Class 7

Sparsest Cuts from Metric Embeddings

Warm-Up

Group Work

Let $G = (V, E)$ be a weighted, undirected graph, on n vertices with edge weights w_{uv} on the edge $\{u, v\} \in E$. Let $d : V \times V \to \mathbb{R}$ be the associated graph metric.

Explain how to efficiently find and apply a map $f : V \to \mathbb{R}^k$, for $k = O(\log^2 n)$, so that

$$\frac{\sum_{\{u,v\}\in E} \|f(u) - f(v)\|_1}{\sum_{\{u,v\}\in \binom{V}{2}} \|f(u) - f(v)\|_1} \le O(\log n) \frac{\sum_{\{u,v\}\in E} d(u,v)}{\sum_{\{u,v\}\in \binom{V}{2}} d(u,v)}$$

holds with high probability. Above, $\binom{V}{2}$ refers to the set of all unordered pairs $\{u, v\}$ for $u, v \in V$ and $u \neq v$.

Announcements:

- HW3 due Friday!
- Please fill out feedback form
- Starting today, I'll try to post some version of my in-class slides on the website. (Please email me or ask on Ed if I forget).

Recap

- Bourgain's embedding!
  - Randomized embedding from *any* X of size n into (R^k, ℓ_1)
  - Distortion $\mathrm{O}(\log n)$
  - $k = O(\log^2 n)$

Questions?

Minilectures, quiz, warmup?

Group Work

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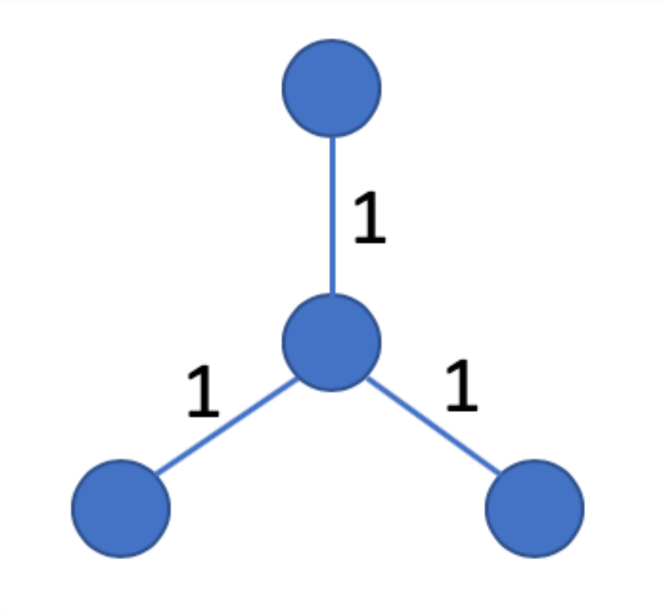
holds with high probability. Above, $\binom{V}{2}$ refers to the set of all unordered pairs $\{u, v\}$ for $u, v \in V$ and $u \neq v$.

Q1 Can it be embedded?

6 Points

Consider the graph metric space (V, d) induced by the following graph:



Into which space can (V, d) be isometrically embedded? Select all that apply.

- [x] $(\mathbb{R}^2, d_1)$
- [ ] $(\mathbb{R}^2, d_2)$
- [x] $(\mathbb{R}^2, d_\infty)$

Q2 An embedding

3 Points

Let (X, d) be a finite metric space with n points, and write $X = \{x_1, x_2, \ldots, x_n\}$. Consider the map $f : X \to \mathbb{R}^n$ given by

$$f(y) = (d(y, x_1), d(y, x_2), \ldots, d(y, x_n))$$

Which of the following are true? Check all that apply.

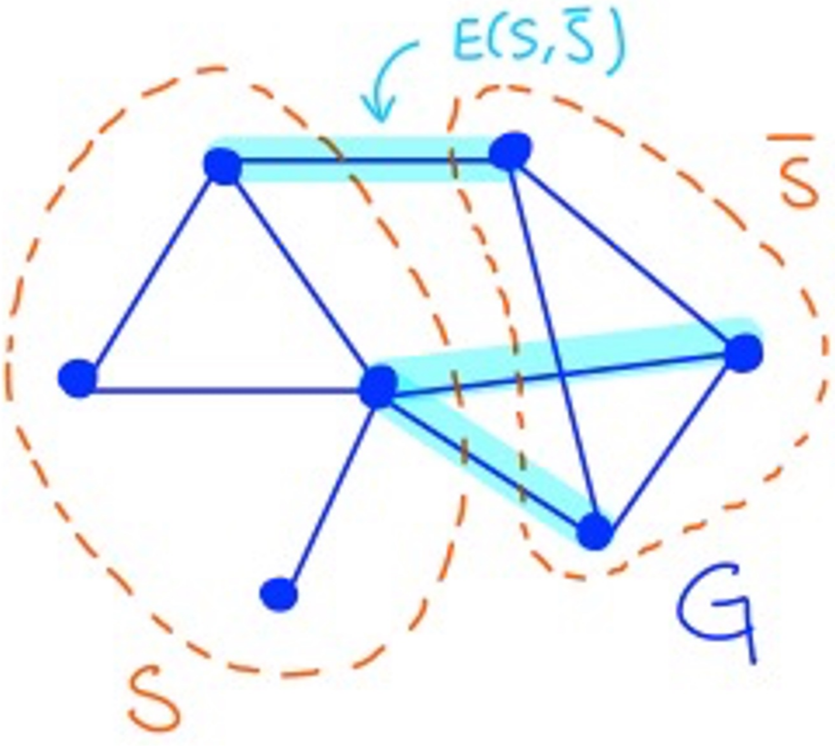
- [x] f is an isometric embedding into $(\mathbb{R}^n, d_\infty)$
- [ ] f is an isometric embedding into $(\mathbb{R}^n, d_1)$

Plan for today

- Application of Bourgain's embedding to sparsest cuts

Sparsest Cuts

- G = (V,E) is an undirected, unweighted graph:



$$\varphi(G,S) = \frac{|E(S,\bar{S})|}{|S|\cdot|\bar{S}|}$$

← Number of edges between S and $\bar{S}$ in G

← Number of edges between S and $\bar{S}$ in the complete graph

"sparsity" of the cut $(S,\bar{S})$.

$$\varphi(G) = \min_{\substack{S\subseteq V \\ S\neq\phi,\, S\neq V}} \varphi(G,S)$$

Goal: Find a sparsest cut

- a.k.a., find S so that $\phi(G, S) = \phi(G)$

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- Problem: this is NP-hard.

Assuming plausible complexity-theoretic assumptions, it's NP-hard even to approximate $\phi(G)$ to within a constant factor.

Goal: Find a sparsest cut

- a.k.a., find S so that $\phi(G,S) = \phi(G)$
- Problem: this is NP-hard.

- Today: randomized algorithm to (probably) find S so that

$$\phi(G,S) \leq O(\log n) \cdot \phi(G)$$

Assuming plausible complexity-theoretic assumptions, it's NP-hard even to approximate $\phi(G)$ to within a constant factor.

Outline

- Group Work 1:

$$\varphi(G) = \min_{f: V \to \mathbb{R}^k} \frac{\sum_{\{u,v\} \in E} \|f(u) - f(v)\|_1}{\sum_{\{u,v\} \in \binom{V}{2}} \|f(u) - f(v)\|_1}$$

- Group Work 2:
  - …use something about metric embeddings to approximate that thing.

Group Work!

1. $$\phi(G) = \min_{f:V \to \{0,1\}} \frac{\sum_{\{u,v\} \in E} |f(u) - f(v)|}{\sum_{\{u,v\} \in \binom{V}{2}} |f(u) - f(v)|},$$

2. $$\phi(G) = \inf_{f:V \to \mathbb{R}} \frac{\sum_{\{u,v\} \in E} |f(u) - f(v)|}{\sum_{\{u,v\} \in \binom{V}{2}} |f(u) - f(v)|},$$

3. $$\phi(G) = \min_{f:V \to \mathbb{R}^k} \frac{\sum_{\{u,v\} \in E} \|f(u) - f(v)\|_1}{\sum_{\{u,v\} \in \binom{V}{2}} \|f(u) - f(v)\|_1},$$

This one is the conceptually important one

Just try to get some intuition for these.

Solution: Problem 1

$$\varphi(G) = \min_{f:V\to\{0,1\}} \frac{\sum_{\{u,v\}\in E} |f(u)-f(v)|}{\sum_{\{u,v\}\in\binom{V}{2}} |f(u)-f(v)|}$$

$$S \subseteq V \longleftrightarrow f: V \to \{0,1\}$$

$$f(u) = \mathbb{1}\{u \in S\} \quad \text{aka} \quad S = \{u \in V : f(u) = 1\}$$

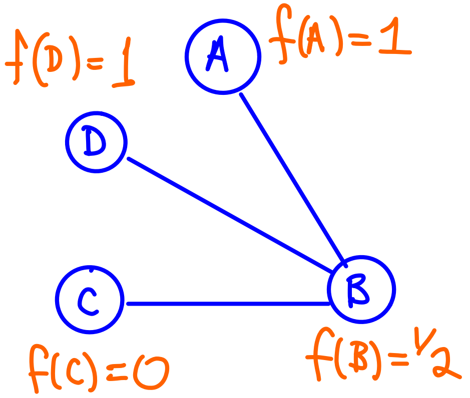
$$\frac{\sum_{\{u,v\}\in E} \overbrace{|f(u)-f(v)|}^{\mathbb{1}\{\{u,v\} \text{ crosses the cut}\}}}{\sum_{\{u,v\}\in\binom{V}{2}} |f(u)-f(v)|} = \frac{\#\text{edges crossing the cut } S,\bar{S} \text{ in } G}{\#\text{edges crossing the cut } S,\bar{S} \text{ in } K_n} = \frac{|E(S,\bar{S})|}{|S|\cdot|\bar{S}|} = \varphi(G,S)$$

Solution: Problem 2

Note: this is just meant as intuition

$$\varphi(G) = \min_{f:V\to\mathbb{R}} \frac{\sum_{\{u,v\}\in E} |f(u)-f(v)|}{\underbrace{\sum_{\{u,v\}\in\binom{V}{2}} |f(u)-f(v)|}_{\text{Call this } R(f)}}$$

EXAMPLE: Say $f: V \longrightarrow \{0, \frac{1}{2}, 1\}$



$$R(f) = \frac{|1-\frac{1}{2}| + |\frac{1}{2}-0| + |\frac{1}{2}-1|}{|1-\frac{1}{2}| + |\frac{1}{2}-0| + |\frac{1}{2}-1| + |1-0| + |1-1| + |1-0|}$$

Solution: Problem 2

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$$\varphi(G) = \min_{f: V \to \mathbb{R}} \underbrace{\frac{\sum_{\{u,v\} \in E} |f(u) - f(v)|}{\sum_{\{u,v\} \in \binom{V}{2}} |f(u) - f(v)|}}_{\text{Call this } R(f)}$$

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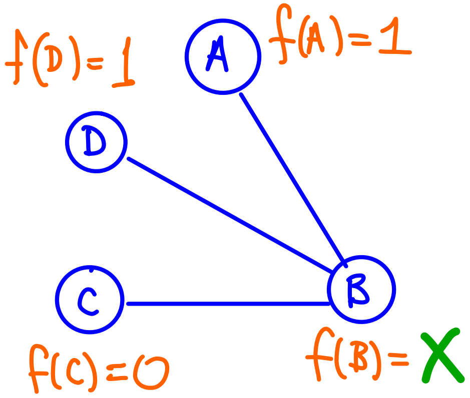

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$$R(x) = \frac{|1 - x| + |x - 0| + |x - 1|}{|1 - x| + |x - 0| + |x - 1| + |1 - 0| + |1 - 1| + |1 - 0|}$$

Solution: Problem 2

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$$\varphi(G) = \min_{f:V\to\mathbb{R}} \underbrace{\frac{\sum_{\{u,v\}\in E} |f(u)-f(v)|}{\sum_{\{u,v\}\in\binom{V}{2}} |f(u)-f(v)|}}_{\text{Call this } R(f)}$$

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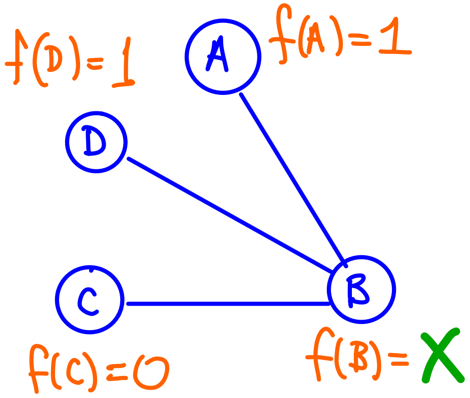

$$R(x) = \frac{|1-x| + |x-0| + |x-1|}{|1-x| + |x-0| + |x-1| + |1-0| + |1-1| + |1-0|}$$

Solution: Problem 2

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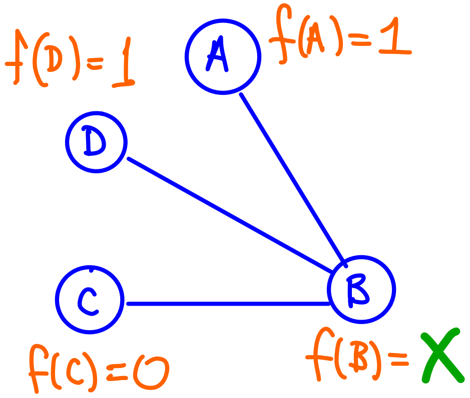

$$R(x) = \frac{|1-x|+|x-0|+|x-1|}{|1-x|+|x-0|+|x-1|+|1-0|+|1-1|+|1-0|}$$

for $x\in[0,1]$...

$$R(x) = \frac{(1-x)+(x-0)+(1-x)}{(1-x)+(x-0)+(1-x)+2} = \frac{2-x}{4-x}$$

Solution: Problem 2

Note: this is just meant as intuition

$$\varphi(G) = \min_{f:V\to\mathbb{R}} \underbrace{\frac{\sum_{\{u,v\}\in E} |f(u)-f(v)|}{\sum_{\{u,v\}\in\binom{V}{2}} |f(u)-f(v)|}}_{\text{Call this } R(f)}$$

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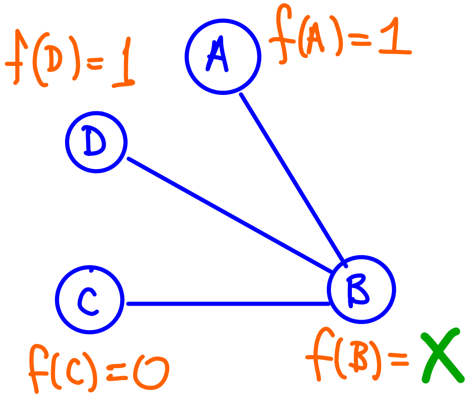

for $x \in [0, 1]$ ... $R(x) = \dfrac{2-x}{4-x}$

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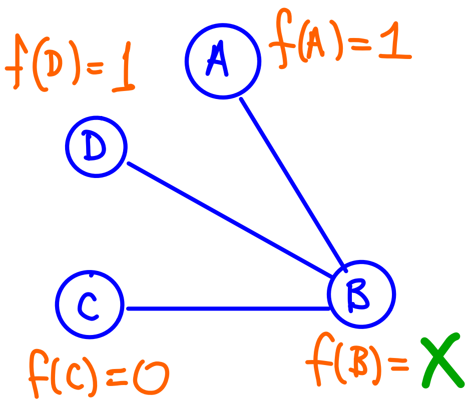
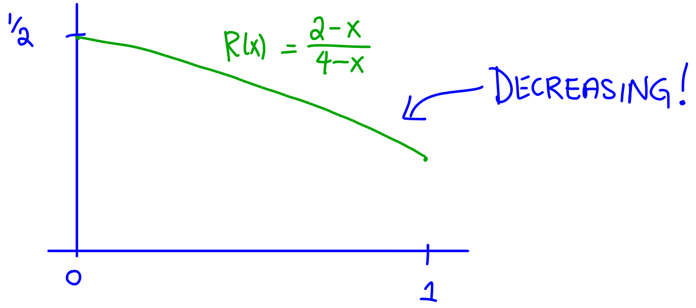


for $x \in [0,1]$... $R(x) = \dfrac{2-x}{4-x}$

- This will always be either (weakly) increasing or decreasing.



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$$\varphi(G) = \min_{f: V \to \mathbb{R}} \underbrace{\frac{\sum_{\{u,v\} \in E} |f(u) - f(v)|}{\sum_{\{u,v\} \in \binom{V}{2}} |f(u) - f(v)|}}_{\text{Call this } R(f)}$$

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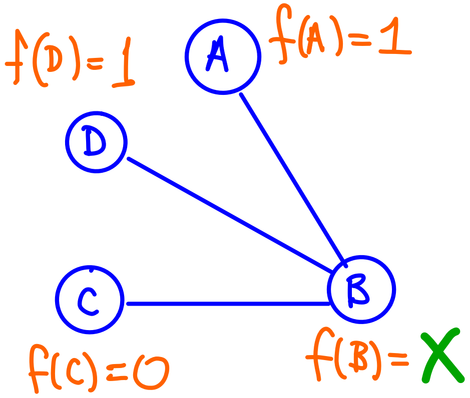
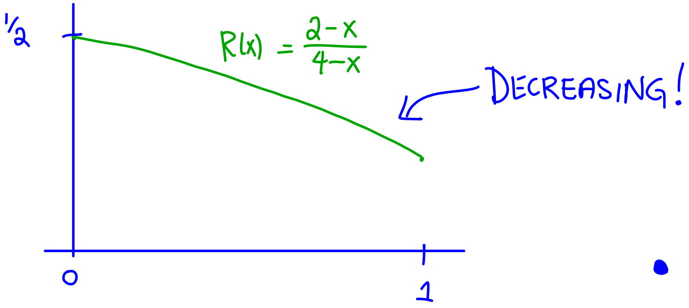
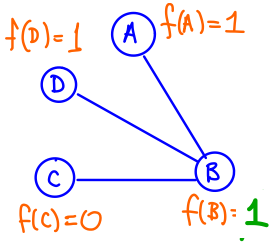


for $x \in [0, 1]$ ... $R(x) = \frac{2 - x}{4 - x}$



- This will always be either (weakly) increasing or decreasing.
- If we replace f with  , $R(f)$ doesn't increase.
- $\Rightarrow$ There is a fn $f: V \to \mathbb{R}$ that minimizes $R(f)$ that takes only two values.

Solution: Problem 2

Note: this is just meant as intuition

$$\varphi(G) = \min_{f:V\to\mathbb{R}} \frac{\sum_{\{u,v\}\in E} |f(u)-f(v)|}{\sum_{\{u,v\}\in\binom{V}{2}} |f(u)-f(v)|}$$

- $\Rightarrow$ There is a fn $f: V \to \mathbb{R}$ that minimizes $R(f)$ that takes only two values.

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From before:

$$\varphi(G) = \min_{f:V\to\{0,1\}} \frac{\sum_{\{u,v\}\in E} |f(u)-f(v)|}{\sum_{\{u,v\}\in\binom{V}{2}} |f(u)-f(v)|}$$

We just showed that the min over $f: V \to \mathbb{R}$ is actually attained by some $f: V \to \{0,1\}$.

Solution: Problem 3

$$\varphi(G) = \min_{f:V\to\mathbb{R}^k} \frac{\sum_{\{u,v\}\in E} \|f(u)-f(v)\|_1}{\sum_{\{u,v\}\in\binom{V}{2}} \|f(u)-f(v)\|_1}$$

If $f: V \to \mathbb{R}^k$, say $f(x) = (f_1(x), \ldots, f_k(x))$,

$$\frac{\sum_{\{u,v\}\in E} \|f(u)-f(v)\|_1}{\sum_{\{u,v\}\in\binom{V}{2}} \|f(u)-f(v)\|_1} = \frac{\sum_{i=1}^k \left(\sum_{\{u,v\}\in E} |f_i(u)-f_i(v)|\right)}{\sum_{i=1}^k \left(\sum_{\{u,v\}\in\binom{V}{2}} |f_i(u)-f_i(v)|\right)} \geq \underbrace{\min_i \frac{\sum_{\{u,v\}\in E} |f_i(u)-f_i(v)|}{\sum_{\{u,v\}\in\binom{V}{2}} |f_i(u)-f_i(v)|}}_{\text{this is the case where } f:V\to\mathbb{R}}$$

So adding more dimensions to f can't make this value any smaller than $f: V \to \mathbb{R}$

Conclusion

$$\varphi(G) = \min_{f: V \to \mathbb{R}^k} \frac{\sum_{\{u,v\} \in E} \|f(u) - f(v)\|_1}{\sum_{\{u,v\} \in \binom{V}{2}} \|f(u) - f(v)\|_1}$$

- Next up: using this to design an algorithm!

Let’s come up with an algorithm!

- Hope: find f to minimize $R(f) := \frac{\sum_{\{u,v\}\in E} \|f(u) - f(v)\|_1}{\sum_{\{u,v\}\in \binom{V}{2}} \|f(u) - f(v)\|_1}$
  - Unfortunately that’s not so easy…

- Instead,

Find values $d_{u,v} \in \mathbb{R}$ for all $u \neq v \in V$ to minimize

$$Q(d) := \sum_{\{u,v\}\in E} d_{u,v}$$

subject to:

- $d_{u,v} = d_{v,u} \geq 0$ for all u, v
- $d_{u,v} + d_{v,w} \geq d_{u,w}$ for all u, v, w
- $\sum_{\{u,v\}\in \binom{V}{2}} d_{u,v} = 1$

This is a **linear program.** Turns out we can solve it efficiently.

Group Work!

2. Suppose that d^* is the minimizer of the problem above.

   Explain why $Q(d^*) \leq \phi(G)$.

3. Find a randomized algorithm to approximate $\phi(G)$. More precisely, give a randomized algorithm that finds $f : V \to \mathbb{R}^k$ so that, with high probability,

$$\frac{\sum_{\{u,v\}\in E} \|f(u) - f(v)\|_1}{\sum_{\{u,v\}\in\binom{V}{2}} \|f(u) - f(v)\|_1} \leq O(\log n)\phi(G).$$

4. Given f as in the previous part, explain how to efficiently find a set $S \subset V$ so that

$$\phi(G, S) \leq O(\log n)\phi(G).$$

Find values $d_{u,v} \in \mathbb{R}$ for all $u \neq v \in V$ to minimize

$$Q(d) := \sum_{\{u,v\}\in E} d_{u,v}$$

subject to:

- $d_{u,v} = d_{v,u} \geq 0$ for all u, v
- $d_{u,v} + d_{v,w} \geq d_{u,w}$ for all u, v, w
- $\sum_{\{u,v\}\in\binom{V}{2}} d_{u,v} = 1$

Solution: Problem 2

$$\varphi(G) = \min_{f: V \to \mathbb{R}^k} \underbrace{\frac{\sum_{\{u,v\} \in E} \|f(u) - f(v)\|_1}{\sum_{\{u,v\} \in \binom{V}{2}} \|f(u) - f(v)\|_1}}_{R(f)}$$

$$Q(d) = \sum_{\{u,v\} \in E} d_{u,v} = \frac{\sum_{\{u,v\} \in E} d_{u,v}}{\sum_{\{u,v\} \in \binom{V}{2}} d_{u,v}}$$

$$\min \quad Q(d) = \sum_{\{u,v\} \in E} d_{u,v}$$

s.t.

- $d_{u,v} = d_{v,u} \geq 0 \quad \forall\, u,v$
- $d_{u,v} + d_{v,w} \geq d_{u,w} \quad \forall\, u,v,w$
- $\sum_{\{u,v\} \in \binom{V}{2}} d_{u,v} = 1$

Let $d_f(u,v) = \dfrac{\|f(u) - f(v)\|_1}{\sum_{\{u,v\} \in \binom{V}{2}} \|f(u) - f(v)\|_1}$

$$Q(d_f) = R(f)$$

Solution: Problem 2

- d_f is a metric, and in particular it satisfies these constraints.
- Thus, for all $f: V \to \mathbb{R}^k$,
$$Q(d^*) \leq Q(d_f) = R(f)$$
- $\Rightarrow Q(d^*) \leq \min_f R(f) = \phi(G)$

$$\min \quad Q(d) = \sum_{\{u,v\} \in E} d_{u,v}$$

s.t.
- $d_{u,v} = d_{v,u} \geq 0 \quad \forall\, u,v$
- $d_{u,v} + d_{v,w} \geq d_{u,w} \quad \forall\, u,v,w$
- $\sum_{\{u,v\} \in \binom{V}{2}} d_{u,v} = 1$

Let $d_f(u,v) = \dfrac{\| f(u) - f(v) \|_1}{\sum_{\{u,v\} \in \binom{V}{2}} \| f(u) - f(v) \|_1}$

$Q(d_f) = R(f)$

Solution: Problem 3

Find a randomized alg. to approximate $\phi(G)$

3. Find a randomized algorithm to approximate $\phi(G)$. More precisely, give a randomized algorithm that finds $f : V \to \mathbb{R}^k$ so that, with high probability,

$$\frac{\sum_{\{u,v\}\in E} \|f(u) - f(v)\|_1}{\sum_{\{u,v\}\in \binom{V}{2}} \|f(u) - f(v)\|_1} \leq O(\log n)\phi(G).$$

Solution: Problem 3

Find a randomized alg. to approximate $\phi(G)$

- This was the warm-up problem!

Technically a pseudo-metric.

- Find d^*, and interpret it as a metric.
- Use Bourgain's embedding on d^* to obtain $f: V \to \mathbb{R}^k$ so that $\frac{k}{b \log n} d^*(u,v) \le \|f(u) - f(v)\|_1 \le k\, d^*(u,v) \quad \forall u,v.$
- Following logic from the warm-up exercise,

$$R(f) \le O(\log n)\, Q(d^*) \le O(\log n)\, \varphi(G)$$

Solution: Problem 4

The final algorithm

4. Given f as in the previous part, explain how to efficiently find a set $S \subset V$ so that

$$\phi(G, S) \leq O(\log n)\phi(G).$$

- Find d^* by solving the linear program.
- Find $f: V \to \mathbb{R}^k$ by applying Bourgain's embedding to d^*
- Write $f(x) = (f_1(x), \ldots, f_k(x))$ and find $i^* = \underset{i}{\operatorname{argmin}} \dfrac{\sum_{\{u,v\} \in E} |f_i(u) - f_i(v)|}{\sum_{\{u,v\} \in \binom{V}{2}} |f_i(u) - f_i(v)|}$
- While $f_i(x)$ takes on ≥ 3 distinct values $a_1 < a_2 < a_3 < \cdots$:
  - Set a_2 to either a_1 or a_3, whichever makes $R(f_i)$ smaller.
- When f_i only takes 2 values, a_1, a_2, set $f_i \leftarrow \dfrac{f_i - a_1}{a_2 - a_1}$
- Return $S = \{u \in V : f_i(u) = 1\}$

Recap

- We can find approximately-sparsest cuts efficiently!

- Step 1: $$\varphi(G) = \min_{f:V\to\mathbb{R}^k} \frac{\sum_{\{u,v\}\in E} \|f(u)-f(v)\|_1}{\sum_{\{u,v\}\in\binom{V}{2}} \|f(u)-f(v)\|_1}$$

- Step 2: Use an LP to to find **some** metric d^* (not necessarily an ℓ_1 metric) so that this quantity is small.
- Step 3: Use Bourgain's embedding to find some f so that $||f(u) - f(v)||_1 \approx d^*(u,v)$, so that this quantity is still pretty small.
- Step 4: Reverse-engineer Step 1 to find an actual cut $S, \bar{S}$.