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EE263 (Introduction to Linear Dynamical Systems)
Homework 5
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## 1 Invariance of the unit square

Consider the linear dynamical system $\dot{x} = Ax$, where $A \in \mathbb{R}^{2\times 2}$. The unit square in $\mathbb{R}^2$ is

$$S = \{x \in \mathbb{R}^2 : -1 \le x_1, x_2 \le +1\}.$$

(a) Find the exact conditions on $A$ under which the unit square $S$ is invariant under $\dot{x} = Ax$. Give the conditions as explicitly as possible.

(b) Consider the following statement: if the eigenvalues of $A$ are real and negative, then $S$ is invariant under $\dot{x} = Ax$. Either prove that this statement is true, or give a counterexample showing that it is false.

## 2 The harmonic oscillator

Consider the linear dynamical system

$$\dot{x} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} x.$$

This system is called a harmonic oscillator.

(a) Find the eigenvalues, resolvent and state-transition matrix for the harmonic oscillator. Express $x(t)$ in terms of $x(0)$.

(b) Sketch the vector field associated with the harmonic oscillator for $\omega = 1$.

(c) The state trajectories of the harmonic oscillators are circular orbits: that is, $\|x(t)\|$ is constant for any trajectory. Verify this fact using your expression for $x(t)$.

(d) Verify that the velocity of the system is orthogonal to the position. Do this using the differential equation directly, without using your expression for $x(t)$.

## 3 Analyzing a continuous-time linear dynamical system

Consider the continuous-time linear dynamical system $\dot{x}(t) = Ax(t)$, where

$$A = \begin{bmatrix} -0.1005 & 1.0939 & 2.0428 & 4.4599 \\ -1.0880 & -0.1444 & 5.9859 & -3.0481 \\ -2.0510 & -5.9709 & -0.1387 & 1.9229 \\ -4.4575 & 3.0753 & -1.8847 & -0.1164 \end{bmatrix}.$$

(a) What are the eigenvalues of $A$? Is the system stable?

(b) Plot a few trajectories of the system. Is the qualitative behavior of the solutions consistent with the eigenvalues of the system?

(c) Find the matrix $Z$ such that $x(t+15) = Zx(t)$.

(d) Find the matrix $Y$ such that $x(t-20) = Yx(t)$.

(e) Comment on the relative magnitudes of the entries of $Y$ and $Z$.

(f) Find an $x(0)$ such that $x(10) = \mathbf{1}$.

## 4 Optimal preheating of an espresso cup

At time $t = 0$, boiling water (that is, water with temperature 100 °C) is poured into an espresso cup; after $P$ seconds (the preheating time), the water is poured out, and espresso, with initial temperature 95 °C, is poured in. (You can assume that these pouring operations occur instantaneously.) The espresso is then consumed exactly 15 s later (yes, instantaneously). The problem is to choose the preheating time $P$ in order to maximize the temperature of the espresso when it is consumed.

We can model this situation as follows. We take the temperature of the liquid in the cup (either water or espresso) as one state; we model the cup using an $n$-state finite-element model. The vector $x(t) \in \mathbb{R}^{n+1}$ gives the temperature distribution at time $t$: $x_1(t)$ is the temperature of the liquid (water or espresso) at time $t$, and $x_2(t), \ldots, x_{n+1}(t)$ are the temperatures of the elements in our model of the cup. All temperatures are in °C, and $t$ is in seconds. The dynamics are

$$\frac{d}{dt}(x(t) - 20\mathbf{1}) = A(x(t) - 20\mathbf{1}),$$

where $A \in \mathbb{R}^{(n+1)\times(n+1)}$ is given. (The vector $20\mathbf{1}$ represents the ambient temperature.) The initial temperature distribution is

$$x(0) = \begin{bmatrix} 100 \\ 20 \\ \vdots \\ 20 \end{bmatrix}.$$

At $t = P$, the temperature of the liquid changes instantaneously from whatever value it had just before time $t = P$ to the temperature 95 °C of the fresh espresso; the other states do not change instantaneously at time $t = P$. Note that the dynamics of the system are the same before and after preheating (because we assume that water and espresso behave in the same way thermally.)

The file `espresso_heating_data.m` defines the following variables.

- `A`, the dynamics matrix
- `n`, the number of states in the finite-element model of the cup
- `Ta`, the ambient temperature, 20 °C
- `Te`, the temperature of the espresso, 95 °C
- `Tw`, the temperature of the water used for preheating, 100 °C

Explain how to find the preheating time $P$ that maximizes the temperature of the espresso when it is consumed. Report the optimal value of $P$, and the corresponding temperature of the espresso when it is consumed. Report both quantities to an accuracy of one decimal place.