

Class 16

Martingales and Azuma-Hoeffding bound

Announcements

- HW6 due Friday!
- HW7 out now (THE LAST ONE!)
 - Due Friday 3/14

Recap

- $\{Z_t\}$ is a **martingale** with respect to $\{X_t\}$ if for all t :
 - Z_t is a function of X_0, \dots, X_t
 - $E[|Z_t|] < \infty$
 - $E[Z_t \mid X_0, X_1, \dots, X_{t-1}] = Z_{t-1}$
- **Doob Martingale:** $Z_t = E[A \mid X_0, \dots, X_t]$

Azuma-Hoeffding

- Let $\{Z_t\}$ be a martingale w.r.t. $\{X_t\}$ and suppose there are constants c_1, \dots, c_n so that for all $i \leq n$, $|Z_i - Z_{i-1}| \leq c_i$.

- For any $\lambda > 0$,

$$\Pr[|Z_n - Z_0| \geq \lambda] \leq 2 \exp\left(\frac{-\lambda^2}{2 \sum_i c_i^2}\right)$$

Questions?

Definition of Martingale? Azuma-Hoeffding? Quiz?

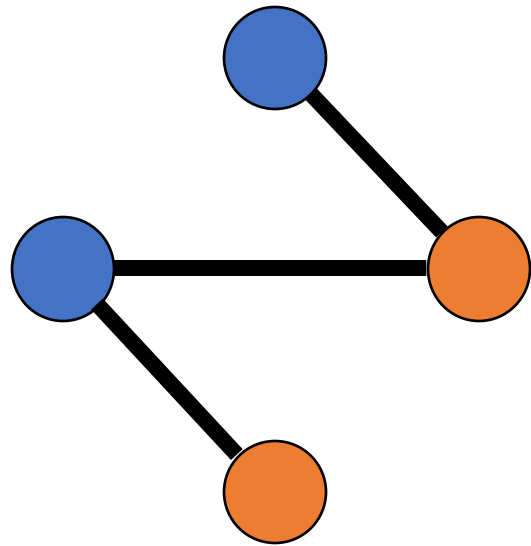
Today: Azuma-Hoeffding in Action

- Example 1: Chromatic number of random graphs
- Example 2: Gambling

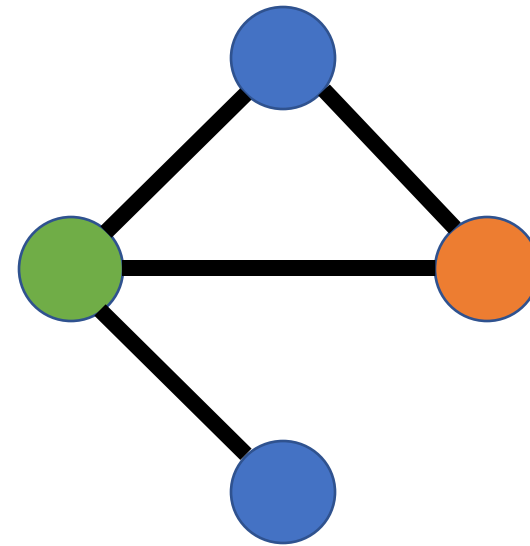
Chromatic number

(We saw this example in the mini-lectures)

- Let $G \sim G(n, p)$
- Let $A = \chi(G)$ be the minimum number of colors needed to properly color G .



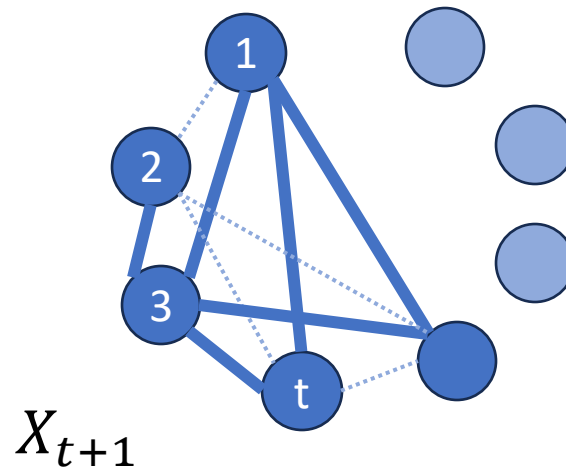
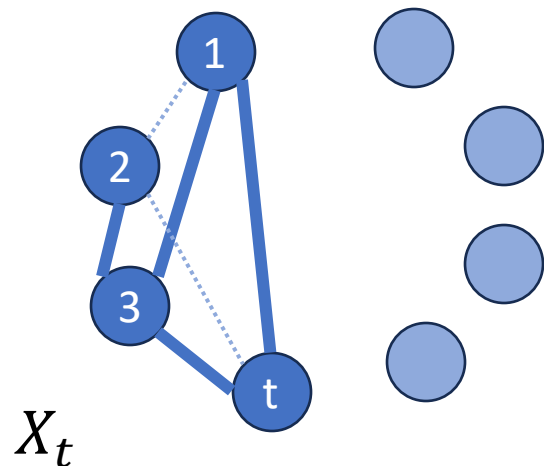
$$\chi(G) = 2$$



$$\chi(G) = 3$$

Vertex exposure martingale

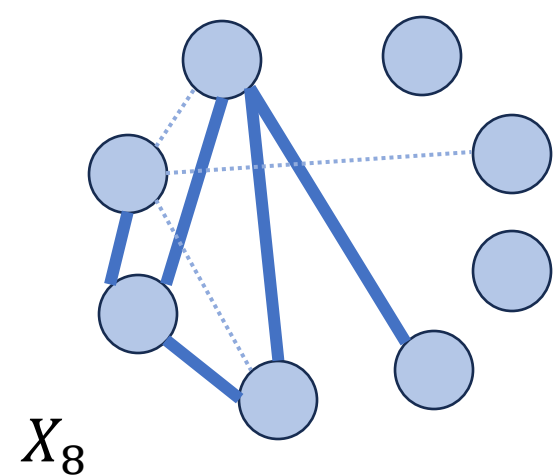
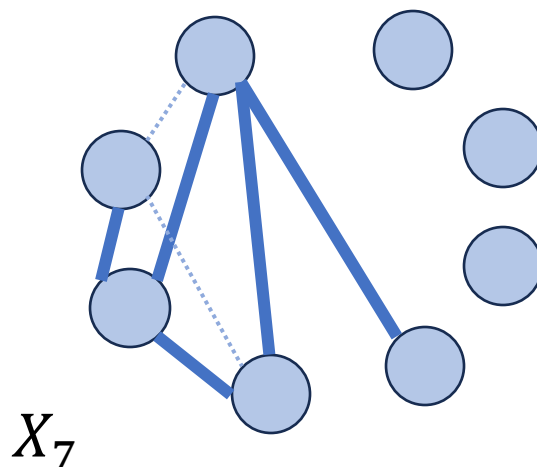
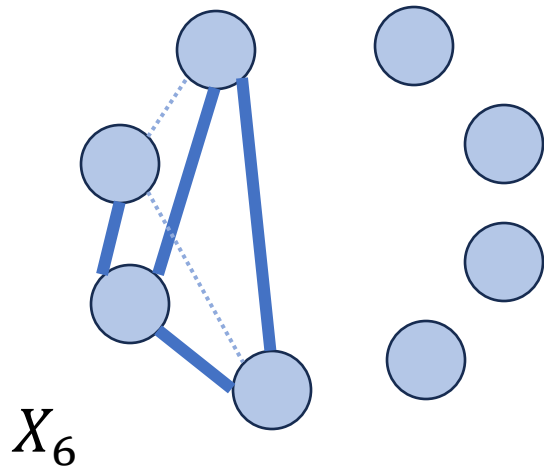
- Let $G \sim G(n, p)$
- Let $A = \chi(G)$
- $X_t =$ status of edges between vertex t and vertices $1, 2, \dots, t - 1$
- $Z_t = \mathbb{E}[A \mid X_1, X_2, \dots, X_t]$



Note: this is slightly different than the lecture notes! Both work fine for this example, this one is maybe more standard.

Edge exposure martingale

- Let $G \sim G(n, p)$
- Let $A = \chi(G)$
- $X_t =$ status of edge t , for $t = 1, \dots, \binom{n}{2}$
- $Z_t = \mathbb{E}[A \mid X_1, X_2, \dots, X_t]$



Group work:

Apply Azuma-Hoeffding both ways!

1. Use Azuma-Hoeffding with the vertex exposure martingale to bound

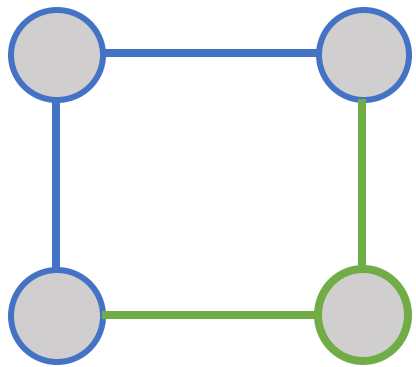
$$\Pr[|A - E[A]| > c\sqrt{n}] \leq 2 \exp\left(-\frac{c^2}{2}\right)$$

2. What happens with the edge exposure martingale?
3. (bonus) what can you say about $E[A]$?

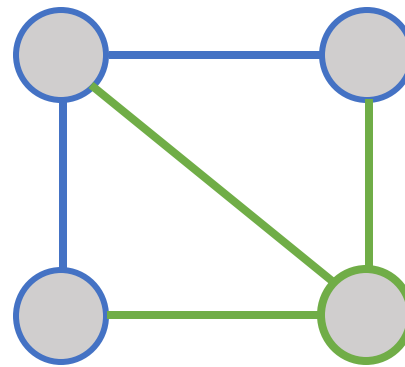
1. Vertex exposure martingale

- Need to bound $|Z_t - Z_{t-1}| \leq \underline{\hspace{2cm}} \mathbf{1}$

Intuitively, if you change the neighborhood of a single vertex, the chromatic number can change by at most 1:



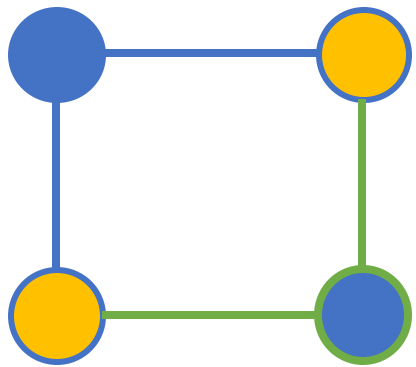
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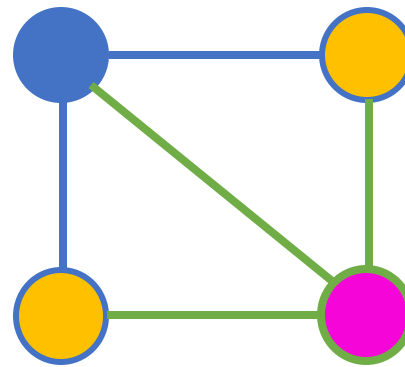
1. Vertex exposure martingale

- Need to bound $|Z_t - Z_{t-1}| \leq \underline{\hspace{2cm}1\hspace{2cm}}$

Intuitively, if you change the neighborhood of a single vertex, the chromatic number can change by at most 1:



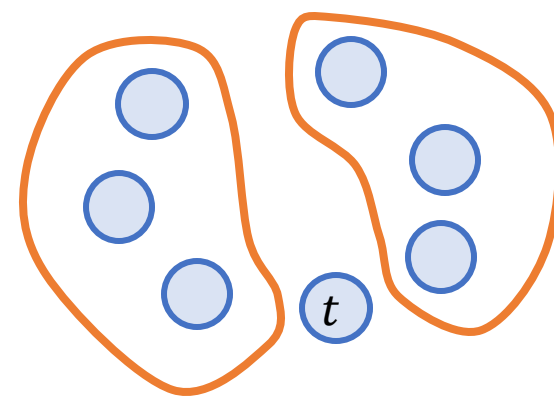
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Formally...

Want to show $|Z_t - Z_{t-1}| \leq 1$

$$Y = X_1, \dots, X_{t-1}$$



$$Y' = X_{t+1}, \dots, X_n$$

- Consider $E[A | Y, X_t = x]$ for any special x

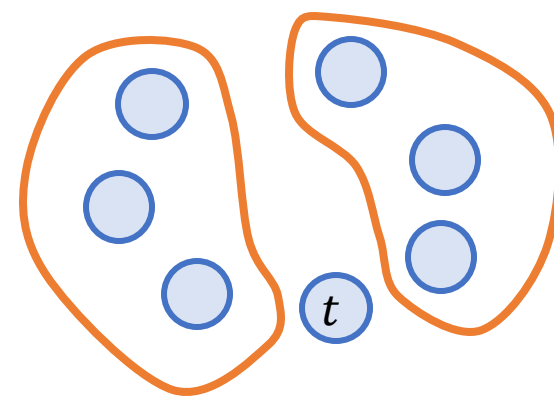
$$E[A | Y, X_t = x] = \sum_{y'} E[A | Y, X_t = x, Y' = y'] \Pr[Y' = y' | Y, X_t = x]$$

Can drop this by independence

Formally...

Want to show $|Z_t - Z_{t-1}| \leq 1$

$$Y = X_1, \dots, X_{t-1}$$



$$Y' = X_{t+1}, \dots, X_n$$

- Consider $E[A \mid Y, X_t = x]$ for any special x

$$E[A \mid Y, X_t = x] = \sum_{y'} E[A \mid Y, X_t = x, Y' = y'] \Pr[Y' = y']$$

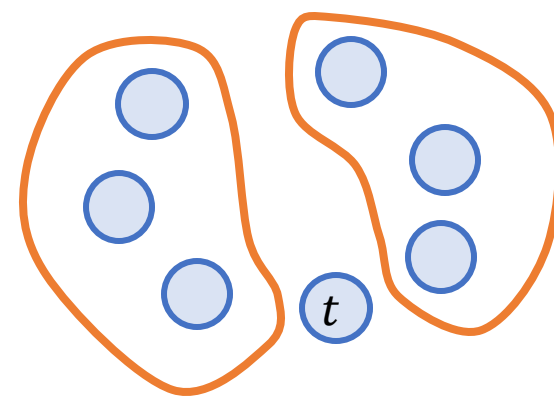
- Similarly,

$$E[A \mid Y, X_t] = \sum_{y'} E[A \mid Y, X_t, Y' = y'] \Pr[Y' = y']$$

Formally...

Want to show $|Z_t - Z_{t-1}| \leq 1$

$$Y = X_1, \dots, X_{t-1}$$



$$Y' = X_{t+1}, \dots, X_n$$

$$E[A | Y, X_t = x] = \sum_{y'} E[A | Y, X_t = x, Y' = y'] \Pr[Y' = y']$$

$$E[A | Y, X_t] = \sum_{y'} E[A | Y, X_t, Y' = y'] \Pr[Y' = y']$$

$$E[A | Y, X_t] - E[A | Y] \quad (\text{This is } Z_t - Z_{t-1})$$

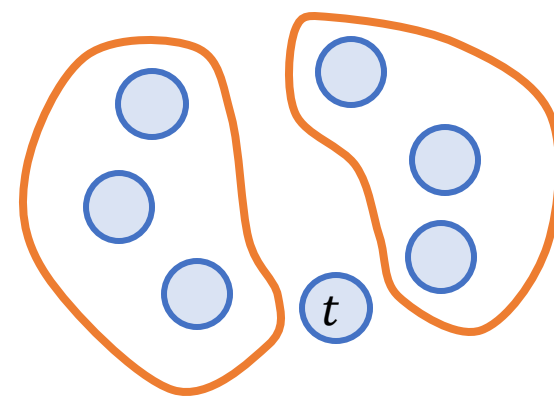
$$= \sum_x (E[A | Y, X_t] - E[A | Y, X_t = x]) \Pr[X_t = x]$$

Win if we can show that this is ≤ 1

Formally...

Want to show $|Z_t - Z_{t-1}| \leq 1$

$$Y = X_1, \dots, X_{t-1}$$



$$Y' = X_{t+1}, \dots, X_n$$

$$E[A | Y, X_t = x] = \sum_{y'} E[A | Y, X_t = x, Y' = y'] \Pr[Y' = y']$$

$$E[A | Y, X_t] = \sum_{y'} E[A | Y, X_t, Y' = y'] \Pr[Y' = y']$$

$$\sum_x (E[A | Y, X_t] - E[A | Y, X_t = x]) \Pr[X_t = x]$$

Win if we can show that this is ≤ 1

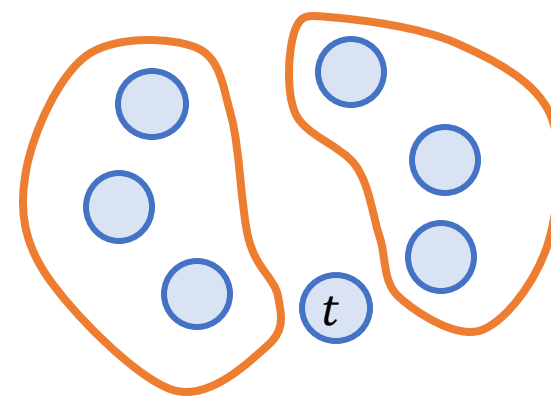
$$\sum_{y'} (E[A | Y, X_t, Y' = y'] - E[A | Y, X_t = x, Y' = y']) \Pr[Y' = y']$$

Win if we can show that this is ≤ 1

Formally...

Want to show $|Z_t - Z_{t-1}| \leq 1$

$$Y = X_1, \dots, X_{t-1}$$



$$Y' = X_{t+1}, \dots, X_n$$

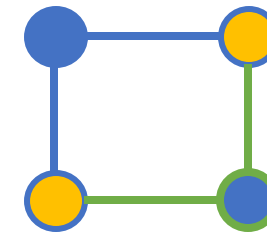
$$\sum_{y'} (E[A | Y, X_t, Y' = y'] - E[A | Y, X_t = x, Y' = y']) \Pr[Y' = y]$$

Win if we can show that this is ≤ 1



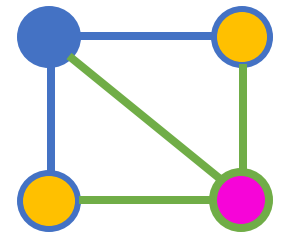
Formally, this is a function of Y and X_t

For any values of those variables we plug in, this differs by at most 1...



Here, $X_t = x$

VS



Here, $X_t =$
(whatever we plug in for X_t)

1. Vertex exposure martingale

- Applying Azuma-Hoeffding:

$$\Pr[|Z_n - Z_0| > c\sqrt{n}] \leq 2 \exp\left(-\frac{c^2 n}{2 \sum_i c_i^2}\right) \quad c_i = 1$$

$$Z_0 = E[A]$$

$$Z_n = E[A|X_1, \dots, X_n] = A$$

$$\Pr[|A - E[A]| > c\sqrt{n}] \leq 2 \exp\left(-\frac{c^2 n}{2n}\right) \leq 2 \exp(-c^2)$$

Want:

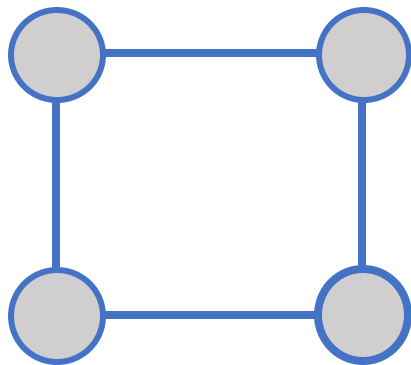
Aside: Method of bounded differences

- We've (essentially) just proved:
- Suppose that $A = A(X_1, \dots, X_n)$ has the property that
$$|A(x_1, \dots, x_{t-1}, \mathbf{x}_t, x_{t+1}, \dots, x_n) - A(x_1, \dots, x_{t-1}, \mathbf{x}_t', x_{t+1}, \dots, x_n)| \leq c_t$$
for all possible values of $x_1, \dots, x_{t-1}, \mathbf{x}_t, \mathbf{x}_t', x_{t+1}, \dots, x_n$.
- Suppose also that X_1, \dots, X_n are independent.
- Then $\Pr[|A - E[A]| > \lambda] \leq 2 \exp\left(-\frac{\lambda^2}{2 \sum_i c_i^2}\right)$

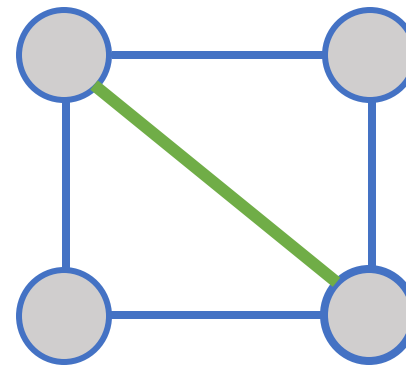
2. Edge exposure martingale

- Need to bound $|Z_t - Z_{t-1}| \leq \underline{\hspace{2cm}} \mathbf{1}$

Intuitively, if you change a single edge, the chromatic number can change by at most 1:



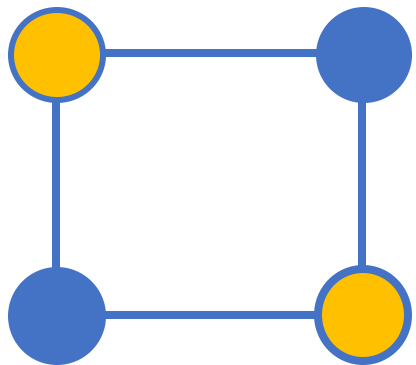
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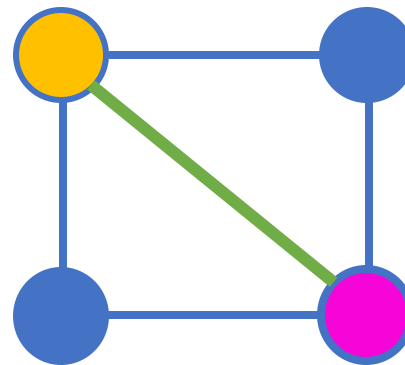
2. Edge exposure martingale

- Need to bound $|Z_t - Z_{t-1}| \leq \underline{\hspace{2cm}} \mathbf{1}$

Intuitively, if you change a single edge, the chromatic number can change by at most 1:



vs



2. Edge exposure martingale

- Applying Azuma-Hoeffding:

$$\Pr[|Z_n - Z_0| > c\sqrt{n}] \leq 2 \exp\left(-\frac{c^2 n}{2 \sum_i c_i^2}\right)$$

Do we get...

- (A) The same? $2\exp(-c^2)$
- (B) Better? (smaller?)
- (C) Worse? (bigger?)

2. Edge exposure martingale

- Applying Azuma-Hoeffding:

$$\Pr[|Z_n - Z_0| > c\sqrt{n}] \leq 2 \exp\left(-\frac{c^2 n}{2 \sum_i c_i^2}\right)$$

$$\Pr[|A - E[A]| > c\sqrt{n}] \leq 2 \exp\left(-\frac{c^2 n}{2 \binom{n}{2}}\right) \approx 2 \exp\left(-\frac{c^2}{n}\right) \text{ Worse!}$$

Next example: gambling



- A fun game:
 - At each step t , you can bet $b_t \in [0, B]$ and guess “heads or tails”
 - Flip a fair coin. If you were right, you win b_t . Otherwise you lose b_t .
- Notice that b_t can depend on everything that’s happened so far.
- Let Z_t be the amount of money you have at time t .
- It’s okay for $Z_t < 0$ (the casino knows you’re good for it...)
- Quick question: **Is Z_t a sum of independent random variables?**

No! The amount we bet can depend on previous rounds.

Group Work

- At each step t , you can bet $b_t \in [0, B]$ and guess “heads or tails”
- Flip a fair coin. If you were right, you win b_t . Otherwise you lose b_t .
- $Z_t =$ amount of money at time t

1. Suppose your betting strategy is deterministic.

Prove $\Pr[|Z_n| \geq cB\sqrt{n}] \leq \underline{\hspace{2cm}}$

2. Does the proof above work if your betting strategy is randomized?

If not, make it work.

1a. Setting up a martingale

- Z_t = amount of \$\$ at time t
- X_t = Outcome of coin flip at time t

- This is a martingale because...



1. Z_t is a function of X_0, \dots, X_t ?

If my betting strategy is deterministic, then X_0, \dots, X_t determines everything that's happened so far, including how much I bet.



2. $E[|Z_t|] < \infty$ For all t , $|Z_t| \leq Bt$



3. $E[Z_t | X_0, \dots, X_{t-1}] = Z_{t-1}$?

Either I win b_t or lose b_t , with equal probability.

1b. Using Azuma-Hoeffding

- $|Z_t - Z_{t-1}| \leq \underline{\hspace{2cm} B \hspace{2cm}}$

- Azuma-Hoeffding:

- $\Pr[|Z_n - Z_0| \geq \lambda] \leq 2 \exp\left(-\frac{\lambda^2}{\sum_i c_i^2}\right)$
 $Z_0 = 0$ $c_i = B$

$$\Pr[|Z_n| \geq cB\sqrt{n}] \leq 2 \exp\left(-\frac{c^2 B^2 n}{nB^2}\right) = 2 \exp(-c^2)$$

Take-away: no matter what betting strategy you use, it's very unlikely you'll win (or lose) **too** much money at this game*.

With a randomized betting strategy?

- Z_t = amount of \$\$ at time t
- X_t = Outcome of coin flip at time t ,
and the bet you decide to place at time $t + 1$
- This is a martingale because...



1. Z_t is a function of X_0, \dots, X_t ?



2. $E[|Z_t|] < \infty$?



3. $E[Z_t | X_0, \dots, X_{t-1}] = Z_{t-1}$?

- We don't have to secretly look into the future to define X_t (since b_{t+1} doesn't depend on $X_{>t}$)
- So we don't have to look into the future to define Z_t either.

Recap

- We got some practice applying Azuma-Hoeffding.
- Moral of the story I:
 - Azuma's inequality can bound sums of random variables that aren't necessarily independent!
- Moral of the story II:
 - One useful case is when you want to establish concentration for a random variable A that doesn't depend too much on any of the underlying variables X_i .
- Moral of the story II:
 - Sometimes there's more than one relevant martingale, and one might be better than the other.

Next time

- More Martingales, and the **Martingale Stopping Theorem!**