Quick Reference: Amplifier Equations

1.0 Introduction

Before developing a deep familiarity with the various characteristics of amplifiers, it’s common to spend a fair amount of time flipping back and forth through your notes and textbook, in search of various formulas. To save you some trouble, here are many of the relevant equations in one place. Because this set of notes is not intended to be a textbook on network theory, only the barest sketches of derivations are usually provided. That said, there should still be enough that you could reconstruct intermediate steps if needed.

Part of what distinguishes an experienced analog designer from others is the ability to apply the simplest approximation that captures the phenomena of interest. Throughout, recognize that many of the equations in this handout simplify considerably if you ignore various factors (e.g. body effect, channel-length modulation, DIBL, etc.). In some cases, we offer those simplifications. In others, they are left for you to finish. In all cases, pause and consider when it may be acceptable to neglect certain of these phenomena, and how that selective and conscious neglect simplifies the equations.

Throughout this handout, we use the following model for the transistor and load/source resistances:

FIGURE 1. Incremental transistor model, plus terminal resistances

As you can see, we are implicitly assuming that the bulk terminal is tied to an incremental ground.

2.0 Gain, and input/output resistances

2.1 Common-source input resistance

The input resistance is effectively infinite (not a function of $g_{mb}$ or CLM/DIBL). In this era of superthin gate oxides, leakage is more than measurable, and the input resistance is
consequently not truly infinite. However, it still remains high enough for most purposes that infinity is a useful approximation.

2.2 Common-source output resistance

Use a test current source, $i_t$. Since $v_{gs} = v_{bs}$, can simply sum the two transconductances into a single one (let’s call it $g_{mtot}$). Compute the resistance looking into the drain, and call it $r_{out}$ The net resistance at the drain will therefore be $r_{out}$ in parallel with any external resistance, $R_L$.

\[ v_{gs} = -i_t R_S; \]  \hfill (1)

\[ v_{r0} = (i_t - g_{mtot} v_{gs})r_0 = i_t(1 + g_{mtot} R_S)r_0; \]  \hfill (2)

\[ V_{test} = i_t R_S + i_t(1 + g_{mtot} R_S)r_0 = i_t[R_S + r_0 + g_{mtot} r_0 R_S]. \]  \hfill (3)

So, at last, the resistance looking into the drain (and therefore not including $R_L$), is

\[ r_{out} = R_S + r_0 + g_{mtot} r_0 R_S = R_S + r_0 + (g_m + g_{mb}) r_0 R_S. \]  \hfill (4)

2.3 Common-source voltage gain

Multiplication of the total resistance at the drain by the effective transconductance will give us the gain. To find the effective transconductance, short the drain to ground (incrementally speaking), and measure the ratio of short-circuit output current to input voltage.

\[ v_{gs} = v_{in} - i_{out} R_S; \]  \hfill (5)

\[ v_{bs} = -i_{out} R_S; \]  \hfill (6)

\[ i_{out} = g_m v_{gs} + g_{mb} v_{bs} - i_{out} \frac{R_S}{r_0} = g_m(v_{in} - i_{out} R_S) + g_{mb}(-i_{out} R_S) - i_{out} \frac{R_S}{r_0}; \]  \hfill (7)

\[ g_{meff} = \frac{i_{out}}{v_{in}} = \frac{g_m}{1 + (g_m + g_{mb}) R_S + \frac{R_S}{r_0}} = \frac{g_m}{1 + (g_m + g_{mb} + g_0) R_S}, \]  \hfill (8)

where we have used $g_0 = 1/r_0$.

The voltage gain is therefore

\[ A_v = -g_{meff} [r_{out} \parallel R_L] = \frac{g_m}{1 + (g_m + g_{mb} + g_0) R_S} \left( [R_S + r_0 + (g_m + g_{mb}) r_0 R_S] \parallel R_L \right) \]  \hfill (9)

which simplifies initially to
After a little more work, the equation’s complexity reduces further, leading to a quite reasonable expression:

\[ A_v = -\frac{g_mR_L}{1 + (g_m + g_{mb} + g_0)R_S} \cdot \frac{R_S + r_0 + (g_m + g_{mb})r_0R_S}{R_s + r_0 + (g_m + g_{mb})r_0R_S + R_L}. \]  

After a little more work, the equation’s complexity reduces further, leading to a quite reasonable expression:

\[ A_v = -\frac{g_mR_L}{1 + (g_m + g_{mb})R_S + g_0(R_L + R_S)}. \]  

The reader should verify that this expression reduces to the expected approximations when the body effect may be neglected, when there is no source degeneration, etc.

### 2.4 Common-gate input resistance

Use a test voltage source. Note that \( v_{gs} = v_{bs} \), again, so we can merge the two transconductances into a single \( g_{mtot} \). Compute the resistance looking into the source first, then accommodate \( R_S \) at the end, if desired or necessary. You may use superposition to simplify the derivation (treat transconductances as independent current sources in this computation, since the control voltage is fixed throughout this particular experiment).

\[ v_{gs} = -v_{test}. \]

With transconductance \( g_{mtot} \) disabled, we compute one contribution to the test current:

\[ i_{t1} = \frac{v_{test}}{r_0 + R_L}. \]

Next, disable (short out) the test voltage source, taking care not to zero out \( g_{mtot} \) and compute the other contributions to the test current. The current from the transconductance is added to the current flowing through \( r_0 \).

\[ i_{t2} = -g_{mtot}v_{gs} + \left( -i_{t2} \frac{R_L}{r_0} \right) = g_{mtot}v_{test} - i_{t2} \frac{R_L}{r_0} \Rightarrow i_{t2} = \frac{g_{mtot}v_{test}}{1 + \frac{R_L}{r_0}} = \frac{g_{mtot}r_0v_{test}}{r_0 + R_L}; \]

\[ i_t = i_{t1} + i_{t2} = \frac{v_{test}}{r_0 + R_L} + \frac{g_{mtot}r_0v_{test}}{r_0 + R_L} = \frac{v_{test}}{r_0 + R_L} \left( 1 + g_{mtot}r_0 \right). \]

So, the resistance looking into the source, \( r_{in} \), is:

\[ \frac{v_{test}}{i_t} = \frac{r_0 + R_L}{1 + g_{mtot}r_0} = \frac{r_0 + R_L}{1 + (g_m + g_{mb})r_0}. \]

Note that, if \( R_L \ll r_0 \), and if the transistor’s “intrinsic voltage gain” \( g_mr_0 \) is much greater than unity, then the resistance looking into the source is approximately
If you desire the total resistance between the source node and ground, simply compute the parallel combination of \( r_{in} \) and \( R_S \).

### 2.5 Common-gate output resistance

This resistance is precisely the same as the output resistance of the degenerated common-source amplifier:

\[
r_{out} = R_S + r_0 + g_{mtot}r_0R_S = R_S + r_0 + (g_m + g_{mb}) r_0 R_S.
\]

If we drive the source terminal directly with a voltage source, then \( R_S = 0 \), and the equation for the resistance looking into the drain collapses to simply \( r_0 \). That remains a reasonably good approximation as long as \( R_S \) is small compared with \( r_0 \), and \( (g_m + g_{mb})R_S \) is small compared with unity.

### 2.6 Common-gate voltage gain

As before, let’s first compute the effective transconductance of the amplifier, defined here as the ratio of short-circuit drain current to source voltage:

\[
v_{gs} = -v_{in};
\]

\[
v_{bs} = v_{gs};
\]

\[
i_{out} = -g_{mtot}v_g + \frac{v_{in}}{r_0} = v_{in}(g_{mtot} + g_0) \Rightarrow g_{meff} = g_{mtot} + g_0 = (g_m + g_{mb} + g_0).
\]

The voltage gain from source to drain is therefore

\[
A_{v0} = g_{meff}(r_{out} \parallel R_L) = (g_m + g_{mb} + g_0)(r_0 \parallel R_L),
\]

where we have used the fact that, in this first computation, we are driving the source directly with a voltage source. The total drain resistance is thus just \( r_0 \parallel R_L \), and the gain in Eqn.22 is therefore that from the source to drain, as stated.

The voltage divider formed at the input must be taken into account to complete the calculation of the overall gain:

\[
A_v = A_{v0} \cdot \frac{r_{in}}{r_{in} + R_S}.
\]

So,
which simplifies a bit to

\[
A_v = \frac{r_0 + R_L}{1 + (g_m + g_{mb})r_0} \cdot \frac{1 + (g_m + g_{mb})r_0}{r_0 + R_L + R_S}.
\]  

(24)

and further to:

\[
A_v = \frac{r_0 + R_L}{r_0 + R_L + R_S + (g_m + g_{mb})r_0R_S}.
\]  

(25)

As expected, the gain expression collapses to more familiar forms when body effect is neglected, the source resistance \(R_S\) is zero, and if the transistor’s output conductance is zero, etc. The reader should verify these statements independently.

### 2.7 Source follower input resistance

Again, we treat it as infinite, independent of body effect and CLM/DIBL.

### 2.8 Source follower output resistance

The source follower’s output resistance is the same as the input resistance of a common-gate stage:

\[
r_{out} = \frac{r_0 + R_L}{1 + g_{mtot}r_0} = \frac{r_0 + R_L}{1 + (g_m + g_{mb})r_0}.
\]  

(27)

In most source followers, \(R_L\) is chosen very small compared with \(r_0\). In such cases, the output resistance simplifies to:

\[
r_{out} \approx \frac{r_0}{1 + (g_m + g_{mb})r_0}.
\]  

(28)

If, in addition, the transistor’s intrinsic voltage gain is high, then the unity factor in the denominator can be neglected, simplifying the output resistance expression even further:

\[
r_{out} \approx \frac{1}{g_m + g_{mb}}.
\]  

(29)
2.9  **Source follower voltage gain**

Again, compute an effective transconductance, and multiply by the output resistance. That output resistance needs to include any external loading that might appear in parallel with the resistance seen when looking into the transistor’s source.

\[ v_{gs} = v_{in}; \]

\[ v_{bs} = v_{gs}; \]  

(for output resistance only; \( v_{bs} = 0 \) for effective transconductance calculations). So,

\[ i_{out} = g_m v_{in} - i_{out} \frac{R_L}{r_0}; \]

\[ g_{meff} = \frac{i_{out}}{v_{in}} = \frac{g_m}{R_L + \frac{r_0}{r_0}}. \]  

Overall voltage gain is therefore:

\[ A_{v0} = g_{meff}(r_{out} \| R_S) = \frac{g_m}{1 + \frac{r_0 + R_L}{r_0}} \cdot \left[ \left( \frac{r_0 + R_L}{1 + (g_m + g_{mb})r_0} \right) \| R_S \right] \]

which, after some effort, simplifies to

\[ A_{v0} = \frac{g_m r_0 R_S}{(g_m + g_{mb}) r_0 R_S + (r_0 + R_L + R_S)}. \]

Alternatively, we may write:

\[ A_{v0} = \frac{g_m R_S}{(g_m + g_{mb}) R_S + 1 + g_0 (R_L + R_S)} = \frac{1}{1 + \frac{(g_{mb} + g_0)}{g_m} + \frac{(1 + g_0 R_L)}{g_m R_S}}. \]

from which it is perhaps somewhat easier to see that the voltage gain can only approach unity, as we would expect from a source follower.

As usual, the reader should verify that the exact equation simplifies to the numerous known approximations in the case where body effect is zero, transistor \( g_0 \) may be neglected, \( R_L \) is zero, etc.

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3.0 Resistances for open-circuit time constant calculations

It’s important to recall that a two-port model is a general and complete description for the terminal behavior of any single-input, single-output amplifier. Hence, the effective resistance facing any capacitor may be evaluated directly from the two-port model if the corresponding parameters are known. We further simplify the situation by assuming that we may safely neglect reverse transmission. This assumption of unilateral behavior is usually well satisfied in practical amplifiers. However, this observation does not mean that you are absolved of verifying whether your particular amplifier satisfies this assumption. Always check assumptions in any case where it matters!

Using a current source in a hybrid model, we find that the effective resistance may be stated in a simple, universal, mnemonic form:

\[ r_{eq} = r_{left} + r_{right} + g_{mf} r_{left} r_{right}, \]  

(37)

where \( r_{left} \) is the resistance seen between the capacitor’s left terminal and ground, \( r_{right} \) is that seen between the right terminal and ground, and \( g_{mf} \) is the effective transconductance, defined as the ratio of short-circuit output current to input voltage.

This equation may also be expressed as

\[ r_{eq} = r_{left} + r_{right} - A_{vf} r_{left}, \]  

(38)

where we have recognized the product of effective transconductance and \( r_{right} \) as (minus) the voltage gain between the two terminals of the capacitor. From this form of the equation, we can see directly the effect of voltage gain between the capacitor’s terminals, a phenomenon first explained by John M. Miller of the National Bureau of Standards, in 1918.

3.0.1 Resistance facing \( c_{gd} \)

For the drain-gate capacitance, the effective resistance is

\[ R_G + r_{out} \parallel R_L + \frac{g_m}{1 + (g_m + g_{mb} + g_0) R_S} (r_{out} \parallel R_L) R_G, \]  

(39)

where

\[ r_{out} \parallel R_L = [R_S + r_0 + (g_m + g_{mb}) r_0 R_S] \parallel R_L. \]  

(40)

The expression for the effective resistance simplifies considerably in the case of a common-gate connection, where \( R_G \) is often zero (or very small). In that case, Eqn.40 is all you need.

In a source-follower connection, where \( R_L \) is typically zero (or very small), an even greater simplification results. In that case, \( R_G \) is perhaps a good approximation to the
resistance facing \( c_{gd} \). So, the expression for the resistance is somewhat complicated only for the case of the common-source amplifier.

### 3.0.2 Resistance facing \( c_{gs} \)

Here, the same basic two-port model applies, but now the model parameters have to be chosen to reflect the fact that the capacitor in question connects between the gate and source terminals. That is, the amplifier under consideration has its input at the gate, and provides an output at the source. If we call \( R_G \) the resistance to the left, the resistance to the right is simply the total output resistance of a source-follower (i.e., the resistance looking into the source terminal, in parallel with any external \( R_S \)). Similarly, the effective transconductance is also merely that of a source-follower. So, we have actually derived all of the pieces already. We just need to put them together:

\[
\begin{align*}
\text{equivalent resistance } R_{eq} &= R_G + \left[ \frac{r_0 + R_L}{1 + (g_m + g_{mb})r_0} \right] R_S \left[ \frac{g_m}{1 + \frac{R_L}{r_0}} \right] R_G \left[ \frac{r_0 + R_L}{1 + (g_m + g_{mb})r_0} \right] R_S \\
&= R_G + \left[ \frac{R_S}{r_0 + R_L + R_S + (g_m + g_{mb})r_0R_S} \right] \left[ (r_0 + R_L) - g_m r_0 R_G \right].
\end{align*}
\]

which we try to simplify as follows:

\[
\begin{align*}
r_{eq} &= R_G + \frac{R_S}{r_0 + R_L + R_S + (g_m + g_{mb})r_0R_S} \left[ (r_0 + R_L) - g_m r_0 R_G \right].
\end{align*}
\]

After some crunching, we finally get something that does look simpler:

\[
\begin{align*}
r_{eq} &= \frac{R_G(r_0 + R_L + R_S + g_{mb}r_0R_S) + R_S(r_0 + R_L)}{r_0 + R_L + R_S + (g_m + g_{mb})r_0R_S}.
\end{align*}
\]

Collecting terms in a slightly different way is also perhaps useful:

\[
\begin{align*}
r_{eq} &= \frac{R_G + R_S + g_{mb}R_G R_S + g_0(R_G R_L + R_G R_S + R_S R_L)}{1 + (g_m + g_{mb})R_S + g_0(R_L + R_S)}.
\end{align*}
\]

Note that, in the limit of no body effect and infinite \( R_0 \), the equivalent resistance facing \( c_{gs} \) indeed simplifies to a result we’ve seen before:

\[
\begin{align*}
r_{eq} &\approx \frac{R_G + R_S}{1 + g_m R_S}.
\end{align*}
\]

### 3.0.3 Resistance facing \( c_{db} \)

In the general case, this resistance is just the resistance looking into the drain of a transistor, in parallel with any additional resistance connected to that drain (call that extra resis-
tance $R_L$ to be consistent with our notation so far). We’ve already found the resistance looking into the drain:

$$r_{out} = R_S + r_0 + g_{mtot}r_0 R_S = R_S + r_0 + (g_m + g_{mb})r_0 R_S.$$  (46)

So, just place that value in parallel with any $R_L$ that happens to be present.

### 3.0.4 Resistance facing $c_{sb}$

Similarly, this resistance is just that looking into the source of a transistor, in parallel with any other resistance connected to it (we’ve been calling it $R_S$). Thus, the source-follower output resistance equation is what we need here:

$$r_{out} = \frac{r_0 + R_L}{1 + (g_m + g_{mb})r_0}.$$  (47)

Just compute the parallel combination of $r_{out}$ and $R_S$ to get the final answer.