

Class 15

Mixing Times and Coupling

Announcements

- HW6 due Friday

Recap: Mixing Times

Distribution of X_t ,
conditioned on $X_0 = s$



$$\Delta(t) = \max_s \|\pi - P_s^t\|$$

$$\tau_{mix} = \min \left\{ t : \Delta(t) < \frac{1}{2e} \right\}$$

Recap: Coupling

- Motivation:

$$\Delta(t) \leq \max_{s,s'} \|P_s^t - P_{s'}^t\| \leq 2\Delta(t)$$

- Def of Coupling:

- X_t, Y_t both individually follow the Markov chain (but may be correlated!)
- If $X_t = Y_t$, then $X_{t+1} = Y_{t+1}$

- Why we care:

$$\Delta(t) \leq \max_{s,s'} \Pr[X_t \neq Y_t \mid X_0 = s, Y_0 = s']$$

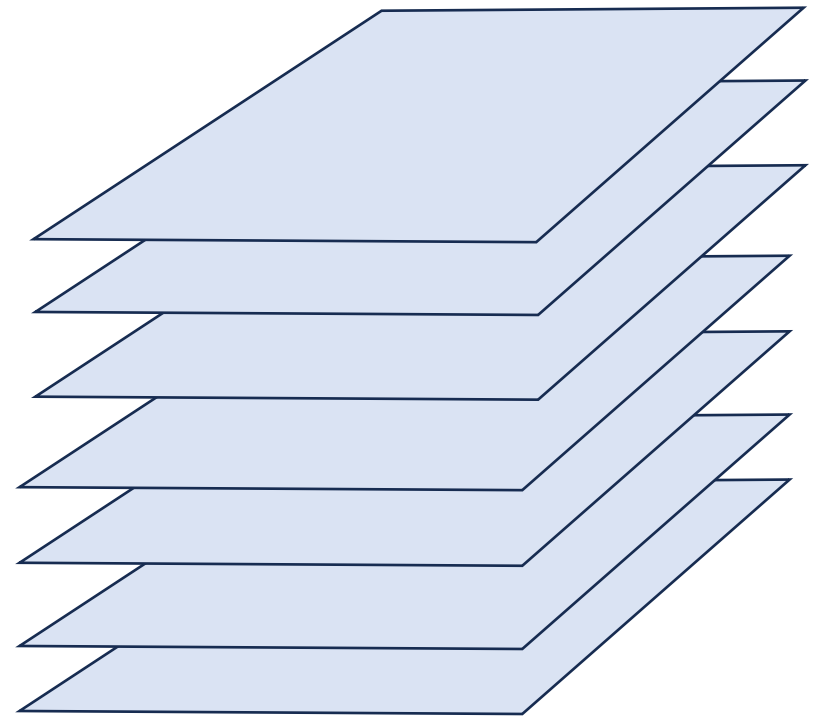
Aka, $\tau_{mix} \leq$ time for $\{X_t\}$ and $\{Y_t\}$ to meet with probability at least $\frac{1}{2e}$.

Questions?

Mixing times, coupling, quiz?

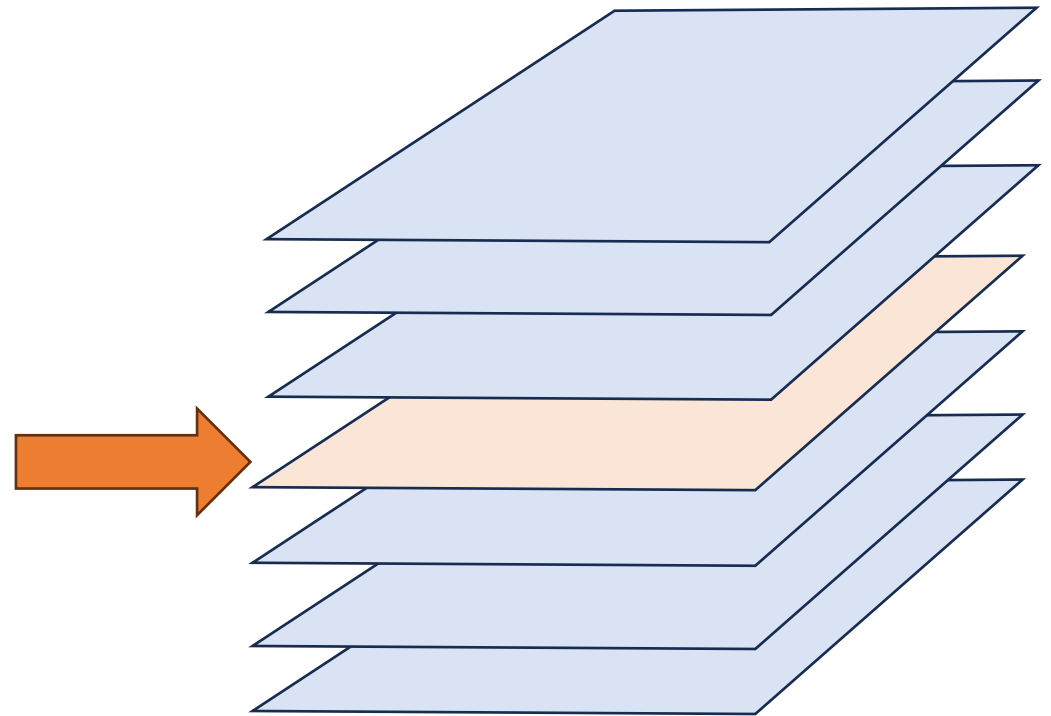
Today: Shuffling!

- Shuffling procedure (for n cards):
 - Pick a random card.
 - Move it to the top of the deck.



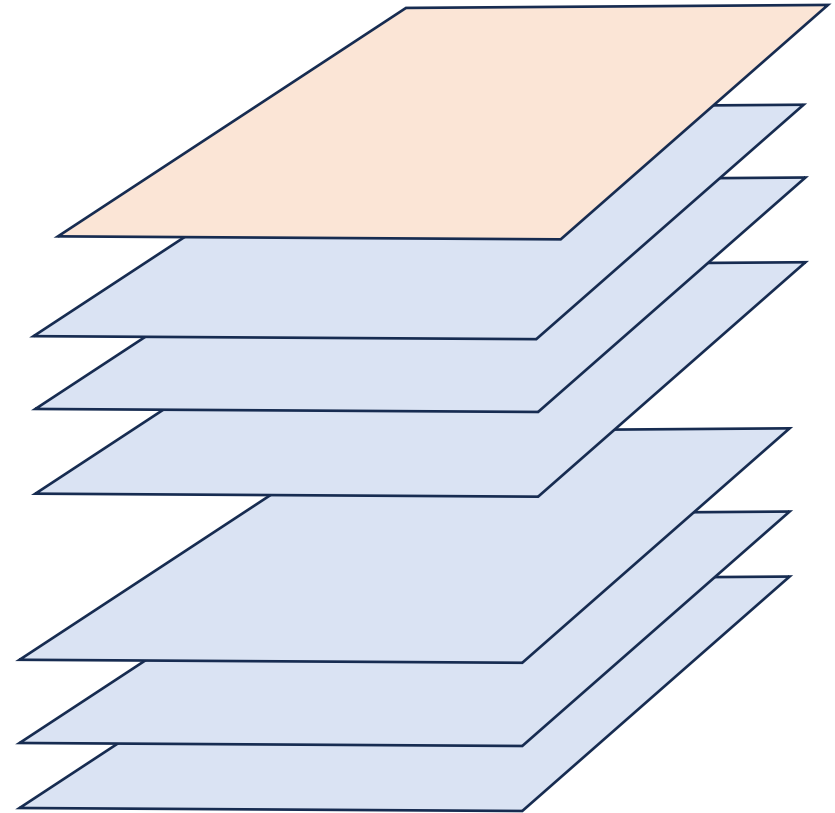
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First Group Work: practice with coupling

- Shuffling procedure (for n cards):
 - Pick a random card.
 - Move it to the top of the deck.
1. Is this irreducible and aperiodic? What's the stationary dist?
 2. Come up with a good coupling.
 3. How long until your coupling couples whp?
 4. Bound the mixing time.

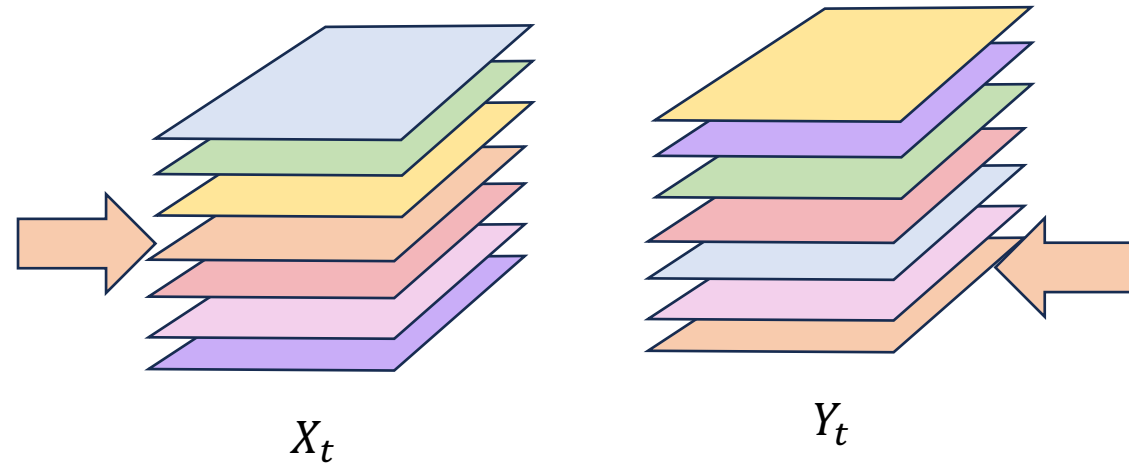
Q1

- This is aperiodic, irreducible:
 - Irreducible since you can get to any ordering by picking cards to move in the order that you want.
 - Aperiodic since there are self-loops.
- The stationary distribution is Uniform!
 - If you start with a uniformly random deck, and move a random card to the top, it's still uniformly random.

Q2: A good coupling?

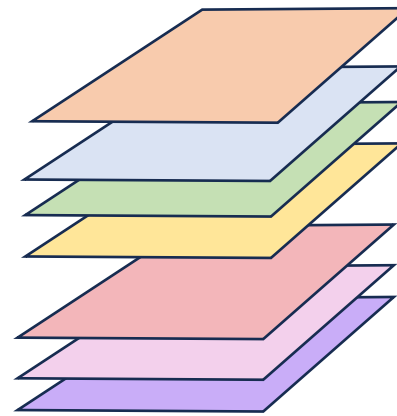
A coupling

- Want to define $(X_0, Y_0), (X_1, Y_1), \dots$
- Choose the **same** card in both decks (both the “X” and the “Y” deck).
 - A.k.a., if we move the Ace of Spades in the X deck, also move the Ace of Spades in the Y deck.

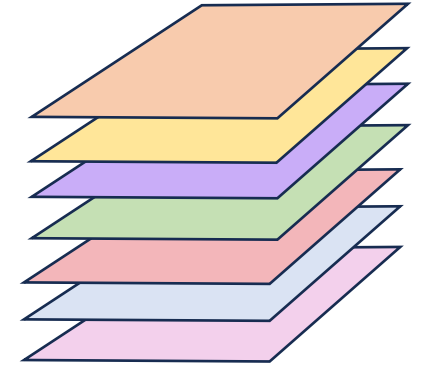


A coupling

- Want to define $(X_0, Y_0), (X_1, Y_1), \dots$
- Choose the **same** card in both decks (both the “X” and the “Y” deck).
 - A.k.a., if we move the Ace of Spades in the X deck, also move the Ace of Spades in the Y deck.
- This is a valid coupling:
 - Looking at just one deck, it looks like we are always choosing a uniformly random card and moving it to the top.
 - Once the two decks are the same, they will always stay the same.



X_{t+1}

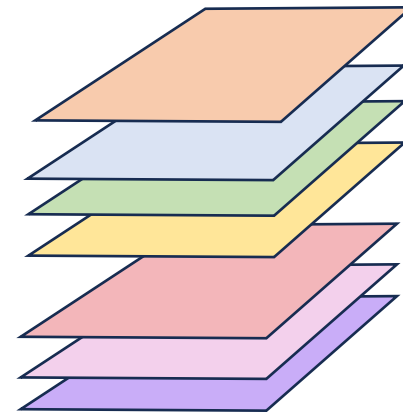


Y_{t+1}

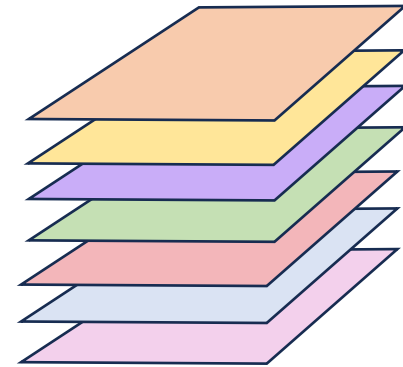
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- Observation: The chains have coupled once every card has been chosen.



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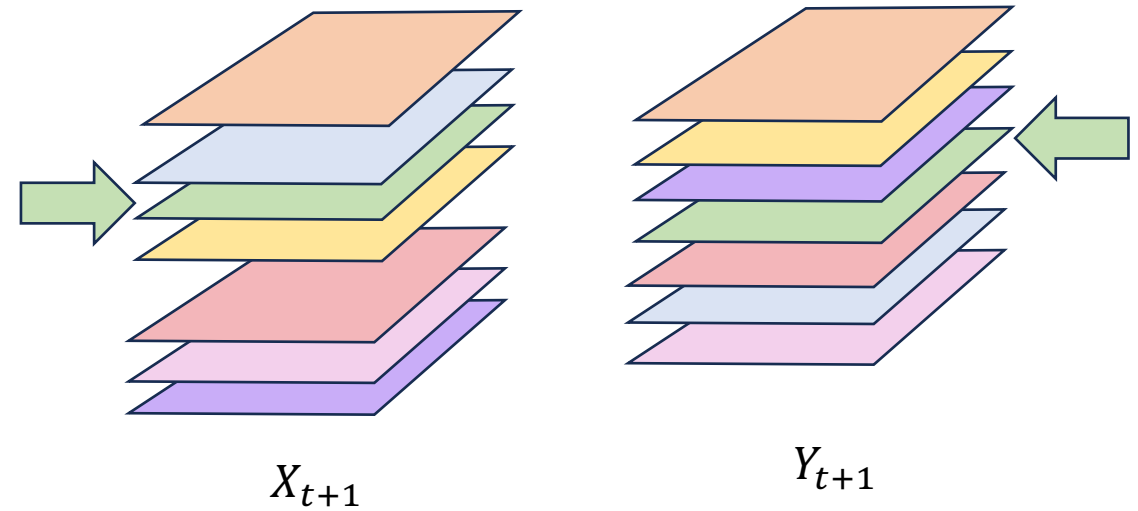



Y_{t+1}

Chosen so far: 

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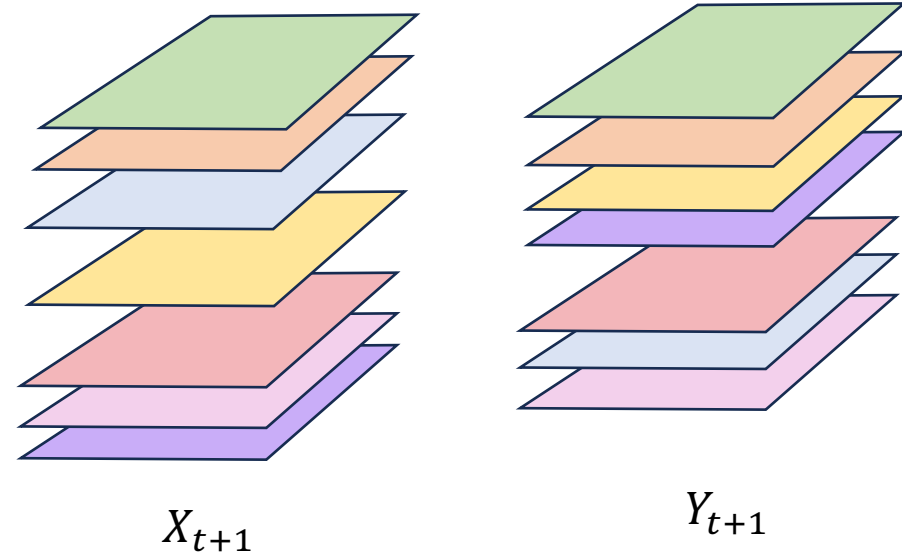
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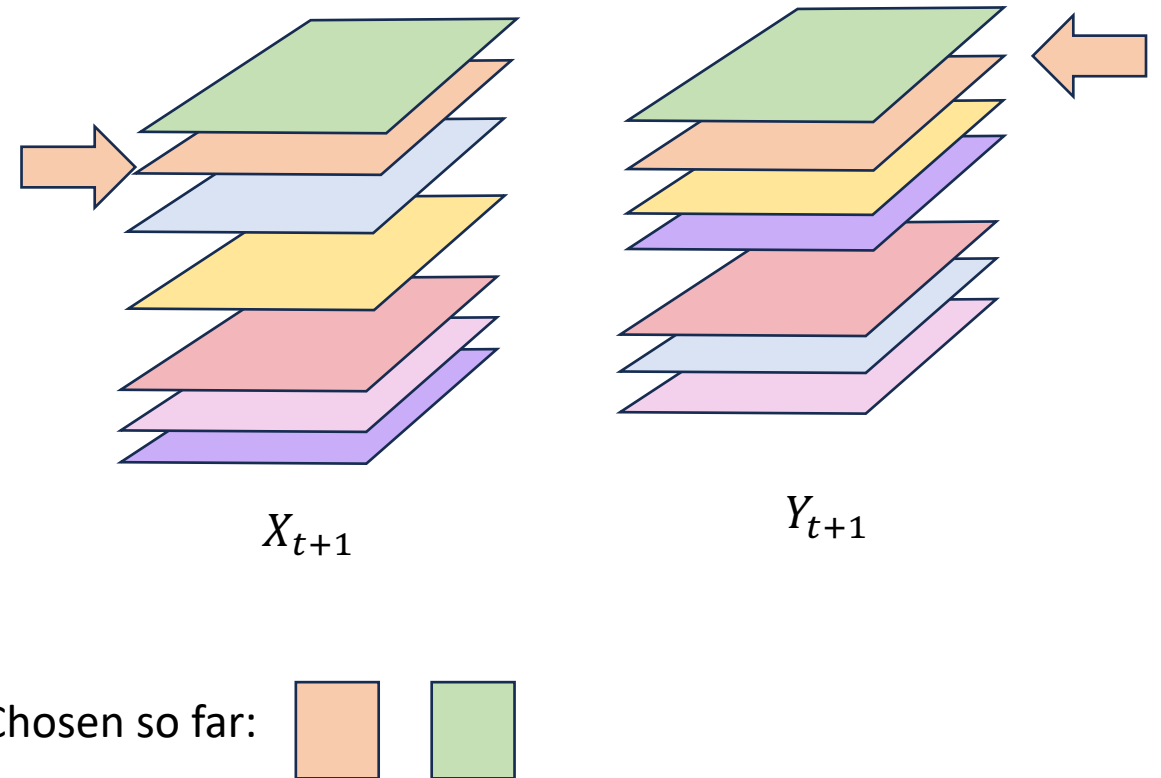
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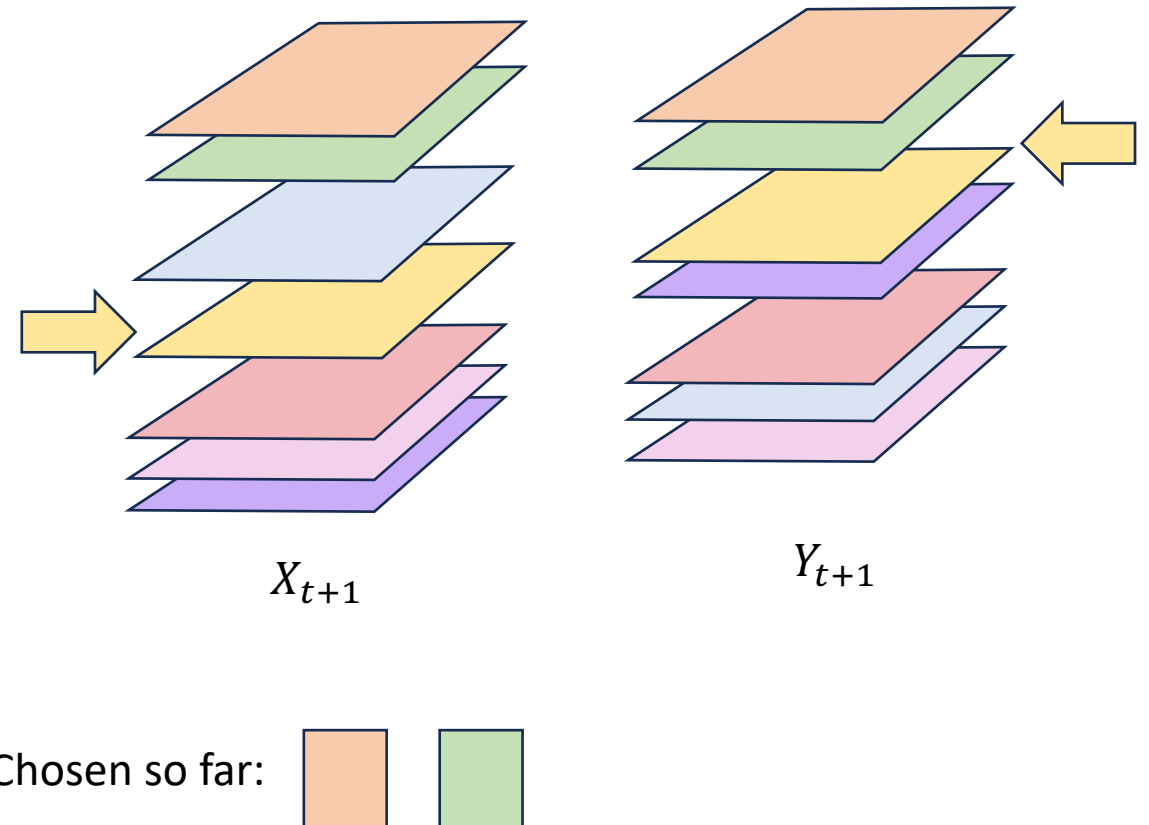
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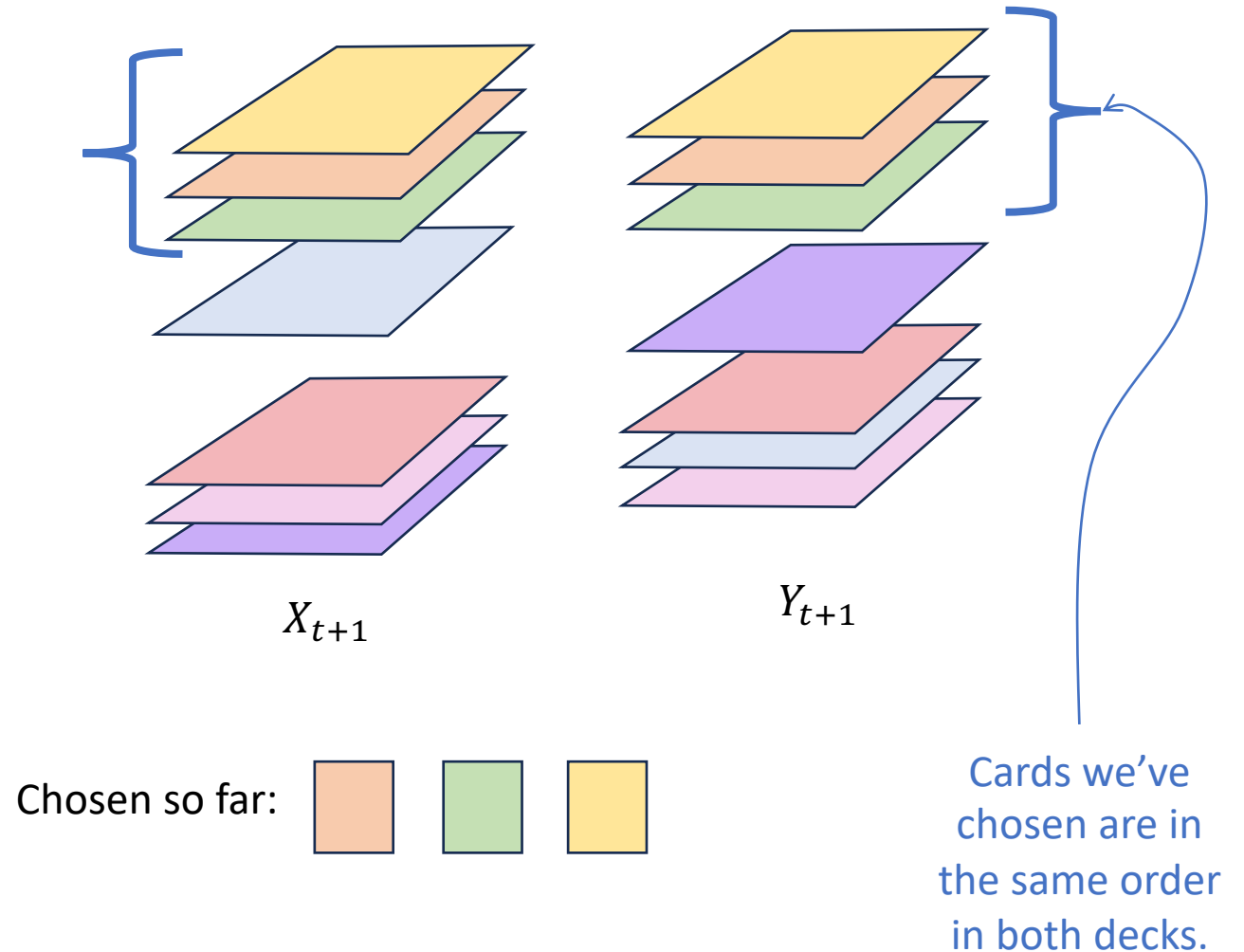
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Q3. How long until the chains couple?

- Observation: The chains have coupled once every card has been chosen.
- How long until every card has been chosen?
 - This is coupon collecting! About $n \log n$.

Formally:

$$\max_{s, s'} \Pr[T_{s, s'} \geq 2n \log n] \leq \Pr[\text{time to choose all } n \text{ cards} \geq 2n \log n]$$
$$= o(1)$$

Time until the two chains (started at s and s') couple

Q4. Translate this to a bound on mixing time

$$\Delta(2n \log n) \leq \max_{s,s'} \Pr[T_{s,s'} \geq 2n \log n] = o(1)$$

$$\Delta(t) = \max_s \|\pi - P_s^t\|$$

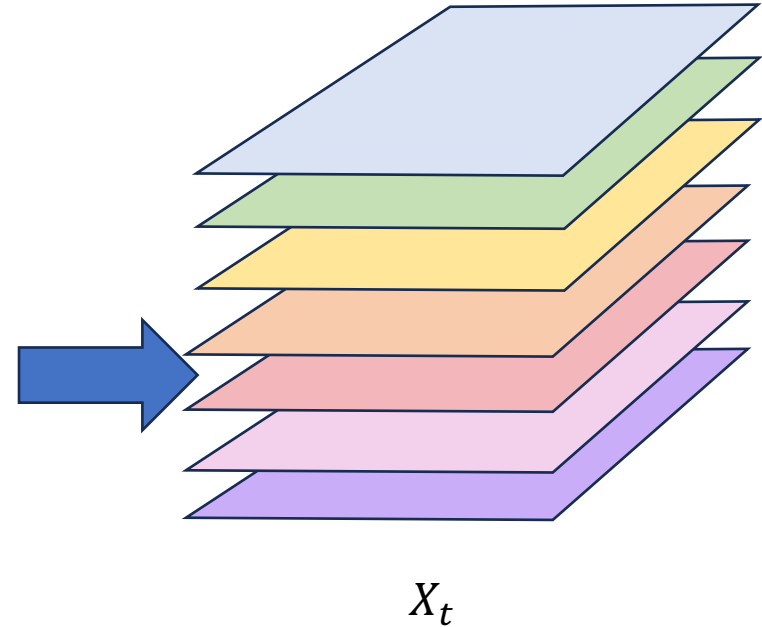
$P_s^t = X_t$ started at s

$$\Rightarrow \tau_{mix} \leq 2n \log n$$

as long as n is big enough that the $o(1)$ term is at most $1/(2e)$.

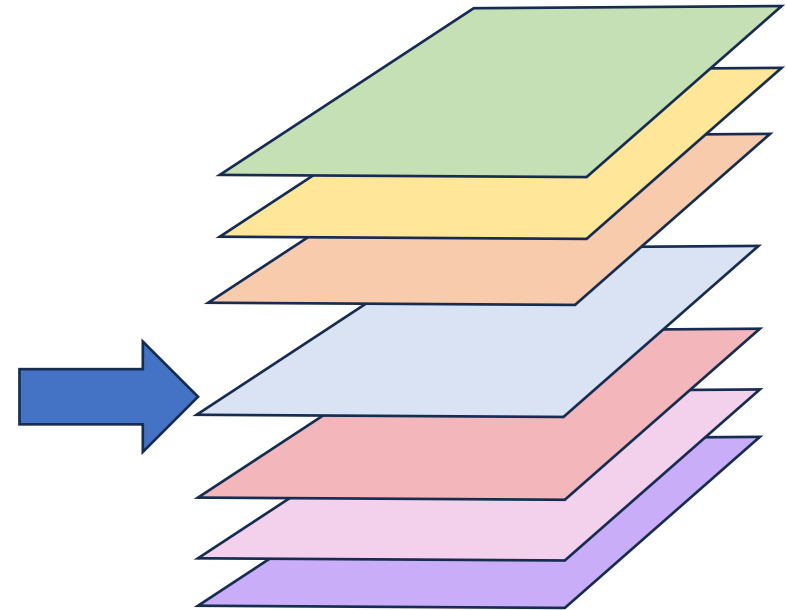
Group Work II: More shuffling!

- Another shuffling procedure:
 - Choose the top card.
 - Put it somewhere random.



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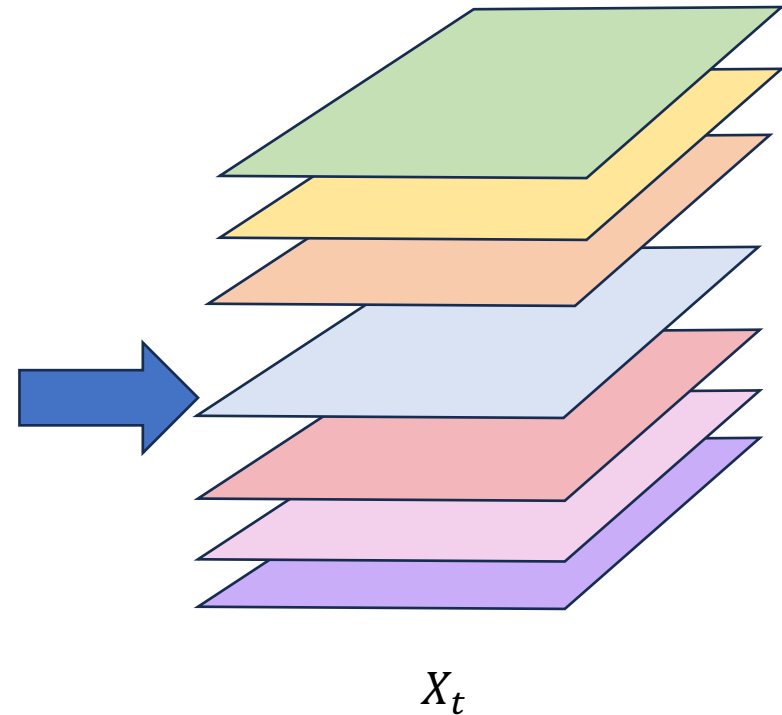
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X_t

Group Work II: More shuffling!

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- We'll analyze this one with a different technique:
Strong stationary stopping times!

Group work!

- Another shuffling procedure:
 - Choose the top card.
 - Put it somewhere random.

1. Convince yourself that this chain is aperiodic and irreducible, with uniform stationary distribution.
2. Let T be the first time the original bottom card is placed randomly somewhere. Show that the deck is completely uniform at any time $t > T$.
3. What is $E[T]$?
4. Bound the mixing time.

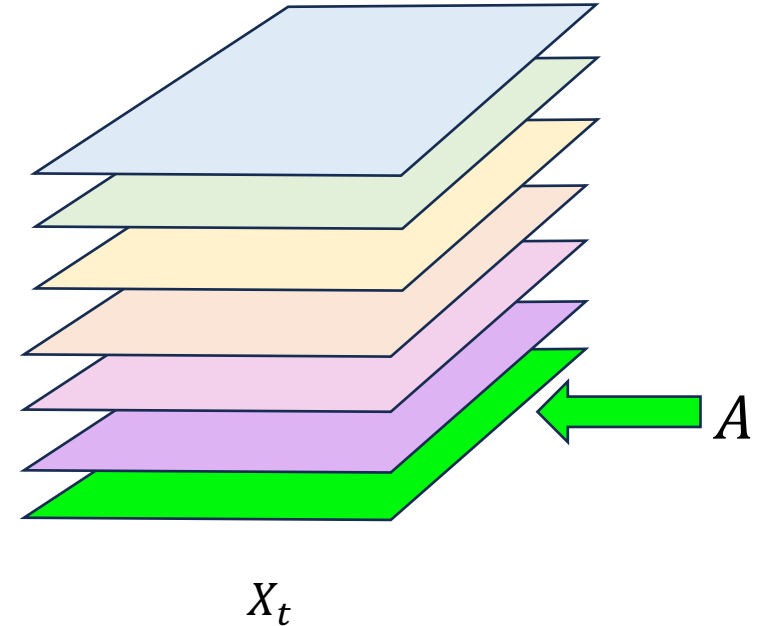
Q1

- This is irreducible and aperiodic.
 - Actually, this was a quiz question last time! It's aperiodic since there are self-loops, and irreducible since you can get any deck you want by building it on the bottom of the deck.
- The stationary distribution is uniform.
 - If you take a uniformly random deck and put the top card somewhere random, it's still a uniformly random deck.

Q2: Show that X_t is uniform if $t > T$

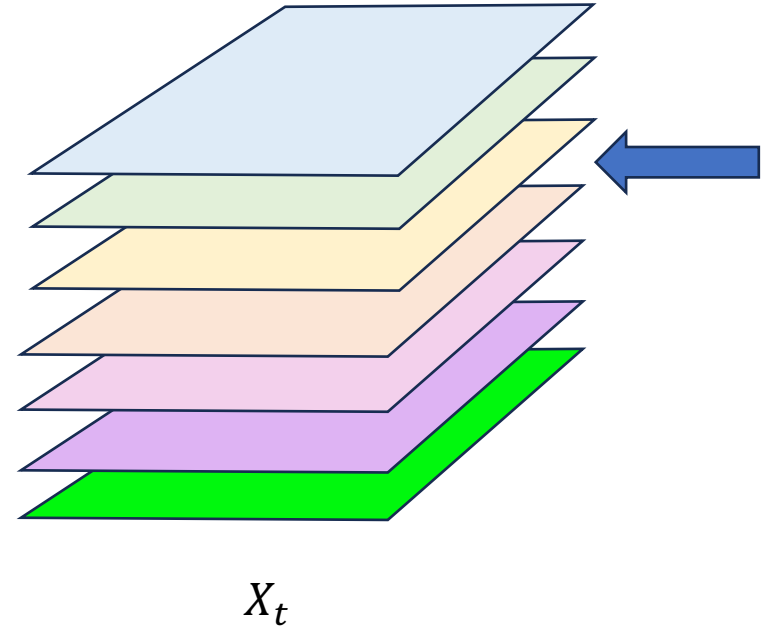
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 - For convenience, from now on “A” refers to “the original bottom card.”
- When $t \geq T$, X_t is uniform.



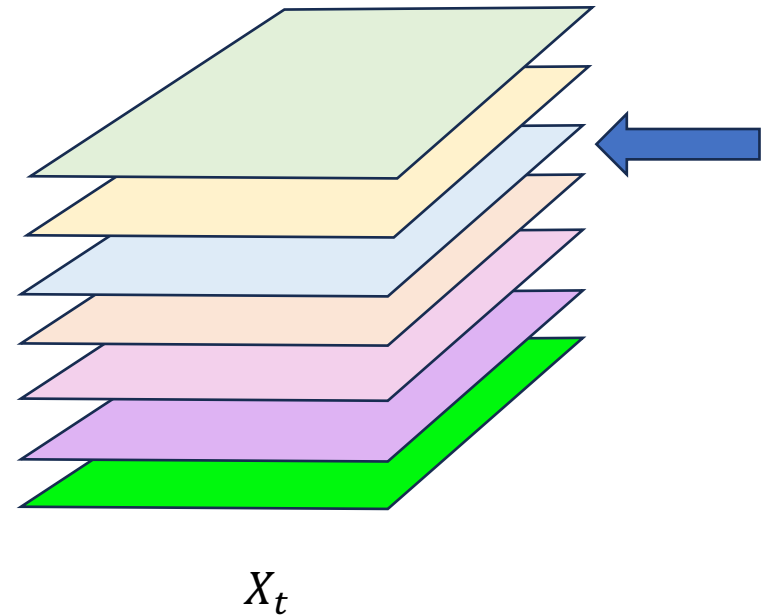
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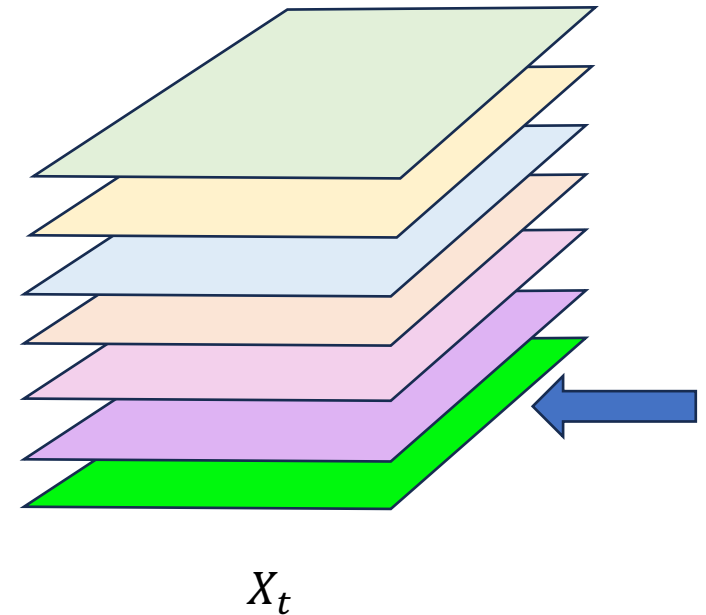
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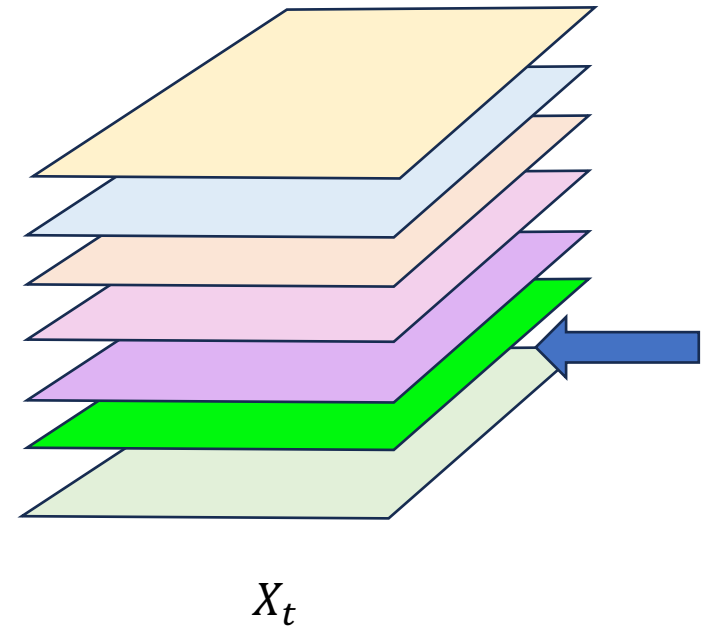
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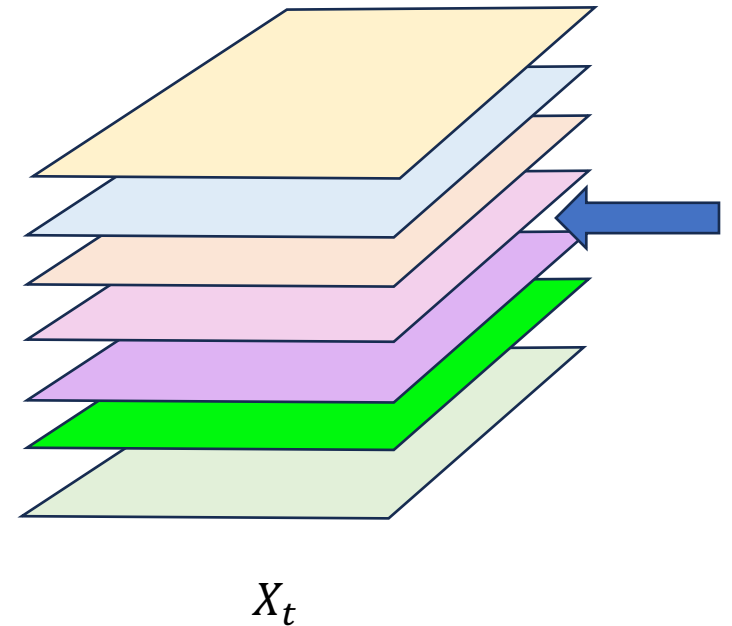
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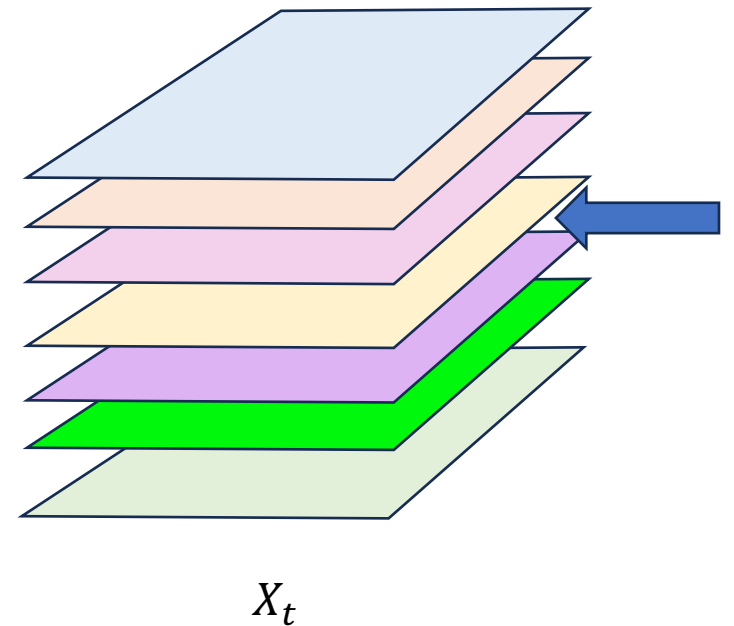
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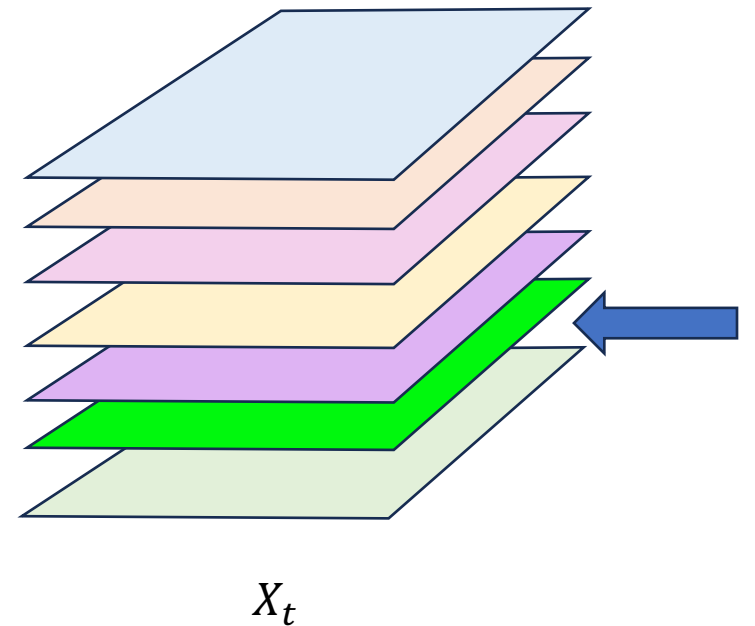
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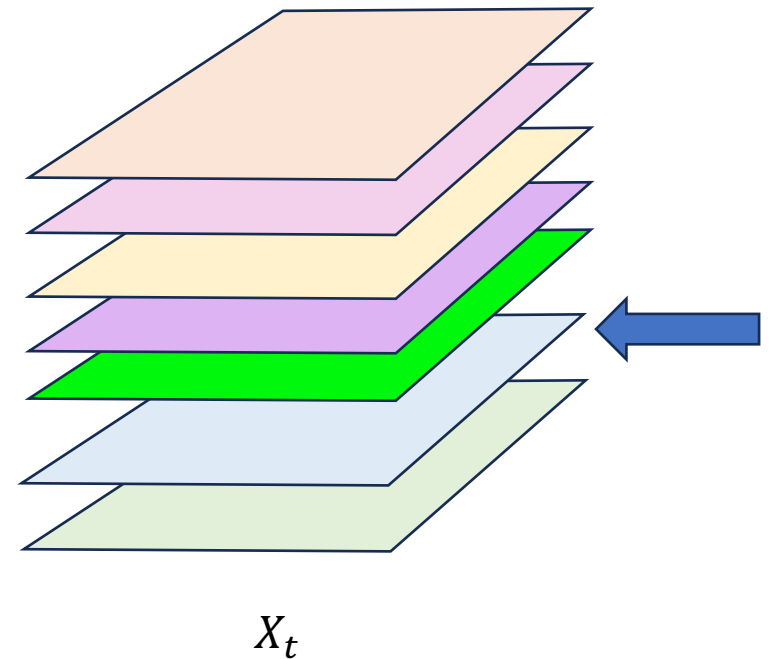
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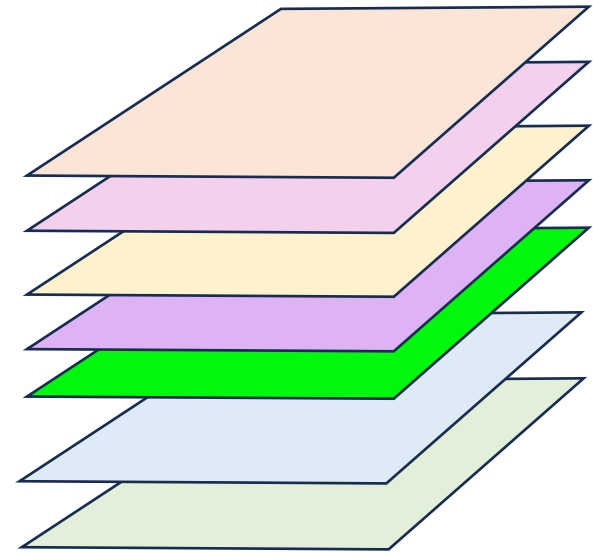
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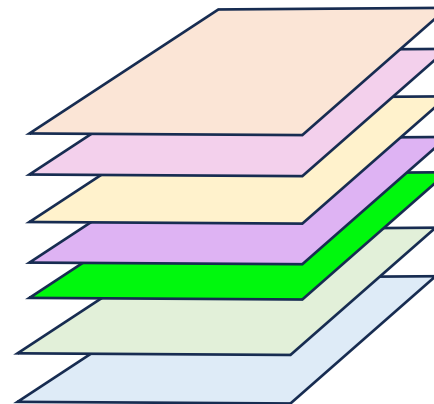
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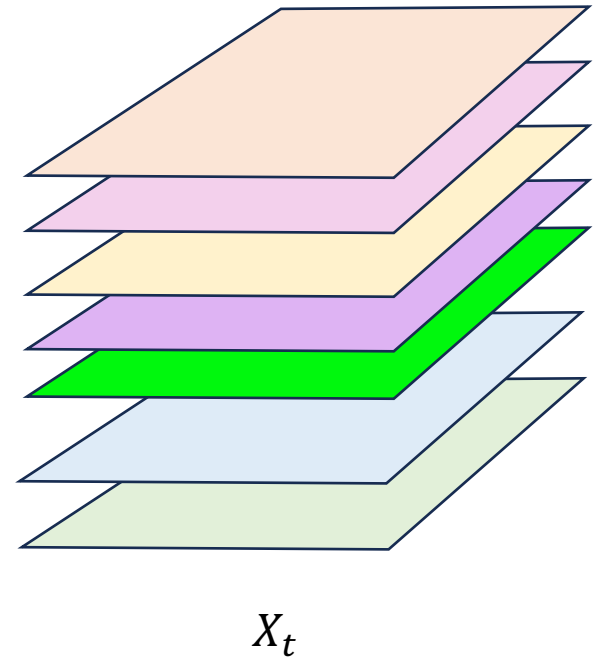
X_t

With equal probability
we would have had:



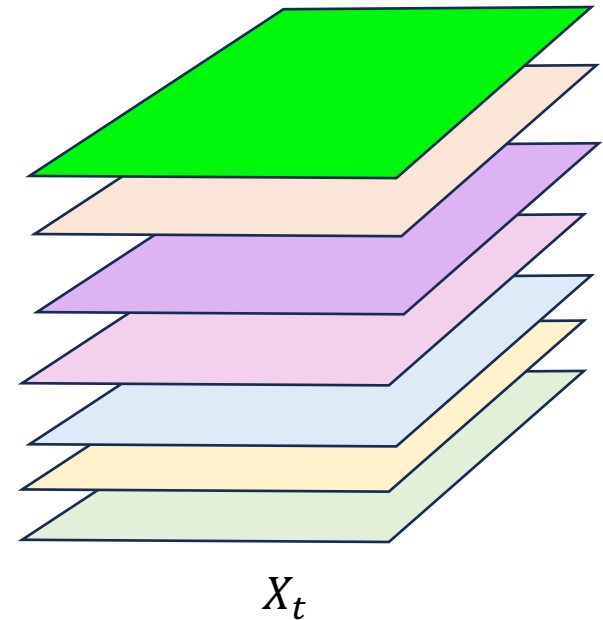
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 - Once the A is at the top, at time $T-1$, the rest of the deck is uniform.



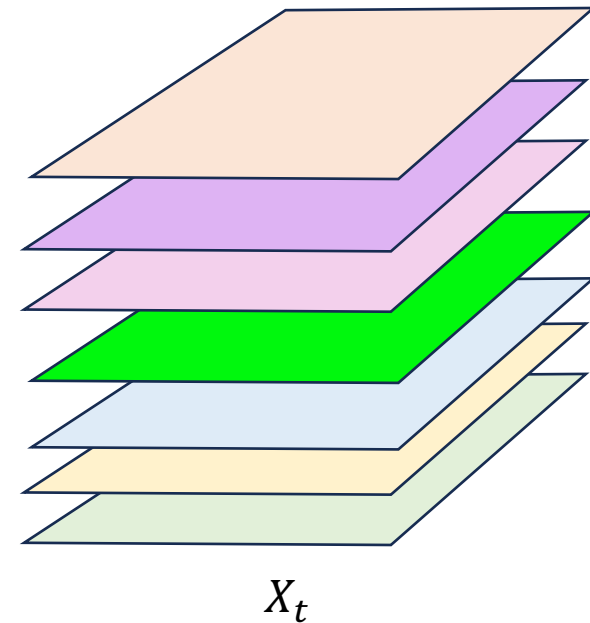
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 - Once you place the A, at time T , the whole deck is now uniform.



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Q3: What is $\mathbb{E}[T]$?

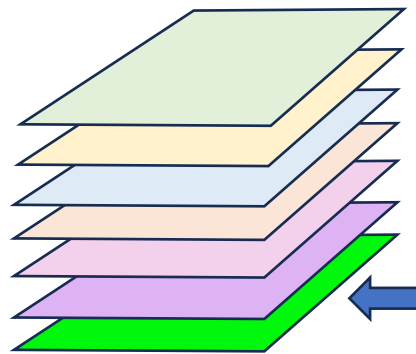
$$\mathbb{E}[T] = \mathbb{E} \begin{array}{l} \text{Time to move} \\ \text{from last to} \\ \text{second-to-last} \end{array} + \mathbb{E} \begin{array}{l} \text{Time to move} \\ \text{from second-} \\ \text{to-last to} \\ \text{third-to-last} \end{array} + \cdots + \mathbb{E} \begin{array}{l} \text{Time to} \\ \text{move from} \\ \text{2}^{\text{nd}} \text{ to } \text{1}^{\text{st}} \end{array} + \mathbb{E} \begin{array}{l} \text{time to} \\ \text{place A} \\ \text{once it's 1}^{\text{st}} \end{array}$$

n

$\frac{n}{2}$

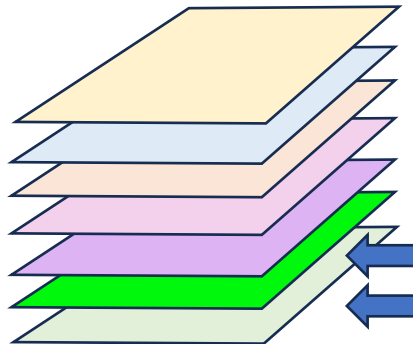
$\frac{n}{n-1}$

1



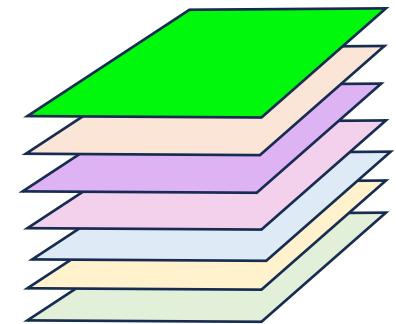
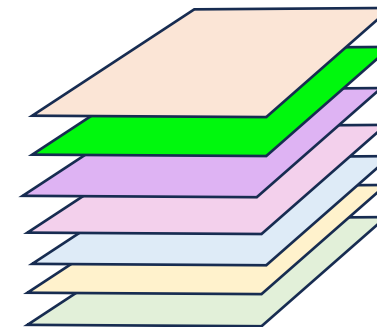
Have to put top card here...

...Probability is $\frac{1}{n}$



Have two options...

...Probability is $\frac{2}{n}$



Q3: What is $\mathbb{E}[T]$?

$$\mathbb{E}[T] = \mathbb{E} \begin{array}{l} \text{Time to move} \\ \text{from last to} \\ \text{second-to-last} \end{array} + \mathbb{E} \begin{array}{l} \text{Time to move} \\ \text{from second-} \\ \text{to-last to} \\ \text{third-to-last} \end{array} + \cdots + \mathbb{E} \begin{array}{l} \text{Time to} \\ \text{move from} \\ \text{2}^{\text{nd}} \text{ to } \text{1}^{\text{st}} \end{array} + \mathbb{E} \begin{array}{l} \text{time to} \\ \text{place A} \\ \text{once it's } \text{1}^{\text{st}} \end{array}$$
$$n \qquad \frac{n}{2} \qquad \frac{n}{n-1} \qquad 1$$

$$\mathbb{E}[T] = \sum_{j=1}^n \frac{n}{j} = \Theta(n \log n)$$

Q4: Bound the mixing time by $O(n \log n)$

- Markov's inequality: $\Pr[T \geq 2e \cdot \mathbb{E}[T]] \leq \frac{1}{2e}$
- Choose $t \geq 2e \mathbb{E}[T] = \Theta(n \log n)$
- **Hint:** $P_s^t = p \cdot \sigma + (1 - p) \cdot \pi$, where $p = \Pr[T > t]$
- $$\begin{aligned} \Delta(t) &= \max_s \| \pi - P_s^t \|_{TV} = \| \pi - \pi(1 - p) - p\sigma \|_{TV} \\ &= \| p\pi - p\sigma \|_{TV} \\ &= p \| \pi - \sigma \|_{TV} \leq p \leq \frac{1}{2e} \end{aligned}$$

Strong Stationary Stopping Times

- A random variable T is a **strong stationary stopping time** if:
 1. The event that $T = t$ depends only on X_0, \dots, X_t
 2. For all states s , $\Pr[X_t = s \mid T \geq t] = \pi(s)$

Strong Stationary Stopping Times

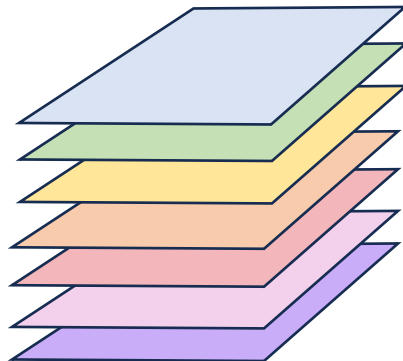
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We essentially just proved:

- **Theorem:** For any strong stationary stopping time, $\Delta(t) \leq P[T > t]$.

Bonus group work if time!

- Shuffling procedure:
 - Assign each card “L” or “R” independently, with probability $1/2$
 - Put all “L” cards to the left, preserving their relative order
 - Put all “R” cards to the right, preserving their relative order
 - Put the “L” stack on top of the “R” stack.

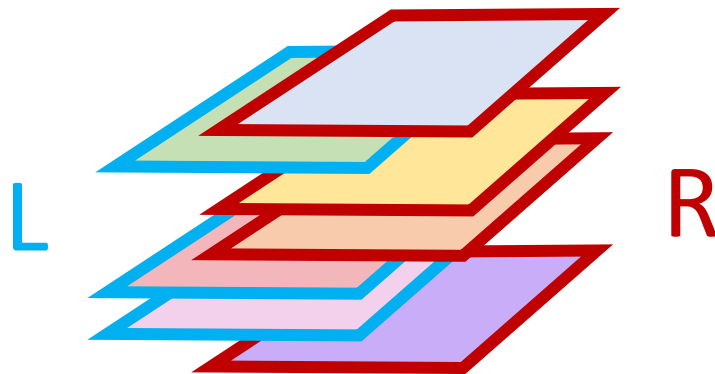


This is the **inverse** of a standard riffle shuffle!



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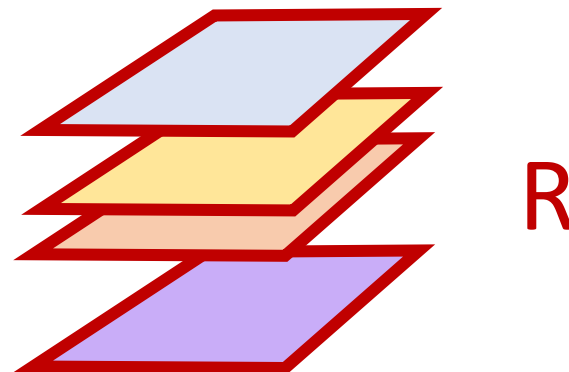
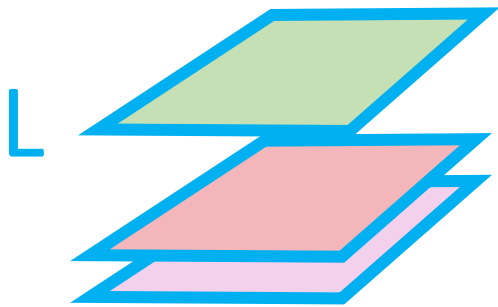


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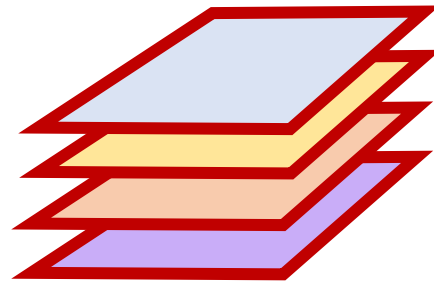
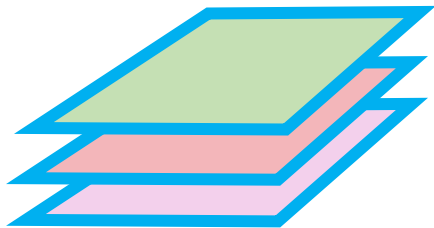
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L



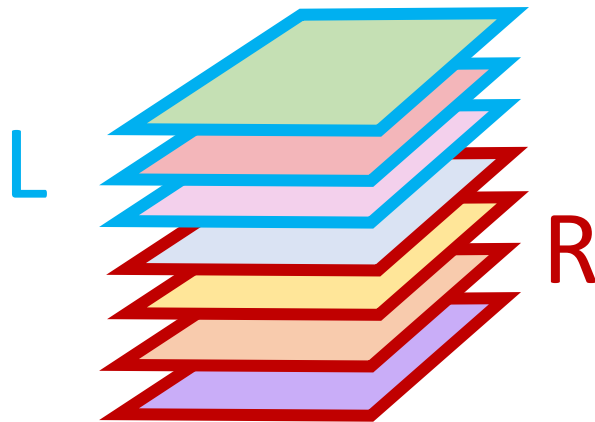
R

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 - Put all "R" cards to the right, preserving their relative order
 - Put the "L" stack on top of the "R" stack.

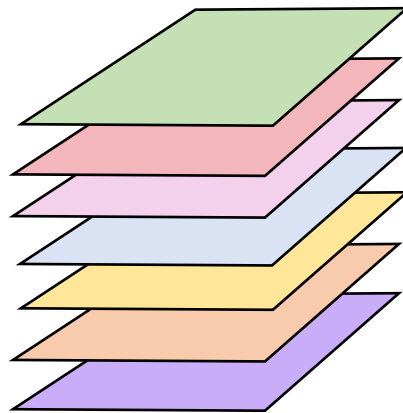


This is the **inverse** of a standard riffle shuffle!



Bonus group work if time!

- Shuffling procedure:
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Group work:

Use the method of strong stationary stopping times to show that the mixing time of this shuffle is $O(\log n)$.

This is the **inverse** of a standard riffle shuffle!



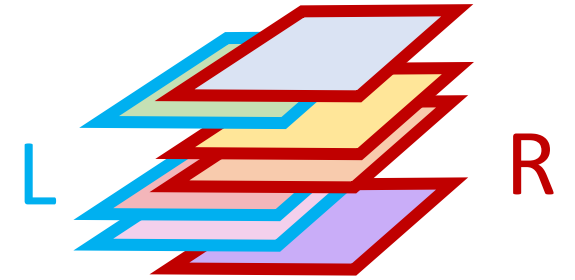
Solution

- Each card gets a string in $\{L, R\}^t$
- If a set of cards have different strings, their order relative to each other will be random.
- Let T be the first time that all the cards have different strings.

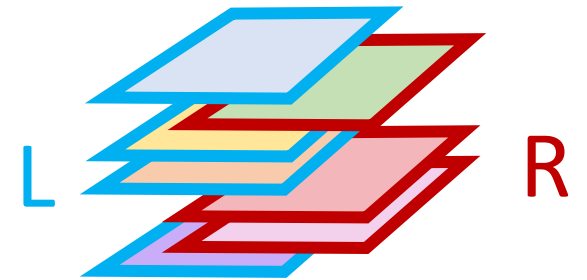
T is a strong stationary stopping time!

You can tell that T has happened at time T

Once T has happened, all of the cards are in random order



is just as likely as



So if two cards have different strings, their order relative to each other is random.

Solution

- Let T be the first time that all the cards have different strings.
- $\Pr[\text{card } x \text{ and card } y \text{ have the same string}] = 1/2^t$
- $\Pr[\text{any two cards have the same string}] \leq n^2 \cdot 2^{-t}$
- Choose $t \geq 3 \log n$ (say) and this is really small.

$$\Delta(3 \log n) \leq P[T > 3 \log n] \leq \text{really small} \leq \frac{1}{2e}$$

$$\Rightarrow \tau_{mix} \leq 3 \log n$$

Recap

- Two ways to bound mixing times:
 1. Coupling
 2. Strong Stationary Mixing Times
- (Plus spectral techniques, which we saw last week)

Next time

- Martingales!