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EE263 (Introduction to Linear Dynamical Systems)
Homework 2
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1 Channel equalizer with disturbance rejection

A communication channel is described by the equation

$$y = Ax + v,$$

where

- $x \in \mathbb{R}^n$ is the (unknown) transmitted signal,
- $y \in \mathbb{R}^m$ is the (known) received signal,
- $v \in \mathbb{R}^m$ is the (unknown) disturbance signal, and
- $A \in \mathbb{R}^{m \times n}$ is a (known) matrix describing the channel.

Although the disturbance v is unknown, we do know that v is a linear combination of some (known) disturbance patterns:

$$d_1, \dots, d_k \in \mathbb{R}^m.$$

We consider linear equalizers for the channel, which have the form $\hat{x} = By$, where $B \in \mathbb{R}^{n \times m}$ is called the equalizer (more precisely, the B_{ij} are the equalizer coefficients). We say that the equalizer B rejects the disturbance pattern d_i if $\hat{x} = x$ for every $x \in \mathbb{R}^n$ when $v = d_i$. Note that if the equalizer rejects a set of disturbance patterns, then it can reconstruct the transmitted signal exactly when the disturbance v is a linear combination of the disturbance patterns that are rejected.

The file `equalizer_disturbance_rejection_data.m` defines the following variables:

- `n`, the size of the transmitted signal,
- `m`, the size of the received signal,
- `k`, the number of disturbance patterns,
- `A`, the $m \times n$ matrix describing the channel, and
- `D`, an $m \times k$ matrix whose columns are the disturbance patterns.

Find an equalizer B that rejects as many disturbance patterns as possible. Give the specific set of disturbance patterns that your equalizer rejects. (You only need to find one equalizer that rejects a set of disturbances of maximum size.) Explain how you know that there is no equalizer that rejects more disturbance patterns than yours does. Include `MATLAB` code verifying all of your claims.

Hint. The function `nchoosek` may be useful.

2 Identifying a point on the unit sphere from spherical distances

In this problem, we consider the unit sphere in $\mathbb{R}^n$: $S^n = \{x \in \mathbb{R}^n : \|x\| = 1\}$. We define the spherical distance between two vectors on the unit sphere as the distance between them, measured along the sphere: that is, the angle between the vectors, measured in radians. Thus, for $x, y \in S^n$, the spherical distance between x and y is

$$\mathrm{sphdist}(x, y) = \angle(x, y),$$

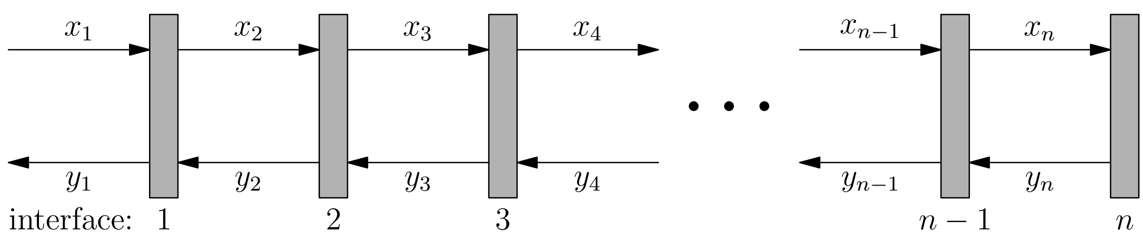
where we take $\angle(x, y)$ to be between 0 and π. (Note that the maximum distance between two points on the unit sphere is π, which occurs only when the points x and y are antipodal: that is, $y = -x$.) Now suppose $p_1, \dots, p_k \in S^n$ are the (known) positions of some beacons, and $x \in S^n$ is an unknown location. We have exact measurements of the spherical distances between the unknown point x and each of the beacons:

$$\rho_i = \mathrm{sphdist}(x, p_i), \qquad i = 1, \dots, k.$$

We would like to exactly determine x from this information. Find conditions on $p_1, \dots, p_k$ such that we can use the measurements $\rho_1, \dots, \rho_k$ to unambiguously determine x, for any $x \in S^n$.

3 Analysis of a layered medium

In this problem we consider a model for (incoherent) transmission in a layered medium. The medium is modeled as a set of n layers, separated by n dividing interfaces, shown as shaded rectangles in the figure below.



For $i = 1, \dots, n$, let $x_i \in \mathbb{R}$ denote the amplitude of the right-traveling wave in layer i, and let $y_i \in \mathbb{R}$ denote the amplitude of the left-traveling wave in layer i. The right-traveling and left-traveling waves in the first layer are called the incident and reflected waves, respectively. The scattering coefficient for the medium is defined to be the ratio $S = y_1/x_1$ (assuming $x_1 \neq 0$).

The right- and left-traveling waves on each side of an interface are related by the transmission and reflection equations:

$$\begin{aligned} x_{i+1} &= t_i x_i + r_i y_{i+1}, \\ y_i &= r_i x_i + t_i y_{i+1}, \end{aligned} \qquad i = 1, \dots, n-1,$$

where $t_i \in [0, 1]$ is the transmission coefficient of the ith interface, and $r_i \in [0, 1]$ is the reflection coefficient of the ith interface. We assume that the last interface is totally reflective, so that $y_n = x_n$.

(a) Explain how to find the scattering coefficient S given the transmission and reflection coefficients for the first $n-1$ layers.

(b) Carry out your method on the instance of the problem with

$$n = 20, \qquad t_1 = \cdots = t_{n-1} = 0.96, \qquad \text{and} \qquad r_1 = \cdots = r_{n-1} = 0.02.$$

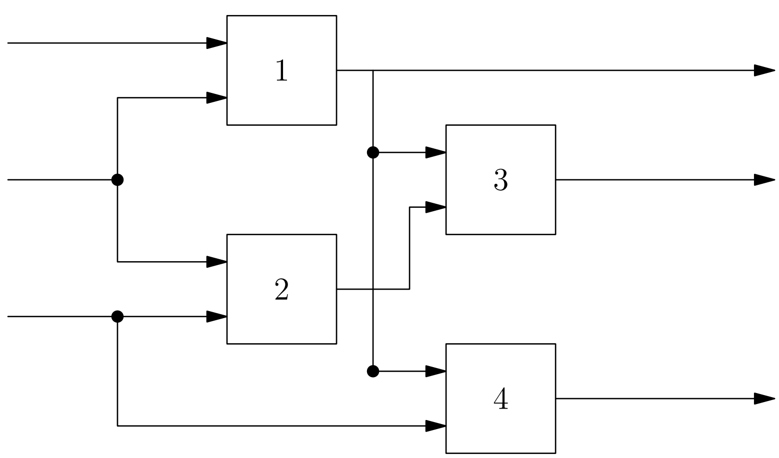
Plot the amplitudes of the left- and right-traveling versus i, and report the value of S that you compute.

(c) *Fault location.* Suppose that there is a fault at interface k, so that $t_k = 0.02$ and $r_k = 0.96$ (the other interfaces still have $t_i = 0.96$ and $r_i = 0.02$). You are told that the scattering coefficient of the medium with the fault is $S = S^{\text{fault}}$. Explain how to determine which interface contains the fault. Carry out your method with $S^{\text{fault}} = 0.70$, and report the interface k that contains the fault. You may assume that the last layer does not contain the fault.

4 Digital circuit gate sizing

A digital circuit consists of n logic gates connected by wires. Each gate has one or more inputs, and one output. The output of a gate is connected via wires to the inputs of other gates, or to external circuitry. If the output of gate i is connected to an input of gate j, then we say that gate i drives gate j, or that gate j is in the fan-out of gate i. We describe the topology of the circuit by the fan-out list for each gate; $\text{FO}(i) \subseteq \{1, \ldots, n\}$ denotes the fan-out list of gate i. Note that it is possible that $\text{FO}(i) = \emptyset$, which indicates that the output of gate i is not connected to any other gates (presumably the output of gate i is connected to some external circuitry). It is common to order the gates in such a way that each gate only drives gates with higher indices – that is, we have that $\text{FO}(i) \subseteq \{i+1, \ldots, n\}$. We will assume that the gates are ordered in this way. (Note that this implies that the gate interconnections form a directed acyclic graph.)

To illustrate the notation, consider the following simple digital circuit with $n = 4$ gates, each with 2 inputs.



The fan-out lists for this circuit are

$$\text{FO}(1) = \{3, 4\}, \qquad \text{FO}(2) = \{3\}, \qquad \text{FO}(3) = \emptyset, \qquad \text{and} \qquad \text{FO}(4) = \emptyset.$$

The three input signals arriving from the left are called primary inputs, and the three output signals emerging from the right are called primary outputs.

Gate i has a scale factor $x_i \in \mathbb{R}$ (x_i is also sometimes called the size of gate i). These scale factors are the design variables in the gate-sizing problem. The gate sizes must satisfy $1 \le x_i \le x^{\max}$, where $x^{\max}$ is a given maximum allowed gate scale. The total area of the circuit is

$$A = \sum_{i=1}^{n} a_i x_i,$$

where the a_i are known positive constants. The input capacitance C_i^{in} of gate i is

$$C_i^{\text{in}} = \alpha_i x_i,$$

where the α_i are known positive constants. The delay of gate i is

$$d_i = \beta_i + \frac{\gamma_i C_i^{\text{load}}}{x_i},$$

where β_i and γ_i are known positive constants, and C_i^{load} is the load capacitance of gate i. Note that the gate delay d_i is always greater than β_i, which we can think of as the minimum possible delay of gate i, achieved in the limit as the gate scale factor becomes large. The load capacitance of gate i is

$$C_i^{\text{load}} = C_i^{\text{ext}} + \sum_{j \in \text{FO}(i)} C_j^{\text{in}},$$

where C_i^{ext} is a known positive constant that accounts for the capacitance of the interconnect wires and external circuitry.

We want to design the circuit so that every gate has delay T for some value of $T > 0$. For a given value of T, there may or may not be a feasible design choice (that is, a choice of $x_1, \ldots, x_n$ such that $1 \le x_i \le x^{\max}$ and $d_i = T$ for $i = 1, \ldots, n$). For example, we must have $T > \max_i \beta_i$.

(a) Explain how to find a design $x^\star \in \mathbb{R}^n$ that minimizes T subject to the area constraint $A \le A^{\max}$, where $A^{\max}$ is a given constant. You may assume that the fan-out lists, and all constants are known; your job is just to find the scale factors x_i. Be sure to explain how you can determine if the design problem is feasible.

(b) Carry out your method on the data given in `gate_sizing_data.m`. The fan-out lists are described by an $n \times n$ matrix `F`, where $F(i, j) = 1$ if $j \in \text{FO}(i)$, and $F(i, j) = 0$ otherwise.

5 Interpolation with rational functions.

Consider a function $f : \mathbb{R} \to \mathbb{R}$ of the form

$$f(x) = \frac{a_0 + a_1 x + \cdots + a_m x^m}{1 + b_1 x + \cdots + b_m x^m},$$

where $a_0, \ldots, a_m$ and $b_1, \ldots, b_m$ are parameters, with either $a_m \neq 0$ or $b_m \neq 0$. Such a function is called a rational function of degree m. We are given data points $x_1, \ldots, x_N \in \mathbb{R}$, and $y_1, \ldots, y_N \in \mathbb{R}$, where $y_i = f(x_i)$.

(a) Explain how to find a rational function of smallest degree that is consistent with the data: that is, explain how to find the smallest value of m, and corresponding values of $a_0, \ldots, a_m$, and $b_1, \ldots, b_m$ such that $f(x_i) = y_i$ for $i = 1, \ldots, N$.

(b) Carry out your method on the data in `rational_interpolation_data.m`. Report your value of m, and the corresponding coefficients $a_0, \ldots, a_m$, and $b_1, \ldots, b_m$. Plot the data and the rational function $f(x)$. Verify that $y_i = f(x_i)$ for $i = 1, \ldots, N$ (possibly with small numerical errors).

6 Orthogonal matrices

In this problem, we prove some properties of orthogonal matrices.

(a) Show that if U and V are orthogonal matrices, then UV is also an orthogonal matrix.

(b) Show that if U is an orthogonal matrix, then U^{-1} is also an orthogonal matrix.

(c) Suppose $U \in \mathbb{R}^{2\times 2}$ is an orthogonal matrix. Prove that U is either a rotation or a reflection. How can you tell whether a 2×2 orthogonal matrix is a rotation or a reflection?