Problem Set 6 CS265/CME309, Winter 2025

Due: Friday 3/7, 11:59pm on Gradescope

Please follow the homework policies on the course website.

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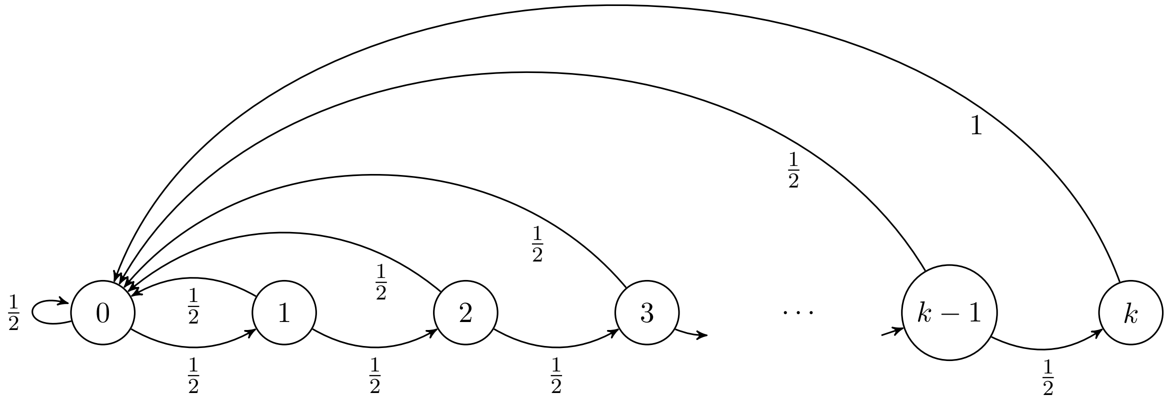
1. **(7 pt.) Coin Tossing Revisited**

   (a) **(4 pt.)** Consider the following Markov chain:



   Prove that this Markov chain has a unique stationary distribution π, and determine what it is.

   [**HINT:** *In case it's helpful,* $\sum_{j=0}^{k} 2^{-j} = 2 - 2^{-k}$*.*]

   (b) **(3 pt.)** Recall that, in HW1, you showed that the expected number of fair coin flips until you see k heads in a row is $2^{k+1} - 2$. Explain how to see this from the stationary distribution π that you computed in part (a).

   Note: Your explanation should contain (i) the connection between that problem and the Markov chain in part (a); and (ii) a short but complete proof of the fact that the answer is $2^{k+1} - 2$*.*

2. **(9 pt.) Fundamental Theorem of Markov Chains: A Special Case**

   Let $X_0, X_1, \ldots$ be a Markov chain over n states (labeled $1, 2, \ldots, n$) with transition matrix $P \in \mathbb{R}^{n \times n}$, i.e., for any $t \geq 0$, $\Pr[X_{t+1} = j | X_t = i] = P_{ij}$. In addition, we assume that $P_{ij} > 0$ for all $i, j \in [n]$, and define $p_{\min} := \min_{i,j \in [n]} P_{ij} > 0$. In this problem, we will prove part of the fundamental theorem of Markov chains for this special case. In particular, we will show that there exists a unique stationary distribution π such that for all $i, j \in [n]$,

$$\lim_{t \to +\infty} \Pr[X_t = j | X_0 = i] = \pi_j.$$

   (a) **(2 pt.)** As a warmup, show that the assumption $P_{ij} > 0$ for all $i, j \in [n]$ implies that the Markov chain is irreducible and aperiodic. Thus, the assumption that we made is not weaker than the one in the original theorem.

(b) **(2 pt.)** Let $a = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$ be a row vector that satisfies $\sum_{i=1}^n a_i = 0$. Prove that $\|aP\|_1 \leq (1 - np_{\min}/2)\|a\|_1$.

[**HINT:** *You can use the following fact: For vectors* $a, b \in \mathbb{R}^n$ *satisfying* $\sum_{i=1}^n a_i = 0$ *and* $\min_{i \in [n]} b_i \geq \epsilon > 0$, $|\sum_{i=1}^n a_i b_i| \leq \sum_{i=1}^n |a_i| b_i - \frac{\epsilon}{2} \sum_{i=1}^n |a_i|$.]

(c) **(3 pt.)** Prove that there exists an n-dimensional row vector $\pi = \begin{bmatrix} \pi_1 & \pi_2 & \cdots & \pi_n \end{bmatrix}$ such that: (1) $\pi = \pi P$; (2) $\sum_{i=1}^n \pi_i = 1$.

[**HINT:** *First prove the existence of a non-zero vector* π *satisfying* $\pi = \pi P$, *and then show that the second condition can be satisfied by scaling* π. *For the first step, you may use the following fact without proof: if* λ *is an eigenvalue of a square matrix* A, λ *is also an eigenvalue of* A^T. *Part 2b might be helpful for the second step.*]

(d) **(2 pt.)** Let $v = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$ be a row vector that satisfies $\sum_{i=1}^n v_i = 1$. Let π be a vector chosen as in Part 2c. Prove that $\lim_{t \to +\infty} vP^t = \pi$. Then, derive that for all $i, j \in [n]$,

$$\lim_{t \to +\infty} \Pr[X_t = j | X_0 = i] = \pi_j.$$

[**HINT:** *Apply Part 2b to* $(v - \pi), (v - \pi)P, (v - \pi)P^2, \ldots$.]

(e) **(0 pt.) [Optional: this won't be graded.]** Extend the proof to the general case, where the Markov chain is irreducible and aperiodic but $P_{ij} > 0$ might not hold.

3. **(7 pt.) Random Walks on the Hypercube**

Earlier this quarter, we studied randomized routing on the hypercube. In this problem, we examine random walks on the hypercube. Let $n > 2$, and consider the Markov chain $\{X_t\}$ defined on the states $\{0, 1\}^n$ where at each step, the chain moves to each neighboring states with equal probability $1/n$. That is, $\{X_t\}$ is defined by the following transition probabilities:

$$\Pr[X_t = x^{(i)} \mid X_{t-1} = x] = \frac{1}{n},$$

where $x^{(i)}$ is the state that differs from x only in the i-th bit.

(a) **(1 pt.)** Is this chain periodic or aperiodic? Is it irreducible? Justify your answers in one sentence each.

(b) **(1 pt.)** Consider the "lazy" version of $\{X_t\}$ that, at every timestep, flips a fair coin and with probability 1/2 stays in its current state, and with probability 1/2 transitions as prescribed above. Call this lazy version $\{\tilde{X}_t\}$. Show that $\{\tilde{X}_t\}$ has a unique stationary distribution.

(c) **(5 pt.)** Show that the mixing time of $\{\tilde{X}_t\}$ is bounded by $O(n \log n)$. [**HINT:** *Define a coupling.*]