

## Class 17: Agenda and Questions

## 1 Warm-Up

**Group Work**

Let  $\{X_t\}$  be i.i.d. random  $\pm 1$  random variables with mean zero, and let  $Z_t = \sum_{j=1}^t X_j$ . Let  $T = \min\{t : Z_t = 10\}$ . Show both of the following:

- $\mathbb{E}[T] = \infty$  (In particular, the Martingale Stopping Theorem does not hold for  $T$ ; we asserted this in the videos but didn't prove it).
- On the other hand,  $\Pr[\exists t, Z_t = 10] = 1$ .

**Extra:** Now suppose that  $Z_t = 1$  with probability  $p$  and  $Z_t = -1$  with probability  $1 - p$ , for  $p \neq 1/2$ . What is  $\Pr[\exists t, Z_t = 10]$ ?

## 2 Announcements

- HW7 due Friday (Last one!)
- Final exam is Friday March 21, 8:30am-11:30pm in Hewlett 201.

## 3 Questions?

Any questions from the minilectures and/or the quiz? (Stopping times, Martingale stopping theorem)

## 4 Wald's equation

In this exercise we'll get some practice applying the martingale stopping theorem, to prove **Wald's equation**.

**Theorem 1** (Wald's equation). *Suppose that  $X_1, X_2, \dots$  are non-negative i.i.d. random variables, distributed according to some random variable  $X$ . Let  $T$  be a stopping time for  $\{X_i\}$ . If  $\mathbb{E}[X]$  and  $\mathbb{E}[T]$  are both bounded, then*

$$\mathbb{E}\left[\sum_{i=1}^T X_i\right] = \mathbb{E}[T] \cdot \mathbb{E}[X]. \quad (1)$$

### Group Work

1. Wald's equation hopefully seems pretty intuitive. But there is something to prove! Come up with an example of some random variables  $X_i$  and  $T$  that don't obey the hypotheses of Theorem 1, so that the (1) does not hold.  
**Note:** To make this more challenging, try to violate as few of the hypotheses as possible.
2. Let  $Z_i = \sum_{j=1}^i (X_j - \mathbb{E}[X])$ . Prove that  $\{Z_i\}$  is a martingale with respect to  $\{X_i\}$ .
3. Argue that the martingale stopping theorem applies to  $\{Z_i\}$  and  $T$ , where  $X, T$  are as in Theorem 1.
4. Use the Martingale stopping theorem to prove Wald's equation.
5. Consider rolling a fair, six-sided die repeatedly. Let  $X$  be the sum of all of the rolls up until the first "6" is rolled, not including that 6. What is  $\mathbb{E}X$ ?

## 5 Ballot Counting

Suppose that there is an election with two candidates,  $A$  and  $B$ , and  $n$  voters; say candidate  $A$  is the winner, receiving  $N_A > N_B$  votes. (So  $N_A + N_B = n$ ). The ballots are counted in a random order. What is the probability that  $A$  remained ahead for the entire count?

Let  $A_t$  be the number of votes for  $A$  at time  $t$ ; let  $B_t$  be the number of votes for  $B$  at time  $t$ .

Let  $Z_t = \frac{A_{n-t} - B_{n-t}}{n-t}$ . That is, we imagine that we've already done the count, and then we "uncount" the votes one-by-one.

### Group Work

1. Let  $T$  be the smallest  $t$  so that  $Z_t = 0$ ; if this never occurs, set  $T = n - 1$ .  
Explain why  $T$  is a stopping time for  $\{Z_t\}$ , and why the Martingale Stopping Theorem applies to it. (Assume for now that  $\{Z_t\}$  is indeed a martingale; you'll show that soon).
2. Apply the Martingale Stopping Theorem to  $\{Z_t\}$  and  $T$ , and use it to compute the probability that candidate  $A$  was ahead throughout the count.
3. Show that  $\{Z_t\}$  is a martingale. (Hint: It might help to think of the process that  $Z_t$  is tracking as follows. Start with two piles of ballots, one of size  $N_A$  and one of size  $N_B$ . Then choose a uniformly random vote to remove from one of the two piles; that will give you two piles corresponding to  $Z_1$ . Continue in this way.)