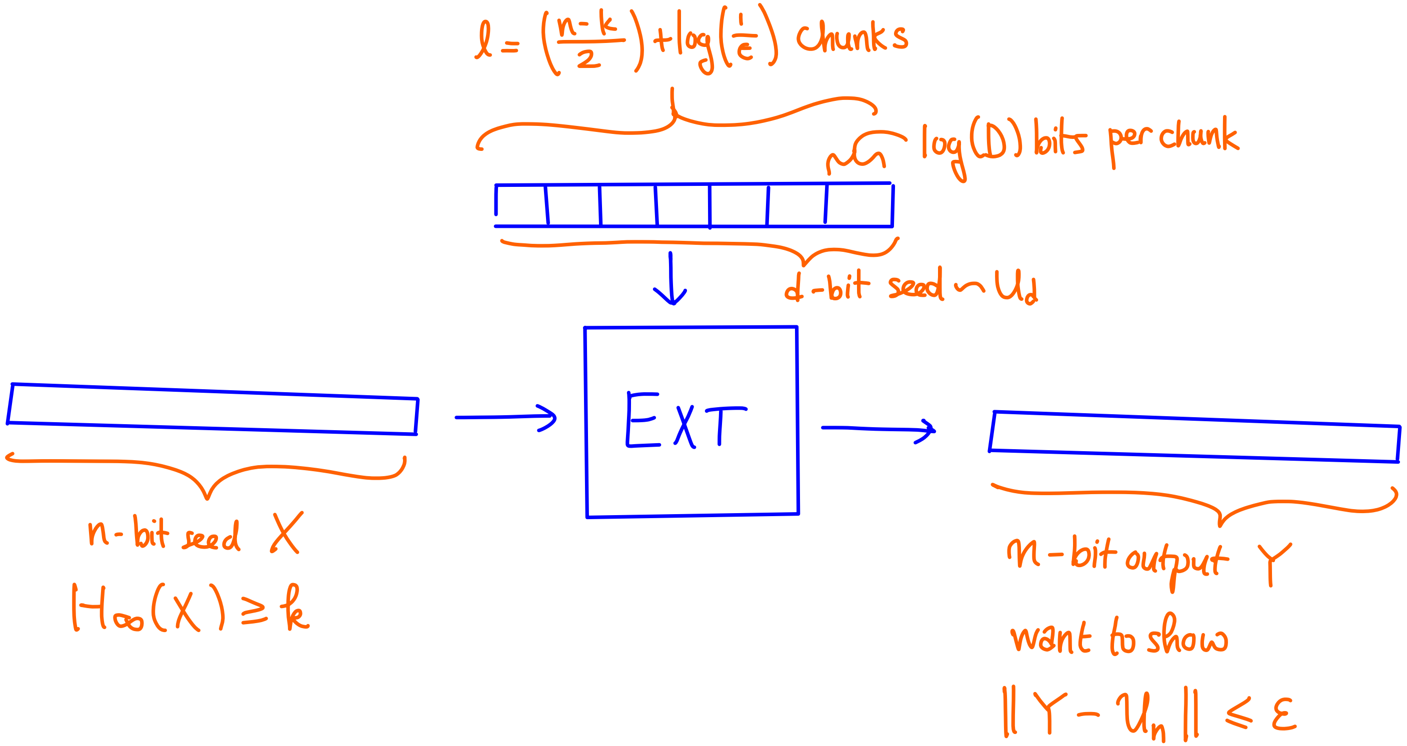

- The source $x \sim X$ tells us a vertex to start at.
- For each step $1, 2, \ldots, \ell$, that chunk of the seed tells us what our next step should be.
- Output the label on the vertex where we are after ℓ steps.

Claim

- If we choose $\ell = \frac{n-k}{2} + \log\left(\frac{1}{\epsilon}\right)$, then this is a (k, ϵ)-extractor.
  - Seed length: $d = \ell \cdot \log(D) = O(\ell)$
  - Output length: n

This is not as good as our existential result, since the seed length is really long unless k is quite large, but it's still non-trivial!

This is pretty good when $k = n - \log n$, for example.

[Backup slide] Comparison to optimal

- $\ell = \frac{n-k}{2} + \log\left(\frac{1}{\epsilon}\right)$
- Seed length $d = \ell \cdot \log(D) = O\left(\frac{n-k}{2} + \log\left(\frac{1}{\epsilon}\right)\right)$
- Output length: n

The seed length is much longer than we'd like unless k is big. However, the output length in that case is pretty good: ideally it wouldn't be much smaller than k + d (the total amount of randomness going in), so it's the right order of magnitude.

—vs—

- Seed length $d = k + d - 2\log\left(\frac{1}{\epsilon}\right) - O(1)$
- Output length: $\log(n-k) + 2\log(1/\epsilon) \ + O(1)$

Group Work: prove the claim!

- If we choose $\ell = \frac{n-k}{2} + \log\left(\frac{1}{\epsilon}\right)$, then this is a (k, ϵ)-extractor.
  - Seed length: $d = \ell \cdot \log(D) = O(\ell)$
  - Output length: n

1. Let $\sigma \in \mathbb{R}^n$ represent the probability mass function of our input X. Explain why $Ext(X, U_d) \sim A^\ell \cdot \sigma$, where A is the normalized adjacency matrix for G.
2. Let $\pi = \frac{1}{N}\mathbf{1}$ correspond to the uniform distribution. Explain why $$\|U_n - Ext(X, U_d)\|_{TV} = \left\|\pi - A^\ell \cdot \sigma\right\|_{TV} \leq \frac{\sqrt{N}}{2}\lambda(G)^\ell \|\pi - \sigma\|_2$$
3. Argue that $\|\pi - \sigma\|_2 \leq 2 \cdot 2^{-k/2}$
4. Conclude that $\|U_n - Ext(X, U_d)\|_{TV} \leq \epsilon$, which means that Ext is a (k, ϵ)-extractor.

Solutions

1. Why is $Ext(X, U_d) \sim A^{\ell} \cdot \sigma$

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- By definition, σ is the distribution of X, the starting distribution for our random walk.
- The normalized adjacency matrix A is the transition matrix for the random walk on G.
- Since U_d is uniformly random, we just take an ℓ-step random walk on G starting from the distribution σ to get the output of Ext.
- The distribution of that is $A^{\ell} \cdot \sigma$, as we saw before when we studied Markov chains.

2. Bounding $||U_n - Ext(X, U_d)||$

- Let $Y = Ext(X, U_d)$
- Let π be the uniform distribution.

$$\| U_n - Y \|_{TV} = \frac{1}{2} \| \pi - A^{\ell} \sigma \|_1$$

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- Let $Y = Ext(X, U_d)$
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$$\| U_n - Y \|_{TV} = \frac{1}{2} \| \pi - A^\ell \sigma \|_1$$

$$\| \pi - A^\ell \cdot \sigma \|_1 = \| A^\ell (\pi - \sigma) \|_1$$

Since $A\pi = \pi$

$$\leq \sqrt{N} \, \| A^\ell (\pi - \sigma) \|_2$$

Cauchy-Schwarz

$$\leq \sqrt{N} \, \lambda(G)^\ell \, \| \pi - \sigma \|_2$$

Since $\pi - \sigma \perp \pi$, and π is the top eigenvector.

3. Bounding $||\pi - \sigma||_2$

$$\|\pi - \sigma\|_2 \leq \|\pi\|_2 + \|\sigma\|_2$$

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$$\|\pi - \sigma\|_2 \leq \|\pi\|_2 + \|\sigma\|_2 \leq 2^{-n/2} + 2^{-k/2} \leq 2^{-\frac{k}{2}+1}$$

$$\|\pi\|_2 = \Big(\sum_{x \in \{0,1\}^n} 1/2^{2n}\Big)^{1/2} = 2^{-n/2}$$

$$\|\sigma\|_2 \leq 2^{-k/2}$$

Warmup!

4. Ext is a (k, ϵ)-extractor

- Let $Y = Ext(X, U_d)$
- Let π be the uniform distribution.

$\|U_n - Y\|_{TV} \leq \epsilon$?

We know:

- $\|U_n - Y\|_{TV} \leq \frac{\sqrt{N}}{2} \cdot \lambda(G)^{\ell} \cdot \|\pi - \sigma\|_2$
- $\|\pi - \sigma\|_2 \leq 2 \cdot 2^{-\frac{k}{2}}$
- $\ell = \frac{n-k}{2} + \log\left(\frac{1}{\epsilon}\right)$
- $\lambda(G) \leq \frac{1}{2}$

4. Ext is a (k, ϵ)-extractor

- Let $Y = Ext(X, U_d)$
- Let π be the uniform distribution.

$$\|U_n - Y\|_{TV} \le \sqrt{N} \cdot \lambda(G)^{\ell} \cdot 2^{-k/2} \le 2^{\frac{n-k}{2}} \cdot \left(\frac{1}{2}\right)^{\ell}$$

$N = 2^n$

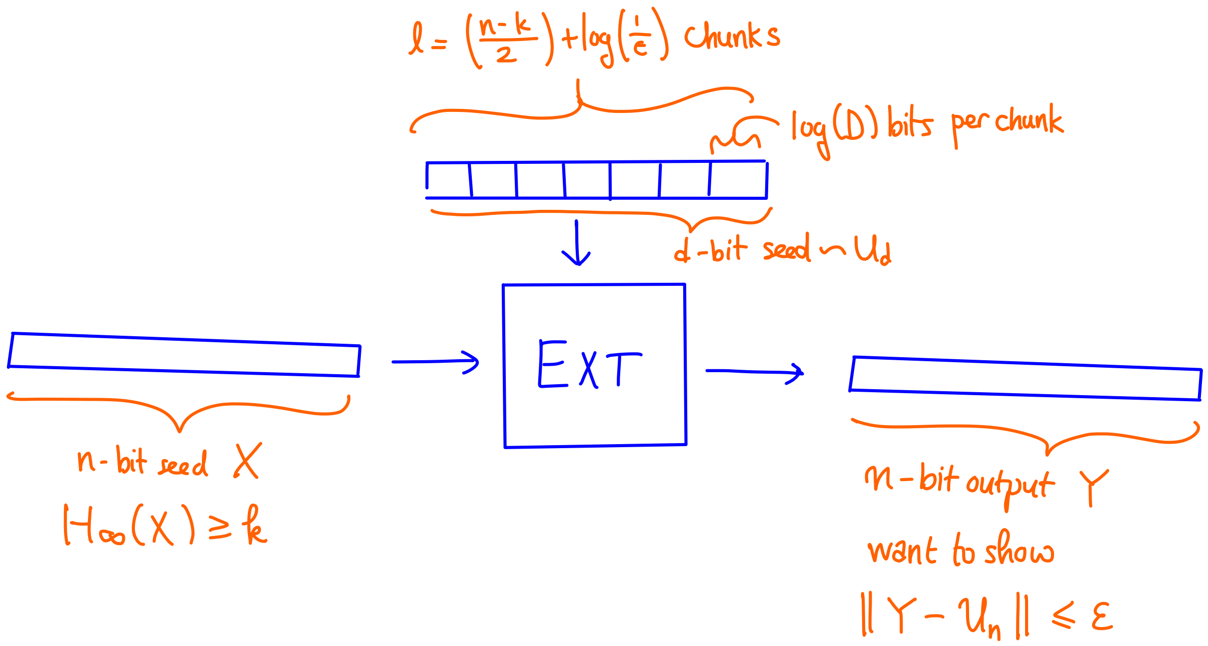
$\lambda(G) \le 1/2$

$\ell = \frac{n-k}{2} + \log(1/\epsilon)$

$$= 2^{\frac{n-k}{2}} \cdot 2^{-\left(\frac{n-k}{2} + \log(1/\epsilon)\right)} = 2^{-\log(1/\epsilon)} = \epsilon$$

Hooray!

- So Ext is a (k, ϵ) extractor.
- It's a pretty good one when $k = n - O(\log n)$, say.
  - In that case the seed length is $O\left(\log\left(\frac{n}{\epsilon}\right)\right)$



- Why do we care? If k is large (as above), then we can actually just exhaust over the seeds! We don't need true randomness!

Recap

- We can use a good spectral expander to get an okay extractor.
- This extractor is pretty good when k is large!