Problem Set 5 CS265/CME309, Winter 2025
Due: Friday 2/28, 11:59pm on Gradescope

Please follow the homework policies on the course website.

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1. **(4 pt.) [Furthering the second moment method]**

   In class, we saw the second moment method to show that a random variable with large expectation and small variance must be non-zero with good probability. Formally, we saw that for a non-negative random variable X,

$$\Pr[X = 0] \le \frac{\mathrm{Var}[X]}{(\mathbb{E}[X])^2}. \tag{1}$$

   While this is often very useful, it does not let us reason about the probability of X being something small but non-zero. In this question, you will prove a similar inequality that **does** let us do such a thing.

   Prove that for a non-negative random variable X and any $0 < t < 1$,

$$\Pr[X \ge t \cdot \mathbb{E}[X]] \ge (1-t)^2 \frac{(\mathbb{E}[X])^2}{\mathbb{E}[X^2]}. \tag{2}$$

   [**HINT:** *Write* X *as* $X \cdot 1_{\{X < t \cdot \mathbb{E}[X]\}} + X \cdot 1_{\{X \ge t \cdot \mathbb{E}[X]\}}$. *Use linearity of expectation to compute* $\mathbb{E}[X]$ *and use the Cauchy-Schwarz inequality to bound the term* $\mathbb{E}\left[X \cdot 1_{\{X \ge t \cdot \mathbb{E}[X]\}}\right]$.]

   Note: By simple rearrangements of (2), one can observe[1] that

$$\Pr[X < t \cdot \mathbb{E}[X]] \le \frac{\mathrm{Var}[X] + (1-(1-t)^2)(\mathbb{E}[X])^2}{\mathbb{E}[X^2]} \le \frac{\mathrm{Var}[X]}{(\mathbb{E}[X^2])} + (1-(1-t)^2). \tag{3}$$

2. **(6 pt.) [Echoing paths]**

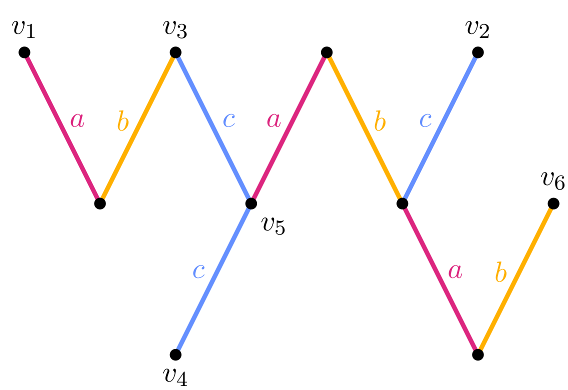

Figure 1: An edge coloring of a graph with some echoing paths.

[1] There is no question you need to answer about this rearrangement. Simply observe this and try to understand how it compares to the inequality (1) from class.

An *edge coloring* of an (undirected) graph $G = (V, E)$ assigns exactly one color to each edge of the graph. We say that a colored path in the graph is *echoing* if the path has an even number of edges, and the second half of the path is colored identically to the first half of the path (i.e. the sequence of colors in the second half of the path is the same sequence as in the first half). For example, in Figure 1, the paths from v_1 and v_2, from v_3 to v_4, and from v_5 to v_6 are all echoing paths. Edges are colored and labeled a, b, or c corresponding to their color.

Throughout this problem, by "path" we refer only to simple paths—i.e. paths that do not re-use any edges.

(a) **(4 pt.)** Prove that for any graph whose maximum degree is d, there exists a coloring using $10 \cdot d^2$ colors such that there are no echoing paths of length 4 (i.e. no echoing paths consisting of 4 distinct edges, like the path from v_5 to v_6 in Figure 1).

[**HINT:** *Use the Lovasz Local Lemma.*]

(b) **(2 pt.)** Given the setup in the previous part, give an algorithm that will find such a coloring in expected time polynomial in the size of the graph, and justify the runtime.

(c) **(0 pt.)** **[This problem is optional.]** Prove that there is some constant C such that for any graph whose maximum degree is d, there exists a coloring using $C \cdot d^2$ colors such that there are no echoing paths (of any length).

3. **(6 pt.)** **[Near-Markov random walks]**

Let G be a d-regular graph with n vertices. Suppose each vertex v has an assigned label $\ell(v)$ of either 0 or 1. Beginning at a fixed vertex v_0, perform a random walk $v_0, v_1, v_2, \ldots$, where v_{t+1} is chosen uniformly at random from the d neighbors of v_t. We say a random walk $v_0, v_1, v_2, \ldots$ is ϵ-**Markov** if for all $t \geq 0$,

$$\left| \mathbb{E}\left[\ell(v_t) \mid \ell(v_{t-1}), \ell(v_{t-2}), \ldots, \ell(v_0)\right] - \frac{1}{2} \right| \leq \epsilon.$$

(a) **(4 pt.)** Let $\epsilon > 0$ be fixed. Show that there is some lower bound d_ϵ such that if $d \geq d_\epsilon$, there exists a labeling for the n vertices such that a random walk beginning at any vertex satisfies the ϵ-Markov property. You do not need to calculate the value of d_ϵ explicitly, but rather simply show that such a d_ϵ exists.

[**HINT:** *Use the Lovasz Local Lemma. For every vertex v, consider the event A_v given by $|P_v - \frac{1}{2}| > \epsilon$, where P_v is the probability that after taking one step starting from v during the walk, we visit a vertex with the label* 1.]

(b) **(2 pt.)** Give an algorithm that will construct such a labeling in time polynomial in n and d.