Class 17: Agenda and Questions

1 Warm-Up

Group Work

Let $\{X_t\}$ be i.i.d. random ± 1 random variables with mean zero, and let $Z_t = \sum_{j=1}^t X_t$. Let $T = \min\{t : Z_t = 10\}$. Show both of the following:

- $\mathbb{E}[T] = \infty$ (In particular, the Martingale Stopping Theorem does not hold for T; we asserted this in the videos).
- On the other hand, $\Pr[\exists t, Z_t = 10] = 1$.

Extra: What if $Z_t = 1$ with probability p and $Z_t = -1$ with probability $1 - p$, for $p \neq 1/2$; whet is $\Pr[\exists t, Z_t = 10]$?

Group Work: Solutions

There are (at least) two ways to show this. The first, we alluded to in an earlier lecture when we showed that the expected time for this walk to get n away from where it started is $\Theta(n^2)$. That argument is that:

- Starting at 0, with equal probability the walk gets to 10 or -10 first (by symmetry), and it takes in expectation about $10^2 = 100$ timesteps to do so. If it gets to 10, then T has occurred. Otherwise, it gets to -10, and we can repeat the argument: with probability 1/2, in expected time about 20^2, the walk gets to 10 and we're done, or it gets to -30. Repeating this forever, we see that

$$\mathbb{E}[T] = \frac{1}{2}10^2 + \frac{1}{4}20^2 + \frac{1}{8}40^2 + \cdots + \frac{1}{2^j}(2^{j-1} \cdot 10)^2 + \cdots$$

  But $\sum_j \frac{1}{2^t}(10 \cdot 2^{j-1})^2 = 25 \sum_j 2^j = \infty$, so $\mathbb{E}[T] = \infty$.
- On the other hand, the same logic as above tells us that the probability that we return to 10 is $\sum_{j=1}^{\infty} 2^{-j} = 1$, since at each of the "stopping points" $-10, -30, -70, \ldots$ considered above, we have a 1/2 probability of returning to 10, and a 1/2 probability of ending up twice as far from 10.

The second uses Theorem 2 from the lecture notes. Let T_k be the first time that Z_t hits *either* $-k$ or 10. Then Thm 2 implies that $\mathbb{E}[T_k] = 10k$, and that $\Pr[Z_{T_k} = 10] = \frac{k}{k+10}$.

- $\mathbb{E}[T] \geq \mathbb{E}[T_k] = 10k$ for all k, so sending $k \to \infty$ we see that $\mathbb{E}[T] = \infty$.

- $Pr[Z_t$ is never $10\ ] \leq \Pr[Z_{T_k} = -k] = \frac{10}{k+10}$, so sending $k \to \infty$, we see that $\Pr[Z_t$ is never $10\ ] = 0$.

The second argument generalizes to the case where $\Pr[X_t = +1] = p$, using Theorem 3. Let $q = \frac{1-c^{10}}{(1/c)^k - c^{10}}$ be the probability that $Z_{T_k} = -k$ from Theorem 3. (Here, $c = 1/p - 1$).

- If $p > 1/2$, then $c < 1$, and

  $$\Pr[Z_t \text{ never returns to } 10\ ] \leq \Pr[Z_{T_k} = -k] = q,$$

  which goes to 0 as $k \to \infty$, since the $(1/c)^k$ term in the denominator of q dominates and gets large. So the walk will return to 10 with probability 1 (which makes sense, since we are only more likely to head in that direction).
- If $p < 1/2$, then $c > 1$. Intuitively, we might think that

  $$\Pr[Z_t \text{ never returns to } 10\ ] = \lim_{k\to\infty} \Pr[Z_{T_k} = -k] = 1 - (1/c)^{10}.$$

  This turns out to be true, although to prove it we should be a bit careful. One way to do it (this is just a sketch) is to say that

  $$\Pr[Z_t \text{ never returns to } 10] = \Pr[Z_{T_k} = -k]\,\Pr[Z_t \text{ never returns to } 10 \text{ starting from } -k\ ]$$

  and then recursively break down the second term to get something like:

  $$\Pr[Z_t \text{ never returns to } 10] = \Pr[Z_{T_k} = -k] \prod_{j=1}^{\infty} \Pr[Z_t \text{ gets to } -k^{j+1} \text{ before } 10, \text{ starting from } -k^j\ ].$$

  We can write the product as

  $$\prod_{j=1}^{\infty}\left(\frac{1-c^{k^j+10}}{(1/c)^{k^{j+1}} - c^{k^j+10}}\right) \approx \prod_{j=1}^{\infty}\left(1 - \frac{1}{c^{k^j+10}}\right) \approx \exp\left(-\sum_{j=1}^{\infty}(1/c)^{k^j+1}\right)$$

  Now, as $k \to \infty$ this term goes to 1 (since $c > 1$). So we get that

  $$\Pr[Z_t \text{ never returns to } 10\ ] = \lim_{k\to\infty} \Pr[Z_{T_k} = -k] = 1 - (1/c)^{10},$$

  which is what our intution said.

2 Announcements

- HW7 due Friday (Last one!)
- Final exam is Friday March 21, 8:30am-11:30pm in Hewlett 201.

3 Questions?

Any questions from the minilectures and/or the quiz? (Stopping times, Martingale stopping theorem)

4 Wald's equation

In this exercise we'll get some practice applying the martingale stopping theorem, to prove **Wald's equation.**

Theorem 1 (Wald's equation)**.** *Suppose that $X_1, X_2, \ldots$ are non-negative i.i.d. random variables, distributed according to some random variable X. Let T be a stopping time for $\{X_i\}$. If $\mathbb{E}[X]$ and $\mathbb{E}[T]$ are both bounded, then*

$$\mathbb{E}\left[\sum_{i=1}^{T} X_i\right] = \mathbb{E}[T] \cdot \mathbb{E}[X]. \tag{1}$$

Group Work

1. Wald's equation hopefully seems pretty intuitive. But there is something to prove! Come up with an example of some random variables X_i and T that don't obey the hypotheses of Theorem 1, so that the (1) does not hold.

   Note: To make this more challenging, try to violate as few of the hypotheses as possible.
2. Let $Z_i = \sum_{j=1}^{i}(X_j - \mathbb{E}[X])$. Prove that $\{Z_i\}$ is a martingale with respect to $\{X_i\}$.
3. Argue that the martingale stopping theorem applies to $\{Z_i\}$ and T, where X, T are as in Theorem 1.
4. Use the Martingale stopping theorem to prove Wald's equation.
5. Consider rolling a fair, six-sided die repeatly. Let X be the sum of all of the rolls up until the first "6" is rolled, not including that 6. What is $\mathbb{E}X$?

Group Work: Solutions

1. There are many examples, but here's a simple one. Let $X_1 = 0$ with probability $1/2$ and 1 with probability $1/2$. Let $T = 1 - X_1$. That is, if $X_1 = 0$, then $T = 1$, and if $X_1 = 1$, then $T = 0$. This violates the hypotheses because T is *not* a stopping time. Indeed, we may find out at time $t = 1$ that the stopping time T was actually

0. To see that this is a counterexample, notice that $\mathbb{E}[T] = \mathbb{E}[X] = 1/2$, while

$$\mathbb{E}[\sum_{i=1}^{T} X_i] = 0.$$

(To see the last thing, notice that in fact this sum is always 0. If $X_1 = 0$, then $T = 1$ and the sum is just $X_1 = 0$. If $X_1 = 1$, then $T = 0$ and the sum is empty.

2. We write

$$\begin{aligned}\mathbb{E}[Z_t|X_1,\dots,X_{t-1}] &= \sum_{j=1}^{t-1}(X_j - \mathbb{E}X) + \mathbb{E}[X_t - \mathbb{E}X|X_1,\dots,X_t] \\ &= \sum_{j=1}^{t-1}(X_j - \mathbb{E}X) = Z_{t-1}.\end{aligned}$$

3. We use the third condition. By the assumption in Wald's thm, $\mathbb{E}T < \infty$, so we just need to show that there is some c so that, for all i, $\mathbb{E}[|Z_{i+1} - Z_i||X_0,\dots X_i] < c$. This conditional expectation is just

$$\mathbb{E}|X_{i+1} - \mathbb{E}X| \le 2\mathbb{E}[X],$$

(using the triangle inequality). And this is again bounded by the assm in Wald's theorem.

4. Applying the Martingale stopping theorem, we have

$$\begin{aligned}0 &= \mathbb{E}Z_0 \\ &= \mathbb{E}Z_T \\ &= \mathbb{E}[\sum_{j=1}^{T}(X_j - \mathbb{E}[X])] \\ &= \mathbb{E}[\sum_{j=1}^{T} X_j] - \mathbb{E}[T]\mathbb{E}[X]\end{aligned}$$

and rearranging proves (1).

5. Let X_i be the outcome of the i'th roll, and let T be the first time we see a six. Then T is a stopping time for X_i and $\mathbb{E}T$, $\mathbb{E}X$ are both bounded. Thus,

$$\mathbb{E}\sum_{i=1}^{T} X_i = \mathbb{E}[T]\mathbb{E}[X] = 6 \cdot \frac{7}{2} = 21.$$

However, what we are after is actually $\sum_{i=1}^{T-1} X_i$, but by definition the last term is 6, so we have

$$\sum_{i=1}^{T-1} X_i = 21 - 6 = 15.$$

5 Ballot Counting

Suppose that there is an election with two candidates, A and B, and n voters; say candidate A is the winner, receiving $N_A > N_B$ votes. (So $N_A + N_B = n$). The ballots are counted in a random order. What is the probably that A remained ahead for the entire count?

Let A_t be the number of votes for A at time t; let B_t be the number of votes for B at time t.

Let $Z_t = \frac{A_{n-t} - B_{n-t}}{n-t}$. That is, we imagine that we've already done the count, and then we "uncount" the votes one-by-one.

Group Work

1. Let T be the smallest t so that $Z_t = 0$; if this never occurs, set $T = n - 1$.

   Explain why T is a stopping time for $\{Z_t\}$, and why the Martingale Stopping Theorem applies to it. (Assume for now that $\{Z_t\}$ is indeed a martingale; you'll show that soon).
2. Apply the Martingale Stopping Theorem to $\{Z_t\}$ and T, and use it to compute the probability that candidate A was ahead throughout the count.
3. Show that $\{Z_t\}$ is a martingale. (Hint: It might help to think of the process that Z_t is tracking as follows. Start with two piles of ballots, one of size N_A and one of size N_B. Then choose a uniformly random vote to remove from one of the two piles; that will give you two piles corresponding to Z_1. Continue in this way.)

Group Work: Solutions

1. Intuitively, T is a stopping time since we don't need to "look into the future" to compute it: we know at time t whether or not $T = t$. With probability 1, $T < n-1$, so the second item of the Martingale Stopping Theorem applies.
2. Applying the Martingale Stopping Theorem, we have

   $$\mathbb{E}[Z_T] = \mathbb{E}[Z_0] = \frac{A_n - B_n}{n} = \frac{N_A - N_B}{n}.$$

   On the other hand, there are two possibilities for how Z_T could end up. Either

$T < n-1$, which means that $Z_T = 0$, or else $T = n-1$, which means that $Z_T = (1-0)/1 = 1$. (Notice that if $Z_T = n-1$, we must have $A_1 = 1$ and $B_1 = 0$, since if $B_1 = 1, A_1 = 0$, we would have had $Z_t = 0$ for some $t < n-1$, since candidate B got ahead somehow.) Thus, if $Z_T = 1$ (and $T = n-1$), then candidate A was ahead for the whole count; otherwise $T < n-1$ and $Z_T = 0$.

Let p be the probability that candidate A was ahead for the whole count. Then the above reasoning shows that

$$\mathbb{E}[Z_T] = (1-p)\cdot 0 + p \cdot 1.$$

Using the above, this shows

$$p = \frac{N_A - N_B}{n}.$$

3. To show that $\{Z_t\}$ is a martingale, we have

$$\mathbb{E}Z_{t+1} = \frac{\mathbb{E}A_{n-t-1}}{n-t-1} - \frac{\mathbb{E}B_{n-t-1}}{n-t-1}.$$

Consider each of these terms separately. By the intuition in the hint, the expectation $\mathbb{E}A_{n-t-1}$ is the probability that we chose our "removed" ballot from pile A (that would be $A_{n-t}/(n-t)$) times $A_{n-t}-1$; plus the probability that we "removed" the ballot from pile B ($B_{n-t}/(n-t)$) times A_{n-t}. We have a similar calculation for the other term. Thus,

$$\begin{aligned}
\mathbb{E}[Z_{t+1}|Z_1,\ldots,Z_t] &= \frac{\mathbb{E}A_{n-t-1}}{n-t-1} - \frac{\mathbb{E}B_{n-t-1}}{n-t-1} \\
&= \frac{1}{n-t-1}\left(\frac{A_{n-t}}{n-t}\cdot(A_{n-t}-1) + \frac{B_{n-t}}{n-t}\cdot A_{n-t}\right) - \\
&\qquad \frac{1}{n-t-1}\left(\frac{B_{n-t}}{n-t}\cdot(B_{n-t}-1) + \frac{A_{n-t}}{n-t}\cdot B_{n-t}\right)
\end{aligned}$$

using the fact that $B_{n-t} + A_{n-t} = n-t$, this simplifies to

$$\begin{aligned}
\cdots &= \frac{A_{n-t}}{n-t+1} - \frac{B_{n-t}}{n-t+1} - \frac{A_{n-t}}{(n-t-1)(n-t)} + \frac{B_{n-t}}{(n-t-1)(n-t)} \\
&= \frac{A_{n-t}}{n-t} - \frac{B_{n-t}}{n-t} \\
&= Z_t.
\end{aligned}$$

This is what we wanted, so Z_t is indeed a martingale.