Class 6

The power of two choices

Recap I

- Balls and bins!
- Powerful tool: Poissonization (Poissonification?)
- $X \sim Poi(\lambda)$:
  - $\Pr[X = k] = \frac{e^{-\lambda}\lambda^k}{k!}$
  - $\mathrm{E}[X] = \mathrm{Var}[X] = \lambda$
  - $\Pr[|X - \lambda| \geq c] \leq 2\exp\left(\frac{-c^2}{2(c+\lambda)}\right)$

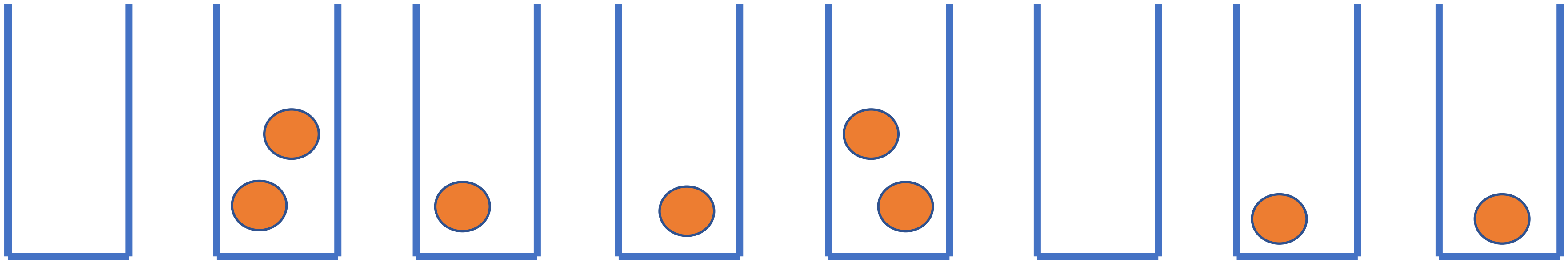
Recap II

- If you drop $k \sim Poi(n)$ balls into m bins, then:
  - Let X_i = #(Balls in bin i)
  - $X_i \sim Poi\left(\frac{n}{m}\right)$
  - The X_i are all independent

- "Poissonization":
  - #(Balls in bin i when you drop n balls into m bins) $\approx X_i$
  - Work with the X_i instead.

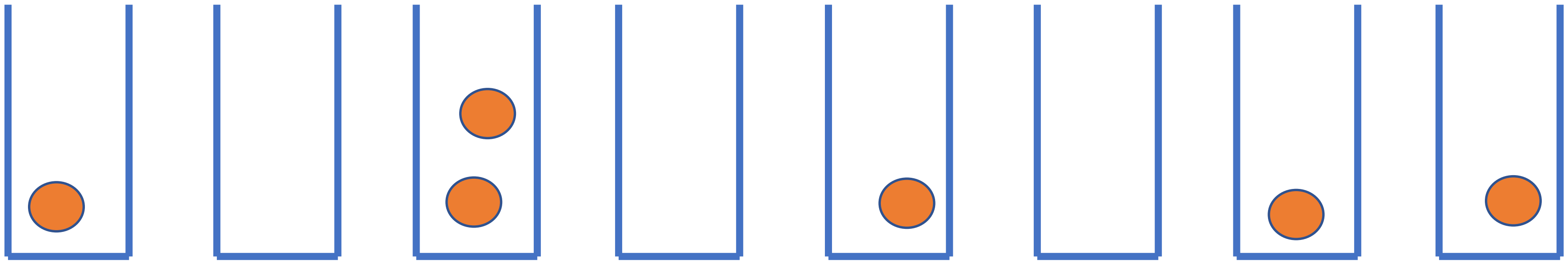
Recap III: Maximum Load

- n balls into n bins.
- Max load is $\Theta\left(\frac{\log n}{\log \log n}\right)$



Today: The power of two choices

- Drop n balls into n bins.
- For each ball, pick two bins at random.
- The ball goes in the less-full bin. (Break ties arbitrarily).



The power of two choices

- n balls into n bins, completely randomly:
  - Max load is $\Theta\left(\frac{\log n}{\log \log n}\right)$

- n balls into n bins, according to the “pick two” scheme:
  - Max load is $\Theta(\log \log n)$

Exponentially smaller!

This is useful, for example, when trying to efficiently assign jobs to processors and wanting to balance the loads.

Group Work

Intuition

Definitions:

$\beta_2 = \frac{n}{2}$ $\quad\quad$ $\beta_i = \frac{\beta_{i-1}^2}{n}$

$B(i,t)$ = number of bins with $\geq i$ balls after step t

1. Explain why $B(2,t) \leq \beta_2$ for all t.

2. Show that

$$\Pr\{\text{Ball } t \text{ is the} \geq \text{3rd ball to land in its bin}\} \leq \left(\frac{B(2,t-1)}{n}\right)^2 \leq \frac{\beta_2^2}{n^2},$$

for all t.

3. Show that, for all t,

$$\mathbb{E}[B(3,t)] \leq \beta_3.$$

4. **Suppose** that $B(3,t) \leq \beta_3$ for all t. That is, suppose that the thing that you showed in expectation before actually held. Show that, for all t,

$$\mathbb{E}[B(4,t)] \leq \beta_4.$$

5. **Suppose** that this logic continued, and you could show that $\mathbb{E}[B(i,t)] \leq \beta_i$ for all t. What would the max load be?

Solutions: Question 1

- $\beta_2 = n/2$.
- There can't be more than 2 buckets with $\geq n/2$ balls in them (since there's only n balls total).
- So $B(2,t) \leq B(2,n) \leq \beta_2$

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Definitions:

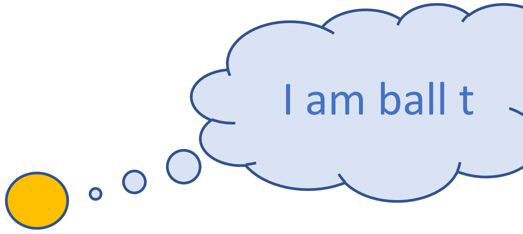
$$\beta_2 = \frac{n}{2} \qquad \beta_i = \frac{\beta_{i-1}^2}{n}$$

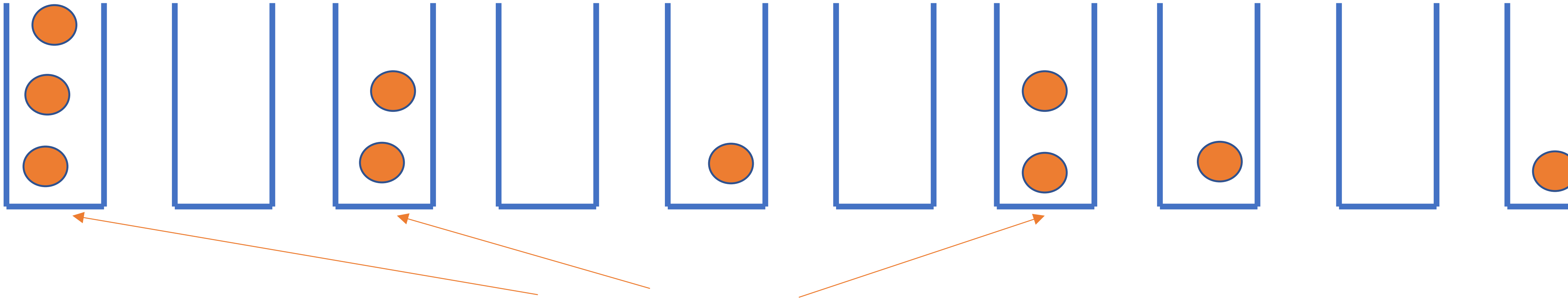
$B(i,t) =$ number of bins with $\geq i$ balls after step t

• Probability that ball t is the third (or greater) ball in its bucket:

Solutions: Question 2

- Probability that ball t is the third (or greater) ball in its bucket:
  - Need to choose two buckets with at least two things in them.
  - The probability of that is at most $\left(\frac{B(2,t-1)}{n}\right)^2 \leq \left(\frac{\beta_2}{n}\right)^2$





$B(2, t-1)$ buckets with ≥ 2 balls in them.

Solutions: Question 3

3. Show that, for all t,
$$\mathbb{E}[B(3,t)] \le \beta_3.$$

$$\mathbb{E}\left[B(3,t)\right] \le \mathbb{E}\left[B(3,n)\right]$$

This is because the number of bins with at least 3 balls is at least the number of balls that were at least 3rd in their bin.

$$\le \mathbb{E}\left[\sum_{t=1}^{n} \mathbb{1}\left\{\begin{array}{l}\text{ball } t \text{ is} \\ \ge 3^{rd} \text{ in its bucket}\end{array}\right\}\right]$$

$$= \sum_{t=1}^{n} \mathbb{P}\left\{\begin{array}{l}\text{ball } t \text{ is} \\ \ge 3^{rd} \text{ in its bucket}\end{array}\right\}$$

$$\le \sum_{t=1}^{n} \left(\frac{\beta_2}{n}\right)^2 = \beta_2^2 / n = \beta_3$$

def. of β_3

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Solutions: Question 4

$$\mathbb{E}[B(4,t)] \leq \mathbb{E}[B(4,n)]$$

$$\leq \mathbb{E}\left[\sum_{t=1}^{n} \mathbb{1}\left\{\begin{array}{l}\text{ball } t \text{ is} \\ \geq 4^{th} \text{ in its bucket}\end{array}\right\}\right]$$

$$= \sum_{t=1}^{n} \mathbb{P}\left\{\begin{array}{l}\text{ball } t \text{ is} \\ \geq 4^{th} \text{ in its bucket}\end{array}\right\}$$

$$\leq \sum_{t=1}^{n} \left(\frac{\beta_3}{n}\right)^2 = \beta_3^2/n \underset{\text{def. of } \beta_4}{=} \beta_4$$

*Note: As per the instructions in the question, we are ignoring anything about conditioning on the event that $B(3,t) \leq \beta_3$

5. **Suppose** that this logic continued, and you could show that $\mathbb{E}[B(i,t)] \le \beta_i$ for all t. What would the max load be?

Solutions: Question 5

- $\beta_i = \frac{n}{2^{2^{i-2}}}$
- You can see this by doing out a bunch and guessing the pattern:
  - $\beta_2 = \frac{n}{2}$
  - $\beta_3 = \frac{1}{n}\left(\frac{n}{2}\right)^2 = \frac{n}{2^2}$
  - $\beta_4 = \frac{1}{n}\left(\frac{n}{2^2}\right)^2 = \frac{n}{2^{2^2}}$
  - $\beta_5 = \frac{1}{n}\left(\frac{n}{2^{2^2}}\right)^2 = \frac{n}{2^{2^3}}$
  - $\beta_6 = \frac{1}{n}\left(\frac{n}{2^{2^3}}\right)^2 = \frac{n}{2^{2^4}}$

(And formally you can prove it by induction.)

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Solutions: Question 5

- $\beta_i = \dfrac{n}{2^{2^{i-2}}}$
- If we believe that $B(i,n) \leq \beta_i$ for all i, then at some point (for large enough i), we have $\beta_i < 1$.
  - That would imply that $B(i,n) = 0$.
  - That would mean that the number of bins with load $\geq i$ is 0, aka the max load is $i - 1$.

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- Set $\frac{n}{2^{2^{i-2}}} < 1$ and solve for i: get $i > \log\log n + 2$.

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- Set $\frac{n}{2^{2^{i-2}}} < 1$ and solve for i: get $i > \log\log n + 2\,.$
- Conclude that max load is $\Theta(\log\log n)$, assuming that the "in expectation" stuff holds exactly.

Here's the outline for an argument

WARNING: This is incorrect in a few ways.

1. Define $\beta_2 = n/2$, $\beta_i = 2(\beta_{i-1})^2/n$

this "2" is new.

number of bins w/ $\geq i$ balls
after all n are tossed.

2. Argue by induction on i that, with probability $\geq 1 - i/n^2$, $B(i,n) \leq \beta_i$:

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- Base case for $i=2$ is by definition.
- Assuming that $B(i-1,n) \leq \beta_{i-1}$ (which holds w/ prob $\geq 1 - \frac{i-1}{n^2}$ by induction)

  the same logic from before implies that

$$\mathbb{E}\left[\sum_{t=1}^{n} \mathbb{1}\left\{ \begin{array}{c} \text{ball } t \text{ is the } \geq i^{th} \\ \text{in its bucket} \end{array} \right\} \right] \leq \beta_{i-1}^2 / n$$

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$$\mathbb{E}\left[\sum_{t=1}^{n} \mathbb{1}\left\{ \begin{array}{c} \text{ball } t \text{ is the } \geq i^{th} \\ \text{in its bucket} \end{array} \right\} \right] \leq \beta_{i-1}^2 / n$$

- A Chernoff bound says that

$$\mathbb{P}\Bigg[\underbrace{\sum_{t=1}^{n} \mathbb{1}\left\{ \begin{array}{c} \text{ball } t \text{ is the } \geq i^{th} \\ \text{in its bucket} \end{array} \right\}}_{\text{we have } B(i,n) \leq \text{ this, as before.}} > \frac{\overbrace{2\beta_{i-1}^2}^{\text{this is } \beta_i \text{, but also twice the expectation}}}{n} \Bigg] \leq \exp\left(-\beta_i/3\right)$$

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- Therefore, if $\beta_i \geq 6 \log n$, this probability is $\leq 1/n^2$. By a union bound w/ the event that $\underbrace{B(i-1,n) > \beta_{i-1}}_{\text{prob. is } \leq \frac{i-1}{n^2}}$, $\mathbb{P}\left[B(i,n) > \beta_i \right] \leq 1 - i/n^2$.

This establishes the inductive hypothesis for i, as long as $\beta_i \geq 6 \log n$

3. Choose i^* so that $\beta_{i^*} \geq 6 \cdot \log n$.

So the argument above shows that, whp, $B(i,n) \leq \beta_i \ \forall\, i \leq i^*$.

You can check that $\beta_{i^*} \approx 6 \log n$ when $i^* = \Theta(\log \log n)$.

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4. We conclude that, whp, $B(i,n) \leq \beta_i$ for all $i \leq i^* = \Theta(\log \log n)$.

Just as before, this implies that the max load is $\Theta(\log \log n)$

Group Work

- What was wrong with this argument/sketch?
- There are at least two or three major problems
    - depending on what you count as “major”

Three problems

1. Can't apply the Chernoff bound – the random variables are not independent!
2. The end of the argument doesn't make any sense! We showed that $B(i, n) \leq \beta_i$ whenever $i \leq i^*$, but $\beta_{i^*} \approx 6 \log n$.
   - So there are still about $6 \log n$ buckets with at least i^* balls in them.
   - (It is true that $i^* = \Theta(\log \log n)$ though).
3. We are not being careful about the conditioning.

Fix for the Chernoff bound problem (sketch)

$$\mathbb{P}\left\{ \sum_{t=1}^{n} \mathbb{1}\left\{ \begin{array}{c}\text{Ball } t \text{ is } \geq i^{th} \\ \text{ball to land in its} \\ \text{bin}\end{array} \right\} \right\}$$

$$= \mathbb{P}\left\{ \underbrace{\sum_{t=1}^{n-1} \mathbb{1}\left\{ \begin{array}{c}\text{Ball } t \text{ is } \geq i^{th} \\ \text{ball to land} \\ \text{in its bin}\end{array} \right\}}_{} + \underbrace{\mathbb{1}\left\{ \begin{array}{c}\text{Ball } n \text{ is the } \geq i^{th} \\ \text{ball to land} \\ \text{in its bin}\end{array} \right\}}_{} \right\}$$

Inductively assume that this sum behaves like Binomial$\left(n-1, \leq \left(\frac{\beta_{i-1}}{n}\right)^2\right)$

Conditioned on balls $1, 2, \ldots, n$, we STILL have $\mathbb{E}\left[\mathbb{1}\left\{ \begin{array}{c}\text{Ball } n \text{ is the } \geq i^{th} \\ \text{ball to land} \\ \text{in its bin}\end{array} \right\}\right] \leq \left(\frac{\beta_{i-1}}{n}\right)^2$, over the randomness in ball n

both of these together behave like Binomial$\left(n, \leq \left(\frac{\beta_{i-1}}{n}\right)^2\right)$

Fix for the "The argument didn't finish!" problem.

Say that $B(i,n) \leq \beta_i \ \forall \ i \leq i^*$, where $i^* = \Theta(\log\log n)$ is such that $\beta_{i^*} = 6\log n$.

Then $\mathbb{P}\{B(i^*+1, n) \geq 1\} \leq \mathbb{P}\{B(i^*, n) \geq 2\}$

This is what we actually claimed to have shown.

because there have to be at least 2 bins w/ i^*+1 things in them for some bin to end up w/ i^* things.

$$\leq \sum_{s<t} \mathbb{P}\left\{\begin{array}{l}\text{BOTH balls } s \text{ and } t \\ \text{are} \geq \text{the } i^{*}\text{'th} \\ \text{thing in their bin}\end{array}\right\}$$

(union bound over all the balls)

$$\leq \binom{n}{2} \mathbb{P}\left\{\begin{array}{l} s \text{ is} \geq i^{*\text{th}} \\ \text{in its bin}\end{array}\right\} \cdot \mathbb{P}\left\{\begin{array}{l} t \text{ is} \geq i^{*\text{th}} \\ \text{in its bin}\end{array} \middle| \begin{array}{l} s \text{ is} \geq i^{*\text{th}} \\ \text{in its bin}\end{array}\right\}$$

this doesn't affect our argument before for t, assuming $B(i^*, n) \leq \beta_{i^*}$.

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this doesn't affect our argument before for t, assuming $B(i^*, n) \leq \beta_{i^*}$.

Fix for the "The argument didn't finish!" problem, ctd.

$$\mathbb{P}\left\{ B(i^*+1, n) \geq 1 \right\} \leq \binom{n}{2} \mathbb{P}\left\{ \begin{array}{l} s \text{ is } \geq i^{* \text{ th}} \\ \text{in its} \\ \text{bin} \end{array} \right\} \cdot \mathbb{P}\left\{ \begin{array}{l} t \text{ is } \geq i^{* \text{ th}} \\ \text{in its bin} \end{array} \middle| \begin{array}{l} s \text{ is } \geq i^{* \text{ th}} \\ \text{in its} \\ \text{bin} \end{array} \right\}$$

this doesn't affect our argument before for t, assuming $B(i^*, n) \leq \beta_{i^*}$.

$$\leq \binom{n}{2} \left(\frac{\beta_{i^*}}{n} \right)^2 \cdot \left(\frac{\beta_{i^*}}{n} \right)^2$$

$$\leq n^2 \cdot \left(\frac{6 \log n}{n} \right)^4 = \frac{6^4 \log^4 n}{n^2} = o(1).$$

Fix for the
"We are being sloppy about the conditioning"
problem.

- Be less sloppy about the conditioning.
  - (It's a bit delicate but not that interesting…)