Lecture 9
Autonomous linear dynamical systems

- autonomous linear dynamical systems
- examples
- higher order systems
- linearization near equilibrium point
- linearization along trajectory

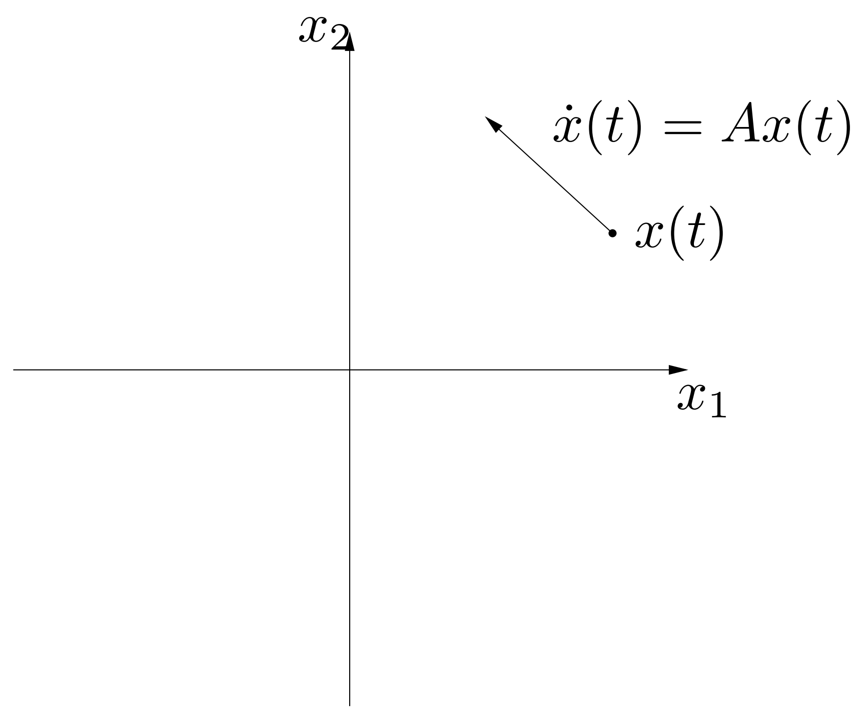
Autonomous linear dynamical systems

continuous-time autonomous LDS has form

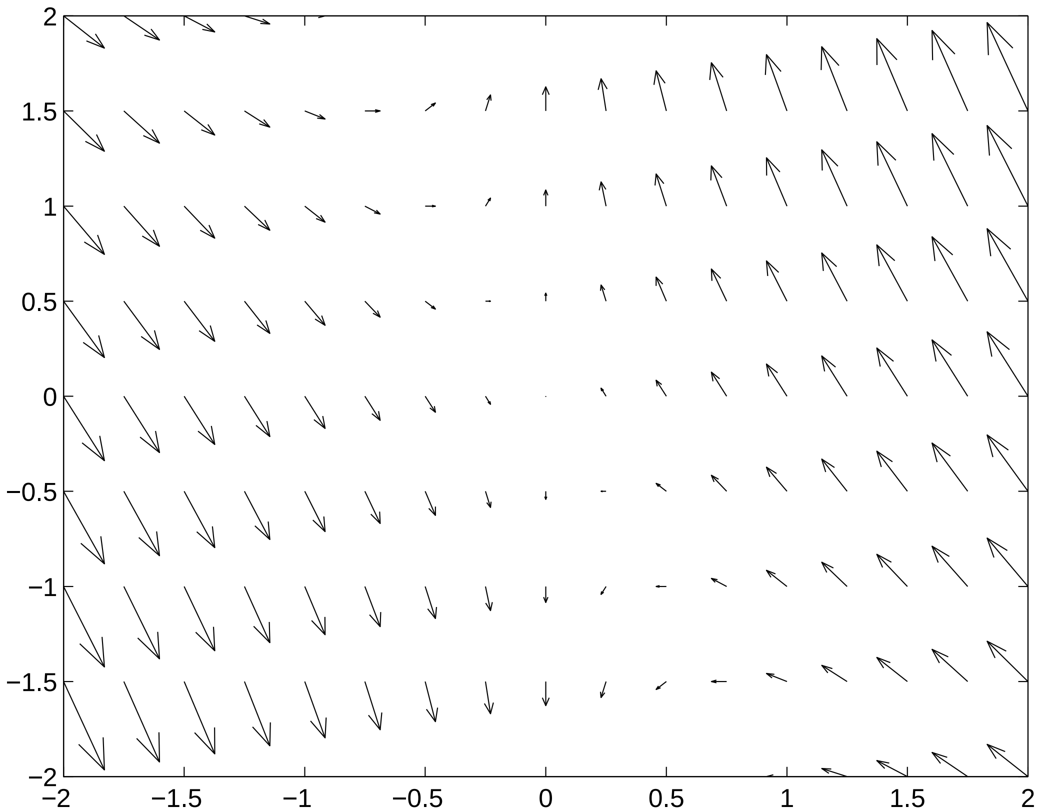
$$\dot{x} = Ax$$

- $x(t) \in \mathbf{R}^n$ is called the state
- n is the *state dimension* or (informally) the *number of states*
- A is the *dynamics matrix*
  (system is *time-invariant* if A doesn't depend on t)

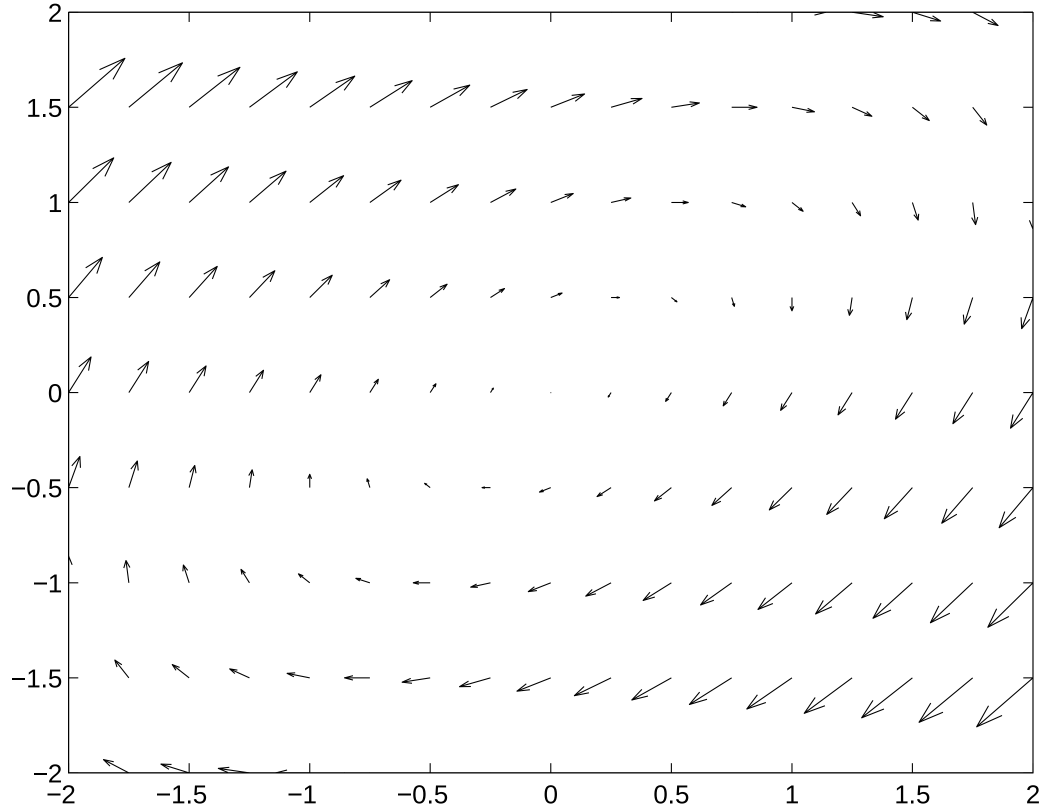
picture (*phase plane*):



example 1: $\dot{x} = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} x$

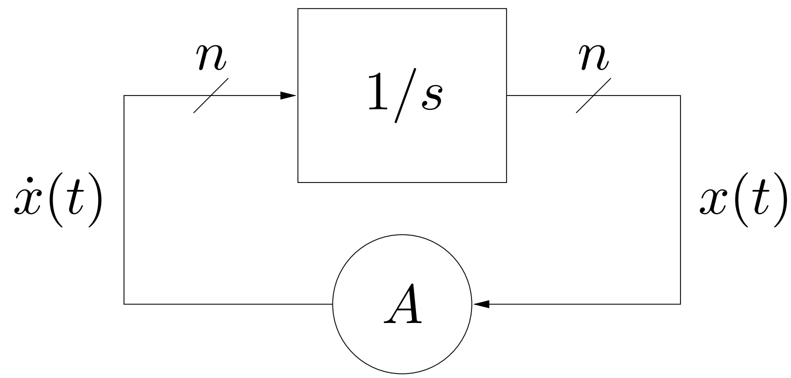


example 2: $\dot{x} = \begin{bmatrix} -0.5 & 1 \\ -1 & 0.5 \end{bmatrix} x$



Block diagram

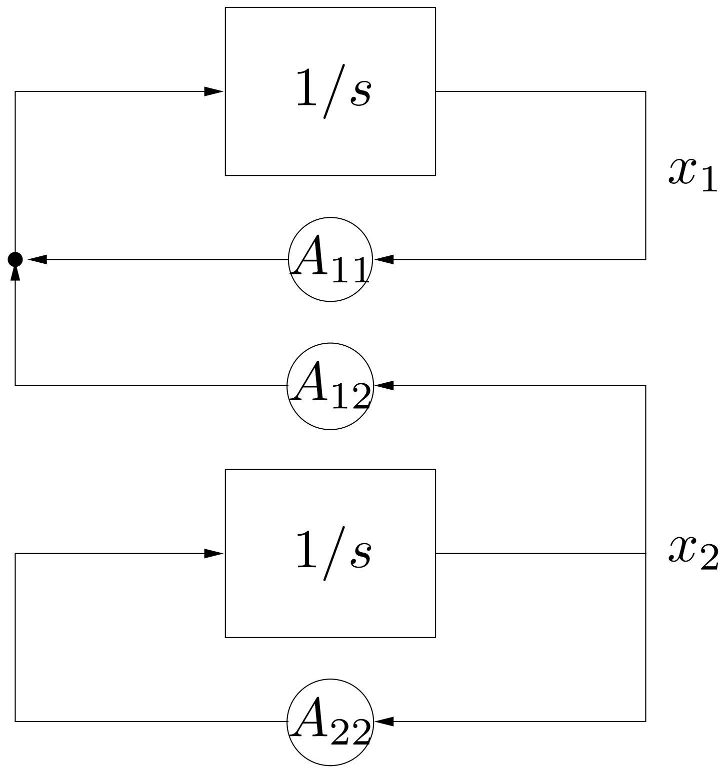
block diagram representation of $\dot{x} = Ax$:



- $1/s$ block represents n parallel scalar integrators
- coupling comes from dynamics matrix A

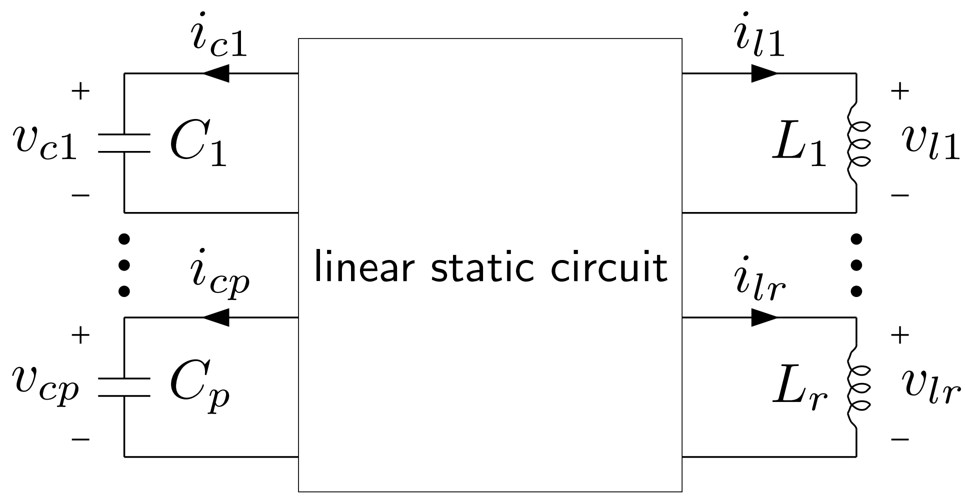
useful when A has structure, $e.g.$, block upper triangular:

$$\dot{x} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} x$$



here x_1 doesn't affect x_2 at all

Linear circuit



circuit equations are

$$C\frac{dv_c}{dt} = i_c, \qquad L\frac{di_l}{dt} = v_l, \qquad \begin{bmatrix} i_c \\ v_l \end{bmatrix} = F \begin{bmatrix} v_c \\ i_l \end{bmatrix}$$

$$C = \mathbf{diag}(C_1, \ldots, C_p), \qquad L = \mathbf{diag}(L_1, \ldots, L_r)$$

with state $x = \begin{bmatrix} v_c \\ i_l \end{bmatrix}$, we have

$$\dot{x} = \begin{bmatrix} C^{-1} & 0 \\ 0 & L^{-1} \end{bmatrix} F x$$

Chemical reactions

- reaction involving n chemicals; x_i is concentration of chemical i
- linear model of reaction kinetics

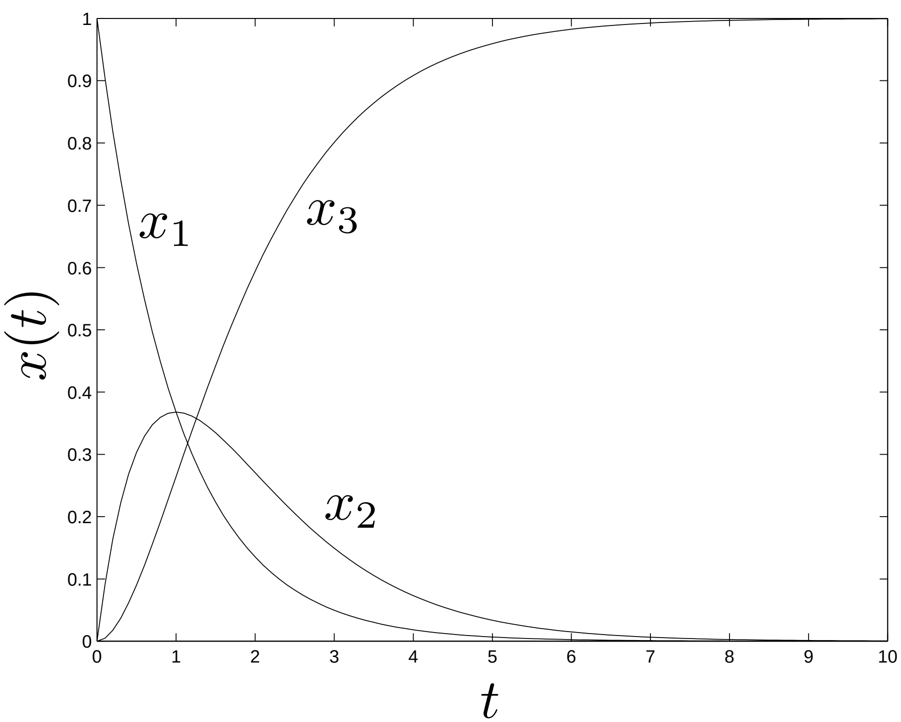
$$\frac{dx_i}{dt} = a_{i1}x_1 + \cdots + a_{in}x_n$$

- good model for some reactions; A is usually sparse

Example: series reaction $A \xrightarrow{k_1} B \xrightarrow{k_2} C$ with linear dynamics

$$\dot{x} = \begin{bmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & 0 \end{bmatrix} x$$

plot for $k_1 = k_2 = 1$, initial $x(0) = (1, 0, 0)$



Finite-state discrete-time Markov chain

$z(t) \in \{1, \ldots, n\}$ is a random sequence with

$$\mathbf{Prob}(\ z(t+1) = i \mid z(t) = j\ ) = P_{ij}$$

where $P \in \mathbf{R}^{n \times n}$ is the matrix of *transition probabilities*

can represent probability distribution of $z(t)$ as n-vector

$$p(t) = \begin{bmatrix} \mathbf{Prob}(z(t) = 1) \\ \vdots \\ \mathbf{Prob}(z(t) = n) \end{bmatrix}$$

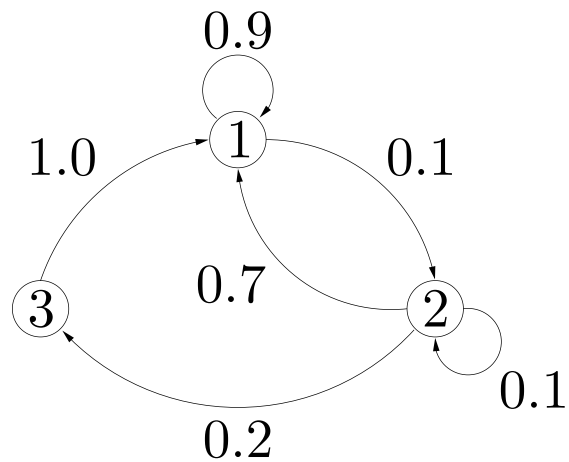
(so, *e.g.*, $\mathbf{Prob}(z(t) = 1, 2, \text{ or } 3) = [1\ 1\ 1\ 0 \cdots 0]p(t)$)

then we have $p(t+1) = Pp(t)$

P is often sparse; Markov chain is depicted graphically

- nodes are states
- edges show transition probabilities

example:



- state 1 is 'system OK'
- state 2 is 'system down'
- state 3 is 'system being repaired'

$$p(t+1) = \begin{bmatrix} 0.9 & 0.7 & 1.0 \\ 0.1 & 0.1 & 0 \\ 0 & 0.2 & 0 \end{bmatrix} p(t)$$

Numerical integration of continuous system

compute approximate solution of $\dot{x} = Ax$, $x(0) = x_0$

suppose h is small time step (x doesn't change much in h seconds)

simple ('forward Euler') approximation:

$$x(t + h) \approx x(t) + h\dot{x}(t) = (I + hA)x(t)$$

by carrying out this recursion (discrete-time LDS), starting at $x(0) = x_0$, we get approximation

$$x(kh) \approx (I + hA)^k x(0)$$

(forward Euler is never used in practice)

Higher order linear dynamical systems

$$x^{(k)} = A_{k-1}x^{(k-1)} + \cdots + A_1 x^{(1)} + A_0 x, \quad x(t) \in \mathbf{R}^n$$

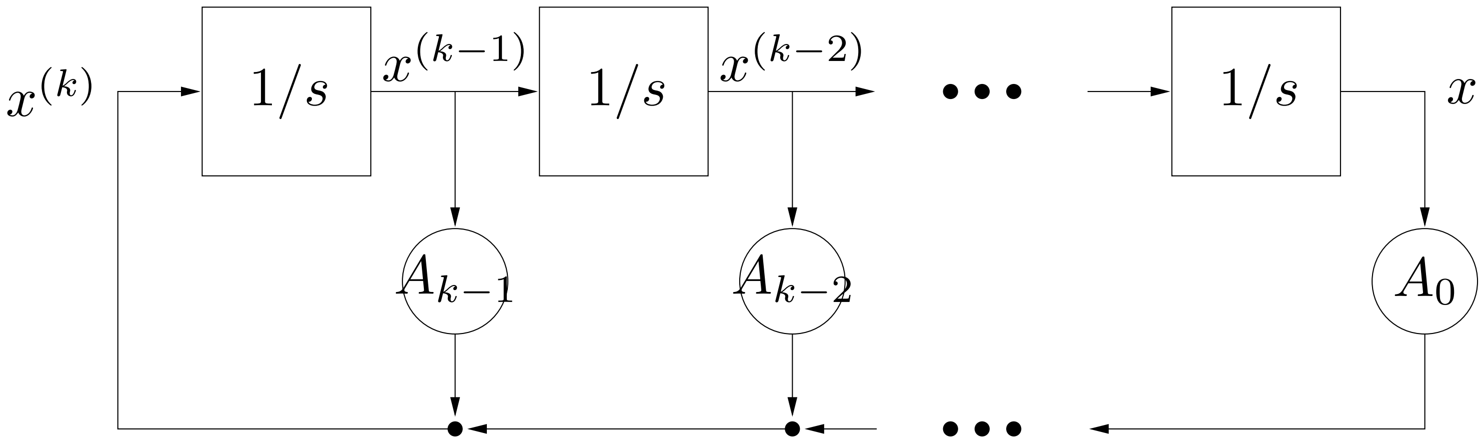
where $x^{(m)}$ denotes mth derivative

define new variable $z = \begin{bmatrix} x \\ x^{(1)} \\ \vdots \\ x^{(k-1)} \end{bmatrix} \in \mathbf{R}^{nk}$, so

$$\dot{z} = \begin{bmatrix} x^{(1)} \\ \vdots \\ x^{(k)} \end{bmatrix} = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & I \\ A_0 & A_1 & A_2 & \cdots & A_{k-1} \end{bmatrix} z$$

a (first order) LDS (with bigger state)

block diagram:



Mechanical systems

mechanical system with k degrees of freedom undergoing small motions:

$$M\ddot{q} + D\dot{q} + Kq = 0$$

- $q(t) \in \mathbf{R}^k$ is the vector of generalized displacements
- M is the *mass matrix*
- K is the *stiffness matrix*
- D is the *damping matrix*

with state $x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$ we have

$$\dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} x$$

Linearization near equilibrium point

nonlinear, time-invariant differential equation (DE):

$$\dot{x} = f(x)$$

where $f : \mathbf{R}^n \to \mathbf{R}^n$

suppose x_e is an *equilibrium point*, $i.e.$, $f(x_e) = 0$

(so $x(t) = x_e$ satisfies DE)

now suppose $x(t)$ is near x_e, so

$$\dot{x}(t) = f(x(t)) \approx f(x_e) + Df(x_e)(x(t) - x_e)$$

with $\delta x(t) = x(t) - x_e$, rewrite as

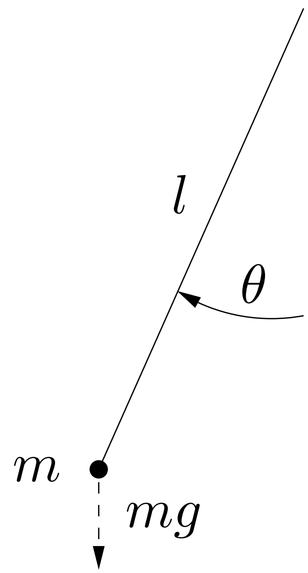
$$\dot{\delta x}(t) \approx Df(x_e)\delta x(t)$$

replacing $\approx$ with $=$ yields *linearized approximation* of DE near x_e

we *hope* solution of $\dot{\delta x} = Df(x_e)\delta x$ is a good approximation of $x - x_e$

(more later)

example: pendulum



2nd order nonlinear DE $ml^2\ddot{\theta} = -lmg\sin\theta$

rewrite as first order DE with state $x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$:

$$\dot{x} = \begin{bmatrix} x_2 \\ -(g/l)\sin x_1 \end{bmatrix}$$

equilibrium point (pendulum down): $x = 0$

linearized system near $x_e = 0$:

$$\dot{\delta x} = \begin{bmatrix} 0 & 1 \\ -g/l & 0 \end{bmatrix} \delta x$$

Does linearization 'work'?

the linearized system usually, but not always, gives a good idea of the system behavior near x_e

example 1: $\dot{x} = -x^3$ near $x_e = 0$

for $x(0) > 0$ solutions have form $x(t) = \left(x(0)^{-2} + 2t\right)^{-1/2}$

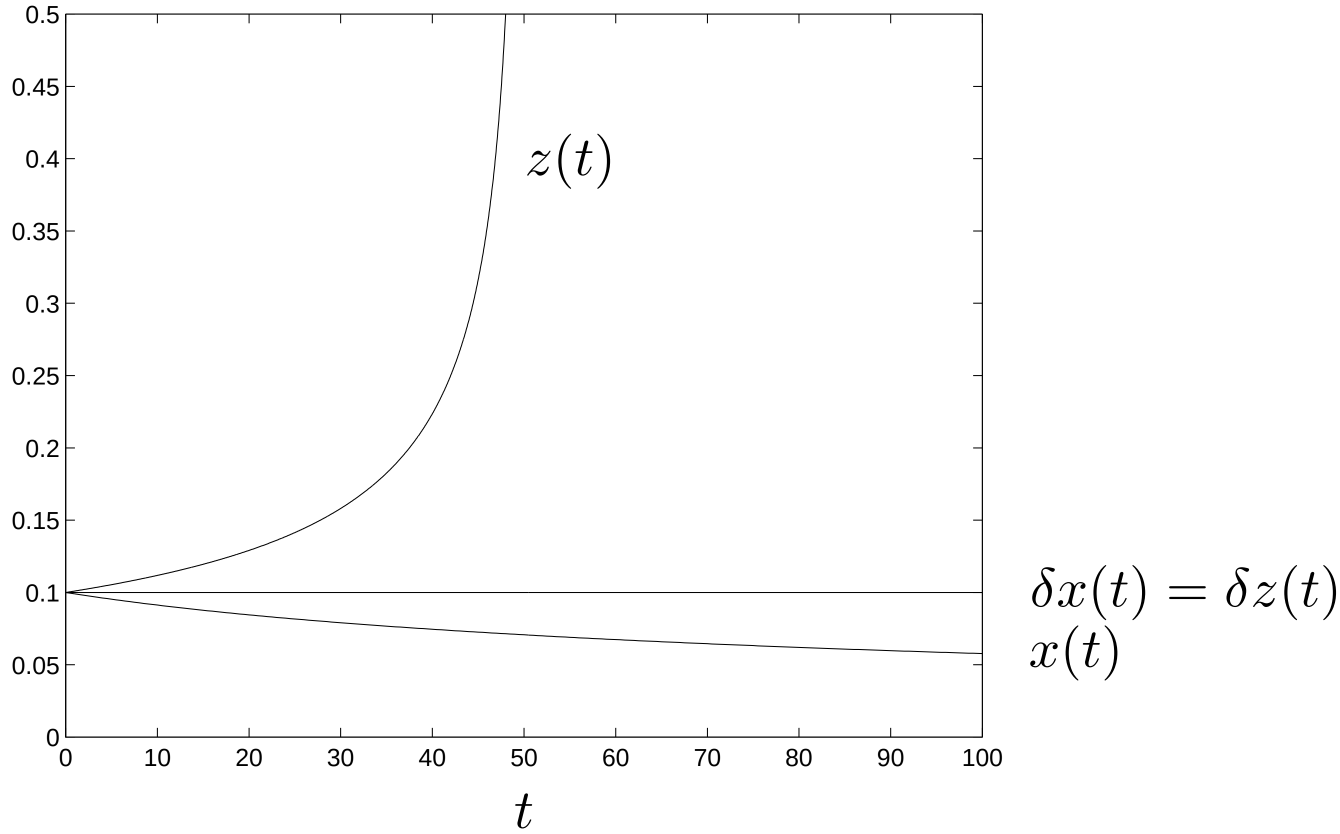
linearized system is $\dot{\delta x} = 0$; solutions are constant

example 2: $\dot{z} = z^3$ near $z_e = 0$

for $z(0) > 0$ solutions have form $z(t) = \left(z(0)^{-2} - 2t\right)^{-1/2}$

(finite escape time at $t = z(0)^{-2}/2$)

linearized system is $\dot{\delta z} = 0$; solutions are constant



- systems with very different behavior have same linearized system

- linearized systems do not predict qualitative behavior of either system

Linearization along trajectory

- suppose $x_{\rm traj} : \mathbf{R}_+ \to \mathbf{R}^n$ satisfies $\dot{x}_{\rm traj}(t) = f(x_{\rm traj}(t), t)$
- suppose $x(t)$ is another trajectory, *i.e.*, $\dot{x}(t) = f(x(t), t)$, and is near $x_{\rm traj}(t)$
- then

$$\frac{d}{dt}(x - x_{\rm traj}) = f(x, t) - f(x_{\rm traj}, t) \approx D_x f(x_{\rm traj}, t)(x - x_{\rm traj})$$

- (time-varying) LDS

$$\dot{\delta x} = D_x f(x_{\rm traj}, t)\delta x$$

is called *linearized* or *variational system* along trajectory $x_{\rm traj}$

example: linearized oscillator

suppose $x_{\rm traj}(t)$ is T-periodic solution of nonlinear DE:

$$\dot{x}_{\rm traj}(t) = f(x_{\rm traj}(t)), \qquad x_{\rm traj}(t+T) = x_{\rm traj}(t)$$

linearized system is

$$\dot{\delta x} = A(t)\delta x$$

where $A(t) = Df(x_{\rm traj}(t))$

$A(t)$ is T-periodic, so linearized system is called T*-periodic linear system*.

used to study:

- startup dynamics of clock and oscillator circuits
- effects of power supply and other disturbances on clock behavior