EE263 homework 6 solutions

9.9 *Analysis of a power control algorithm.* In this problem we consider again the power control method described in homework problem 2.1. Please refer to this problem for the setup and background. In that problem, you expressed the power control method as a discrete-time linear dynamical system, and simulated it for a specific set of parameters, with several values of initial power levels, and two target SINRs. You found that for the target SINR value $\gamma = 3$, the powers converged to values for which each SINR exceeded γ, no matter what the initial power was, whereas for the larger target SINR value $\gamma = 5$, the powers appeared to diverge, and the SINRs did not appear to converge. You are going to analyze this, now that you know alot more about linear systems.

(a) *Explain the simulations.* Explain your simulation results from the problem 2.1(b) for the given values of G, α, σ, and the two SINR threshold levels $\gamma = 3$ and $\gamma = 5$.

(b) *Critical SINR threshold level.* Let us consider fixed values of G, α, and σ. It turns out that the power control algorithm works provided the SINR threshold γ is less than some critical value $\gamma_{\rm crit}$ (which might depend on G, α, σ), and doesn't work for $\gamma > \gamma_{\rm crit}$. ('Works' means that no matter what the initial powers are, they converge to values for which each SINR exceeds γ.) Find an expression for $\gamma_{\rm crit}$ in terms of $G \in \mathbf{R}^{n\times n}$, α, and σ. Give the simplest expression you can. Of course you must explain how you came up with your expression.

Solution:

(a) In the homework we found that the powers propagate according to a linear system. The power update rule for a single transmitter can be found by manipulating the definitions given in the problem.

$$
\begin{aligned}
p_i(t+1) &= \frac{\alpha\gamma p_i(t)}{S_i(t)} = \frac{\alpha\gamma p_i(t) q_i(t)}{s_i(t)} = \frac{\alpha\gamma p_i(t)\left[\sigma + \sum_{j\neq i} G_{ij} p_j(t)\right]}{G_{ii} p_i(t)} \\
&= \frac{\alpha\gamma\left[\sigma + \sum_{j\neq i} G_{ij} p_j(t)\right]}{G_{ii}}
\end{aligned}
$$

In matrix form the equations represent a linear dynamical system with constant input, $p(t+1) = Ap(t) + b$.

$$\underbrace{\begin{bmatrix} p_1(t+1) \\ p_2(t+1) \\ p_3(t+1) \\ \vdots \\ p_n(t+1) \end{bmatrix}}_{p(t+1)} = \alpha\gamma \underbrace{\begin{bmatrix} 0 & \frac{G_{12}}{G_{11}} & \frac{G_{13}}{G_{11}} & \cdots & \frac{G_{1n}}{G_{11}} \\ \frac{G_{21}}{G_{22}} & 0 & \frac{G_{23}}{G_{22}} & \cdots & \frac{G_{2n}}{G_{22}} \\ \frac{G_{31}}{G_{33}} & \frac{G_{32}}{G_{33}} & 0 & \cdots & \frac{G_{3n}}{G_{33}} \\ \vdots & & & \ddots & \\ \frac{G_{n1}}{G_{nn}} & \frac{G_{n2}}{G_{nn}} & \frac{G_{n3}}{G_{nn}} & \cdots & 0 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \\ \vdots \\ p_n(t) \end{bmatrix}}_{p(t)} + \underbrace{\begin{bmatrix} \frac{\alpha\gamma\sigma}{G_{11}} \\ \frac{\alpha\gamma\sigma}{G_{22}} \\ \frac{\alpha\gamma\sigma}{G_{33}} \\ \vdots \\ \frac{\alpha\gamma\sigma}{G_{nn}} \end{bmatrix}}_{b}.$$

where $A = \alpha\gamma P$. This is a discrete LDS, and is stable if and only if $|\lambda_i| < 1$ for all $i = 1, \ldots, n$, where λ_i are the eigenvalues of A. When $\gamma = 3$ the eigenvalues of A are 0.6085, -0.3600, and -0.2485, so the system is stable; for all initial conditions, the powers converge to their equilibrium values.

Also, the SINR at each receiver i, given by S_i, converges to the same constant value $\alpha\gamma$, which is enough for a successful signal reception. This can be shown by observing that at equilibrium $p_i(t+1) = p_i(t) = \bar{p}_i$, and the power update equation gives

$$\bar{p}_i = \bar{p}_i(\alpha\gamma/S_i(t)).$$

After cancellation, we obtain the constant value for each SINR, $S_i = \alpha\gamma$.

When $\gamma = 5$, the eigenvalues of A are 1.0141, -0.6000, and -0.4141. This system is unstable because of the first eigenvalue, so this means there are initial conditions from which the powers diverge.

```
>> inv(v)*b
-0.0670
-0.0000
-0.0182
```

(b) The critical SINR threshold level is a function of dominant system eigenvalue. We will assume that matrix P is diagonizable and that its eigenvalues are ordered by their magnitude when forming Λ matrix. Using the property that scaling of any matrix scales its eigenvalues by the same constant, we can derive:

$$\begin{aligned} A &= \alpha\gamma P = \alpha\gamma T\Lambda T^{-1} \\ &= T\,\mathbf{diag}(\alpha\gamma\lambda_1, \ldots, \alpha\gamma\lambda_n)T^{-1} \end{aligned}$$

For a marginally stable system we need to have $|\alpha\gamma\lambda_1| \le 1$. Manipulating equation $\alpha\gamma_{\text{crit}}|\lambda_1| = 1$, we obtain the critical SINR threshold level,

$$\gamma_{\text{crit}} = \frac{1}{\alpha|\lambda_1|}.$$

10.5 *Determinant of matrix exponential.*

(a) Suppose the eigenvalues of $A \in \mathbf{R}^{n\times n}$ are $\lambda_1, \dots, \lambda_n$. Show that the eigenvalues of e^A are $e^{\lambda_1}, \dots, e^{\lambda_n}$. You can assume that A is diagonalizable, although it is true in the general case.

(b) Show that $\det e^A = e^{\mathbf{Tr}\, A}$. *Hint:* $\det X$ is the product of the eigenvalues of X, and $\mathbf{Tr}\, Y$ is the sum of the eigenvalues of Y.

Solution:

(a) Suppose that A is diagonalizable with eigenvalues $\lambda_1, \dots, \lambda_n$. Therefore, the invertible matrix T exists such that

$$A = T\mathbf{diag}(\lambda_1, \dots, \lambda_n)T^{-1}$$

and we get

$$e^A = Te^{\mathbf{diag}(\lambda_1,\dots,\lambda_n)}T^{-1} = T\mathbf{diag}(e_1^\lambda, \dots, e_n^\lambda)T^{-1}.$$

As a result

$$e^A T = T\mathbf{diag}(e^{\lambda_1}, \dots, e^{\lambda_n})$$

which shows that the eigenvalues of e^A are $e^{\lambda_1}, \dots, e^{\lambda_n}$. Note that this also shows that the eigenvectors of A (the columns of T) and e^A are the same.

(b) The determinant of a matrix is equal to the product of its eigenvalues and therefore

$$\det e^A = e^{\lambda_1}e^{\lambda_2}\cdots e^{\lambda_n} = e^{\lambda_1+\lambda_2+\cdots+\lambda_n}.$$

But $\lambda_1 + \lambda_2 + \cdots + \lambda_n$ is the sum of the eigenvalues of A which is equal to $\mathbf{Tr}\, A$. Thus

$$\det e^A = e^{\mathbf{Tr}\, A}.$$

10.6 *Linear system with a quadrant detector.* In this problem we consider the specific system

$$\dot{x} = Ax = \begin{bmatrix} 0.5 & 1.4 \\ -0.7 & 0.5 \end{bmatrix} x.$$

We have a detector or sensor that gives us the sign of each component of the state $x = [x_1\ x_2]^T$ each second:

$$y_1(t) = \mathrm{sgn}(x_1(t)), \quad y_2(t) = \mathrm{sgn}(x_2(t)), \quad t = 0, 1, 2, \dots$$

where the function $\mathrm{sgn} : \mathbf{R} \to \mathbf{R}$ is defined by

$$\mathrm{sgn}(a) = \begin{cases} 1 & a > 0 \\ 0 & a = 0 \\ -1 & a < 0 \end{cases}$$

There are several ways to think of these sensor measurements. You can think of $y(t) = [y_1(t)\ y_2(t)]^T$ as determining which quadrant the state is in at time t (thus the name *quadrant detector*). Or, you can think of $y(t)$ as a one-bit quantized measurement of the state at time t. Finally, the problem. You observe the sensor measurements

$$y(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad y(1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Based on these measurements, what values could $y(2)$ possibly take on? In terms of the quadrants, the problem can be stated as follows. $x(0)$ is in quadrant IV, and $x(1)$ is also in quadrant IV. The question is: which quadrant(s) can $x(2)$ possibly be in? You do not know the initial state $x(0)$. Of course, you must completely justify and explain your answer. *Solution:*
$x(2)$ is related to $x(0)$ and $x(1)$ through the relationships

$$x(2) = e^{2A}x(0), \quad x(2) = e^{A}x(1).$$

We are given that $x(0)$ and $x(1)$ are in the fourth quadrant. Therefore, from the first equation $x(2) = e^{2A}x(0)$ we conclude that $x(2)$ can only be in the image of the fourth quadrant under the linear transformation e^{2A}. Similarly, from the second equation $x(2) = e^{A}x(1)$, it follows that $x(2)$ can only be in the image of the fourth quadrant under the linear transformation e^{A}. Hence, the set of all possible values for $x(0)$ is equal to the intersection of the images of the fourth quadrant under e^{A} and e^{2A}. To find the image of the fourth quadrant under a linear mapping T we only need to find the transformation of the vectors e_1 and $-e_2$ under T. The image of the fourth quadrant under T is the (conic) region formed by the vectors Te_1 and $-Te_2$ (see Figure 1). Using Matlab:

```
>> A=[0.5 1.4;-0.7 0.5]
A =
0.5000    1.4000
-0.7000    0.5000
>> T1=expm(A)
T1 =
0.9047    1.9493
-0.9746    0.9047
>> T2=expm(2*A)
T2 =
-1.0813    3.5270
-1.7635   -1.0813
>> T1*[1 0;0 -1]
ans =
0.9047   -1.9493
-0.9746   -0.9047
>> T2*[1 0;0 -1]
ans =
```

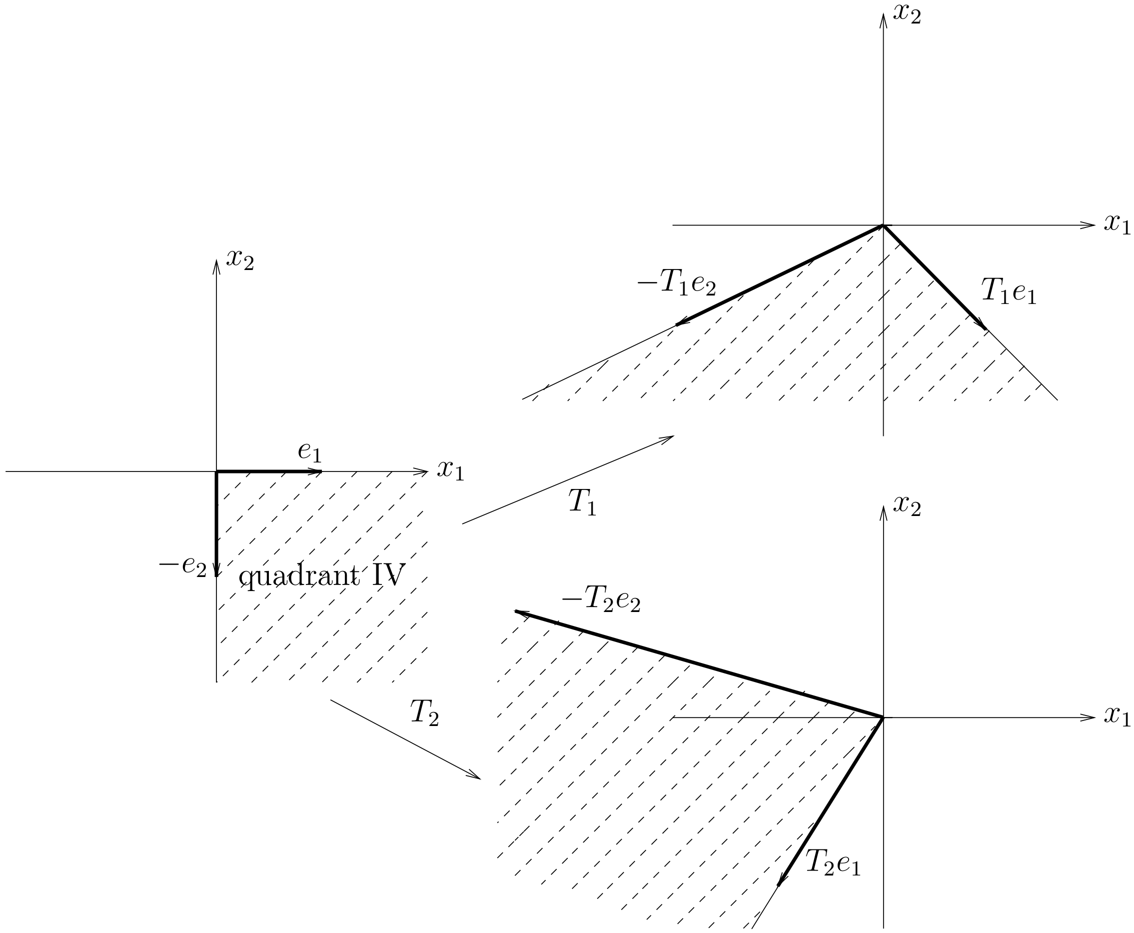

Figure 1: image of quadrant IV under $T_1 = e^A$ and $T_2 = e^{2A}$

```
-1.0813   -3.5270
-1.7635    1.0813
>>
```

The image of the fourth quadrant under $T_1 = e^A$ and $T_2 = e^{2A}$ is given in Figure 1. The intersection of the two regions is given in Figure 2. The valid region for $x(2)$ is the open set that lies between the two lines

$$\{ \ t(-1.95, -0.90) \mid t \in \mathbf{R}_+ \ \}, \quad \{ \ t(-1.08, -1.76) \mid t \in \mathbf{R}_+ \ \}.$$

Clearly, $x(2)$ is in the third quadrant and therefore $y_1(2) = y_2(2) = -1$.

10.8 *Some basic properties of eigenvalues.* Show that

(a) the eigenvalues of A and A^T are the same

(b) A is invertible if and only if A does not have a zero eigenvalue

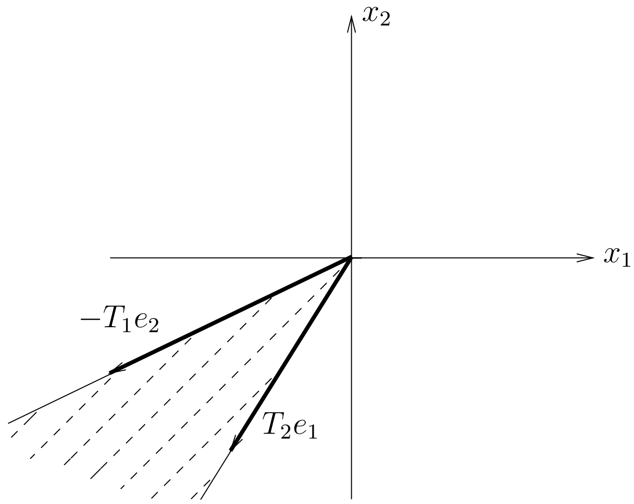

Figure 2: valid region for $x(2)$

(c) if the eigenvalues of A are $\lambda_1, \ldots, \lambda_n$ and A is invertible, then the eigenvalues of A^{-1} are $1/\lambda_1, \ldots, 1/\lambda_n$,

(d) the eigenvalues of A and $T^{-1}AT$ are the same.

Hint: you'll need to use the facts $\det AB = \det A \det B$ and $\det A^{-1} = 1/\det A$ (provided A is invertible). *Solution:*

(a) The eigenvalues of a matrix A are given by the roots of the polynomial $\det(sI - A)$. From determinant properties we know that $\det(sI - A) = \det(sI - A)^T = \det(sI - A^T)$. We conclude that the eigenvalues of A and A^T are the same.

(b) First we recall that A is invertible if and only if $\det(A) \neq 0$. But $\det(A) \neq 0 \iff \det(-A) \neq 0$.

   i. If 0 is an eigenvalue of A, then $\det(sI - A) = 0$ when $s = 0$. It follows that $\det(-A) = 0$ and thus $\det(A) = 0$, and A is not invertible. From this fact we conclude that if A is invertible, then 0 is not an eigenvalue of A.

   ii. If A is not invertible, then $\det(A) = \det(-A) = 0$. This means that, for $s = 0$, $\det(sI - A) = 0$, and we conclude that in this case 0 must be an eigenvalue of A. From this fact it follows that if 0 is not an eigenvalue of A, then A is invertible.

(c) From the results of the last item we see that 0 is not an eigenvalue of A. Now consider the eigenvalue/eigenvector pair (λ_i, x_i) of A. This pair satisfies $Ax_i = \lambda_i x_i$. Now, since A is invertible, λ_i is invertible. Multiplying both sides by A^{-1} and λ_i^{-1} we have $\lambda_i^{-1} x_i = A^{-1} x_i$, and from this we conclude that the eigenvalues of the inverse are the inverse of the eigenvalues.

(d) First we note that $\det(sI - A) = \det(I(sI - A)) = \det(T^{-1}T(sI - A))$. Now, from determinant properties, we have $\det(T^{-1}T(sI - A)) = \det(T^{-1}(sI - A)T)$. But this is also equal to $\det(sI - T^{-1}AT)$, and the conclusion is that the eigenvalues of A and $T^{-1}AT$ are the same.

10.14 *Some Matlab exercises.* Consider the continuous-time system $\dot{x} = Ax$ with

$$A = \begin{bmatrix} -0.1005 & 1.0939 & 2.0428 & 4.4599 \\ -1.0880 & -0.1444 & 5.9859 & -3.0481 \\ -2.0510 & -5.9709 & -0.1387 & 1.9229 \\ -4.4575 & 3.0753 & -1.8847 & -0.1164 \end{bmatrix}.$$

(a) What are the eigenvalues of A? Is the system stable? You can use the command `eig` in Matlab.

(b) Plot a few trajectories of $x(t)$, *i.e.*, $x_1(t)$, $x_2(t)$, $x_3(t)$ and $x_4(t)$, for a few initial conditions. To do this you can use the matrix exponential command in Matlab `expm` (*not* `exp` which gives the element-by-element exponential of a matrix), or more directly, the Matlab command `initial` (use `help initial` for details.) Verify that the qualitative behavior of the system is consistent with the eigenvalues you found in part (0a).

(c) Find the matrix Z such that $Zx(t)$ gives $x(t + 15)$. Thus, Z is the '15 seconds forward predictor matrix'.

(d) Find the matrix Y such that $Yx(t)$ gives $x(t - 20)$. Thus Y reconstructs what the state was 20 seconds ago.

(e) Briefly comment on the size of the elements of the matrices Y and Z.

(f) Find $x(0)$ such that $x(10) = [1\ 1\ 1\ 1]^T$.

Note: The A matrix is available on the class web page. *Solution:*

(a) The eigenvalues are $\{-0.1 \pm j5, -0.15 \pm j7\}$. Since all the eigenvalues have negative real part, we conclude that the system is stable.

(b) The plots are given in figure 3. We can see that they agree with the eigenvalues found in item (a). These eigenvalues indicate an oscillatory response that is lightly damped, since the imaginary parts are almost two orders of magnitude greater than the real part.

(c) This matrix is given by

$$Z = e^{15A} = \begin{bmatrix} 0.2032 & -0.0068 & -0.0552 & -0.0708 \\ 0.0340 & 0.0005 & -0.0535 & 0.1069 \\ 0.0173 & 0.1227 & 0.0270 & 0.0616 \\ 0.0815 & 0.0186 & 0.1151 & 0.1298 \end{bmatrix}$$

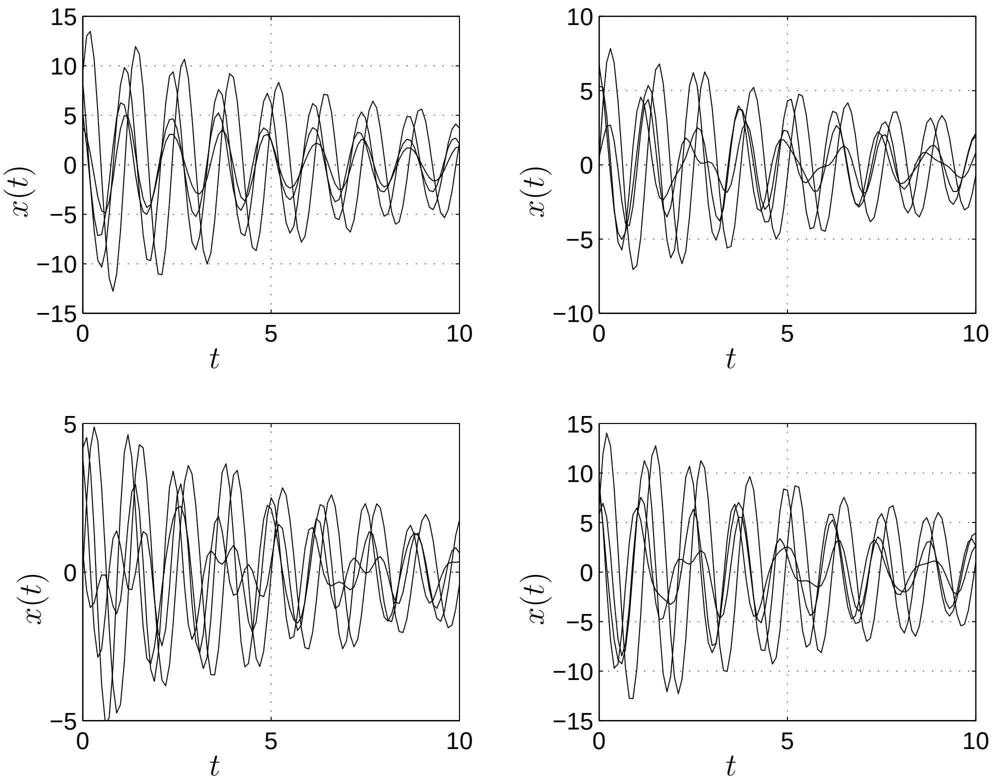

Figure 3: System trajectories for four random initial conditions

(d) This matrix is given by

$$Y = e^{-20A} = \begin{bmatrix} 6.2557 & 3.3818 & 1.7034 & 2.2064 \\ -2.1630 & -2.8107 & -14.2950 & 12.1503 \\ -3.3972 & 17.3931 & -1.6257 & -2.8004 \\ -1.7269 & -6.5353 & 10.7081 & 2.9736 \end{bmatrix}$$

(e) Since the system is stable, we know that all the components of e^{At} will go to zero as $t \to \infty$. We can notice this already in Z: the entries are small, all less than one. This reflects the fact that 15 seconds in the future, the coefficients of the state are, roughly speaking, smaller than the current coefficients. In contrast the entries of Y are all larger than one, which shows that 20 seconds ago the state was 'larger' than it is now.

(f) It suffices to compute

$$x(0) = e^{-10A} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.9961 \\ 1.0650 \\ 3.8114 \\ 1.7021 \end{bmatrix}$$

11.3 *Another formula for the matrix exponential.* You might remember that for any complex number $a \in \mathbf{C}$, $e^a = \lim_{k\to\infty}(1+a/k)^k$. You will establish the matrix analog: for any $A \in \mathbf{R}^{n\times n}$,

$$e^A = \lim_{k\to\infty}(I + A/k)^k.$$

To simplify things, you can assume A is diagonalizable. *Hint:* diagonalize. *Solution:* Assuming $A \in \mathbf{R}^{k\times k}$ is diagonalizable, there exists an invertible matrix $T \in \mathbf{R}^{n\times n}$ such that $A = T\mathbf{diag}(\lambda_1,\dots,\lambda_n)T^{-1}$ where $\lambda_1,\dots,\lambda_n$ are the eigenvalues of A. Therefore

$$\begin{aligned}(I+A/k)^k &= (TT^{-1} + T\mathbf{diag}(\lambda_1/k,\dots,\lambda_n/k)T^{-1})^k \\ &= \left(T(I+\mathbf{diag}(\lambda_1/k,\dots,\lambda_n/k))T^{-1}\right)^k \\ &= T(I+\mathbf{diag}(\lambda_1/k,\dots,\lambda_n/k))^kT^{-1}.\end{aligned}$$

But $(I+\mathbf{diag}(\lambda_1/k,\dots,\lambda_n/k))$ is diagonal and therefore its kth power is simply a diagonal matrix with diagonal entries equal to the kth power of the diagonal entries of $(I+\mathbf{diag}(\lambda_1/k,\dots,\lambda_n/k))$. Thus

$$(I+A/k)^k = T\mathbf{diag}((1+\lambda_1/k)^k,\dots,(1+\lambda_n/k)^k)T^{-1}$$

and taking the limit as $k\to\infty$ gives

$$\begin{aligned}\lim_{k\to\infty}(I+A/k)^k &= \lim_{k\to\infty} T\mathbf{diag}((1+\lambda_1/k)^k,\dots,(1+\lambda_n/k)^k)T^{-1} \\ &= T\mathbf{diag}(\lim_{k\to\infty}(1+\lambda_1/k)^k,\dots,\lim_{k\to\infty}(1+\lambda_n/k)^k)T^{-1} \\ &= T\mathbf{diag}(e^{\lambda_1},\dots,e^{\lambda_n})T^{-1} \\ &= e^A,\end{aligned}$$

and we are done.

11.6 *Rate of a Markov code.* Consider the Markov language described in exercise 2.13, with five symbols $1,2,3,4,5$, and the following symbol transition rules:

- 1 must be followed by 2 or 3
- 2 must be followed by 2 or 5
- 3 must be followed by 1
- 4 must be followed by 4 or 2 or 5
- 5 must be followed by 1 or 3

(a) *The rate of the code.* Let K_N denote the number of allowed sequences of length N. The number

$$R = \lim_{N\to\infty}\frac{\log_2 K_N}{N}$$

(if it exists) is called the *rate* of the code, in bits per symbol. Find the rate of this code. Compare it to the rate of the code which consists of all sequences from an alphabet of 5 symbols (*i.e.*, with no restrictions on which symbols can follow which symbols).

Solution:
Let A denote the transition matrix of the code. We'll consider the general case in the solution, and then specialize to the particular matrix A for our problem. We already know that the ij entry of the matrix A^{N-1} is exactly equal to the number of sequences of length N, that start with symbol j and end with symbol i. Therefore to get the total number of sequences we add these up:

$$K_N = \sum_{i,j} \left(A^{N-1}\right)_{ij} = \mathbf{1}^T A^{N-1} \mathbf{1},$$

where $\mathbf{1}$ is a vector all of whose coefficients are one. Similarly, we have

$$G_{N,i} = e_i^T A^{N-1} \mathbf{1}, \quad F_{N,i} = \mathbf{1}^T A^{N-1} e_i,$$

where e_i is the ith unit vector. In words: K_N is the sum of all entries of A^{N-1}, $G_{N,i}$ is the sum of the ith row of A^{N-1}, and $F_{N,i}$ is the sum of the ith column of A^{N-1}. Assuming the matrix A is diagonalizable (which is the case for our particular matrix A), we can express A in terms of its eigenvector-eigenvalue decomposition:

$$A = \sum_{i=1}^{n} \lambda_i v_i w_i^T,$$

where λ_i are the eigenvalues, v_i are the eigenvectors, and w_i are the left eigenvectors (normalized by $v_i^T w_j = \delta_{ij}$). Let λ_p be the eigenvalue of largest magnitude (*i.e.*, the dominant eigenvalue of the corresponding discrete-time system); in other words, we have $|\lambda_p| > |\lambda_k|$ for $k \neq p$. Here we assume there is only one eigenvalue of largest magnitude, which is true for our particular A, and in fact, for any A that comes from a Markov code. Then we have

$$A^{N-1} = \sum_{i=1}^{n} \lambda_i^{N-1} v_i w_i^T.$$

Now as N grows, the term $\lambda_p v_p w_p^T$ becomes larger and larger, relative to the other terms (which is why we call λ_p the dominant eigenvalue). In other words we have

$$A^{N-1} \approx \lambda_p^{N-1} v_p w_p^T$$

for large N. Therefore we have, for large N,

$$K_N \approx \mathbf{1}^T \left(\lambda_p^{N-1} v_p w_p^T\right) \mathbf{1} = \lambda_p^{N-1} (\mathbf{1}^T v_p)(w_p^T \mathbf{1}),$$

and, similarly,

$$G_{N,i} \approx \lambda_p^{N-1} (e_i^T v_p)(w_p^T \mathbf{1}), \quad F_{N,i} \approx \lambda_p^{N-1} (\mathbf{1}^T v_p)(w_p^T e_i).$$

Now we can answer the specific questions.

(a) *The rate of the code.* The rate of the code is

$$
\begin{aligned}
R &= \lim_{N\to\infty} \frac{\log_2 K_N}{N} \\
&= \lim_{N\to\infty} \frac{\log_2 \left(\lambda_p^{N-1}(\mathbf{1}^T v_p)(w_p^T \mathbf{1})\right)}{N} \\
&= \log_2 \lambda_p + \lim_{N\to\infty} \frac{\log_2 \left((\mathbf{1}^T v_p)(w_p^T \mathbf{1})\right)}{N} \\
&= \log_2 \lambda_p.
\end{aligned}
$$

In other words: the rate of the code is the logarithm (base 2) of the dominant eigenvalue, which is a nice, simple result. For our code, the rate is 0.81137 bits/symbol. If you think about it, this result makes perfect sense. The rate of the code is a measure of the exponential rate of increase of the number of sequences, as length increases. The dominant eigenvalue of a discrete-time system, roughly speaking, gives the asymptotic exponential rate of increase in size of the state with time. For a code with K symbols and no restrictions, we have exactly K^N sequences of length N. Hence the rate is $\log_2 K$ bits per symbol; for $K = 5$ this gives $R = 2.32193$ bits per symbol. This makes good sense!