

## Class 16: Agenda/Questions

## 1 Announcements

- HW6 due Friday!
- HW7 out now! Due Friday 3/14 (also  $\pi$ -day!)

## 2 Questions?

Any questions from the minilectures and/or the quiz? (Martingales, the Doob martingale, Azuma-Hoeffding)

## 3 Chromatic numbers

In this exercise we'll practice using Azuma-Hoeffding

### Group Work

Let  $G \sim G_{n,p}$  be an Erdos-Renyi random graph (so there are  $n$  vertices, and each edge is present independently with probability  $p$ ). Let  $A = \chi(G)$  be the chromatic number of  $G$ . That is,  $A$  is the minimum number of colors necessary to properly color  $G$  (ie color the nodes of the graph such that no pair of neighboring nodes are assigned the same color).

1. Consider the Doob *vertex exposure* martingale. That is:
  - For  $i \in \{1, \dots, n\}$ , let  $X_i$  denote the status of the edges between vertex  $i$  and vertices  $\{1, \dots, i-1\}$ .
  - $Z_i = \mathbb{E}[A | X_1, \dots, X_i]$

[Note: this is a slightly different definition of the vertex exposure martingale than was in the lecture notes. Both work fine for this example.]

Use the Azuma-Hoeffding inequality to show that

$$\Pr[|A - \mathbb{E}[A]| > c\sqrt{n}] \leq 2 \exp(-c^2/2).$$

(Notice that you may not know what  $\mathbb{E}[A]$  is—that's okay!)

**Hint:** To use Azuma-Hoeffding, you need to bound  $|Z_i - Z_{i-1}|$ . How much can your expectation of the chromatic color change if I tell you additional information about a single vertex?

**Hint:** Bounding  $|Z_i - Z_{i-1}|$  really formally is actually a bit tricky. Try to come up with an intuitive bound, and if you have time, try to work it out formally.

2. Repeat the same exercise with the *edge exposure* martingale:

- Let  $X_i$  denote the status of the  $i$ 'th edge, for  $i \in \{1, \dots, \binom{n}{2}\}$ .
- $Z_i = \mathbb{E}[A|X_1, \dots, X_i]$

Do you get the same thing? Do you get something better? Worse?

3. (**Challenging**, but something to think about if you finish early.) What can you say about  $\mathbb{E}[A]$ ?

**Note:** If you're interested, check out <https://arxiv.org/abs/0706.1725> for a surprisingly strong statement about the chromatic number of random graphs!!

### Group Work: Solutions

See lecture notes (at the end of the notes).

## 4 Gambling

In this exercise, we'll get yet more practice applying Azuma-Hoeffding.

### Group Work

Consider the following gambling game:

- At time  $t$ , you can choose to bet *any* amount you like in  $[0, B]$ , where  $B$  is a house limit.
- A fair coin is flipped. If it's heads, you win the amount that you bet; if tails, you lose the amount that you bet.

You're allowed to be in debt; you don't stop when you run out of money.

1. Suppose that the amount you bet is a deterministic function of everything that's happened so far. Let  $Z_t$  be the amount of money you have at time  $t$ . Prove a bound of the form:

$$\Pr[|Z_t| \geq cB\sqrt{n}] \leq \text{-----}$$

**Hint:** Set up a martingale  $\{Z_t\}$  (with respect to some sequence  $\{X_t\}$  that you have to define)...

2. Now suppose that you can use *any* betting strategy you like, even a randomized one. Does your proof still work? If not, repeat the previous part with a possibly randomized betting strategy.

## **Group Work: Solutions**

See lecture notes (at the end of the notes).