Alex Lemon
EE263 (Introduction to Linear Dynamical Systems)
Homework 6
August 1, 2013

1 The rate of a Markov code.

Consider a Markov language with $n = 5$ symbols, and the following transition rules:

- 1 must be followed by 2 or 3,
- 2 must be followed by 2 or 5,
- 3 must be followed by 1,
- 4 must be followed by 2, 4 or 5,
- 5 must be followed by 1 or 3.

(a) *The rate of the code.* Let K_N denote the number of allowed sequences of length N. The rate of the code in bits per symbol is defined to be

$$R = \lim_{N \to \infty} \frac{\log_2(K_N)}{N}$$

(assuming the limit exists). Find the rate of the code described above. Compare with the rate of the code that consists of all sequences from an alphabet with 5 symbols (that is, the code with no restrictions on which symbols can follow which symbols).

(b) *Asymptotic fraction of sequences with a given starting or ending symbol.* Let $F_{N,i}$ denote the number of allowed sequences of length N that start with symbol i, and let $G_{N,i}$ denote the number of allowed sequences of length N that end with symbol i. Note that

$$F_{N,1} + \cdots + F_{N,5} = G_{N,1} + \cdots + G_{N,5} = K_N.$$

Find the asymptotic fractions

$$f_i = \lim_{N \to \infty} \frac{F_{N,i}}{K_N}, \qquad \text{and} \qquad g_i = \lim_{N \to \infty} \frac{G_{N,i}}{K_N}.$$

2 Tax policies

In this problem we explore a dynamic model of an economy, including the effects of government taxes and spending, which we assume (for simplicity) take place at the beginning of each year. Suppose the economy has n sectors, and let $x(t) \in \mathbb{R}^n$ be the pre-tax economic activity at the beginning of year t, where $x_i(t)$ is the pre-tax activity level in sector i. Let $\tilde{x}(t) \in \mathbb{R}^n$ denote the post-tax economic activity at the beginning of year t, where $\tilde{x}_i(t)$ is the post-tax activity level in sector i. We assume that all entries of $x(0)$ are positive, which will imply that all entries of $x(t)$ and $\tilde{x}(t)$ are positive for all $t \geq 0$.

The pre- and post-tax activity levels are related as follows. The government taxes the sector activities at rates given by $r \in \mathbb{R}^n$, where r_i is the tax rate for sector i. These rates all satisfy $0 \le r_i \le 1$. The total government revenue is $R(t) = r^\mathsf{T} x(t)$. Government spending is described by the vector $s \in \mathbb{R}^n$, where s_i gives the fraction of total government spending that is spent in sector i. We assume that $s_i \ge 0$ for $i = 1, \ldots, n$, and $\sum_{i=1}^n s_i = 1$. Thus, the amount of government spending in sector i is $s_i R(t)$. The post-tax economic activity in sector i, which accounts for the government taxes and spending, is then given by

$$\tilde{x}_i(t) = x_i(t) - r_i x_i(t) + s_i R(t), \qquad i = 1, \ldots, n, \qquad t = 0, 1, 2, \ldots.$$

Economic activity propagates from year to year according to the equation

$$x(t+1) = E\tilde{x}(t),$$

where $E \in \mathbb{R}^{n\times n}$ is the input/output matrix of the economy. You may assume that all entries of E are positive. Let $S(t) = \sum_{i=1}^n x_i(t)$ denote the total economic activity in year t, and let

$$G = \lim_{t\to\infty} \frac{S(t+1)}{S(t)}$$

denote the (asymptotic) growth rate of the economy. (If $G < 1$, then the economy is shrinking.)

(a) Express the growth rate G in terms of the problem data r, s, and E. State any assumptions that are needed for your analysis to work. In particular, explain why G does not depend on $x(0)$ for a typical value of $x(0)$. When does G depend on $x(0)$? Why is this situation essentially impossible?

(b) Consider the problem instance with data

$$E = \begin{bmatrix} 0.3 & 0.4 & 0.1 & 0.6 \\ 0.2 & 0.3 & 0.7 & 0.2 \\ 0.1 & 0.2 & 0.2 & 0.1 \\ 0.4 & 0.2 & 0.3 & 0.2 \end{bmatrix}, \qquad r = \begin{bmatrix} 0.45 \\ 0.25 \\ 0.10 \\ 0.10 \end{bmatrix}, \qquad \text{and} \qquad s = \begin{bmatrix} 0.15 \\ 0.30 \\ 0.40 \\ 0.15 \end{bmatrix}.$$

Find the growth rate for this example. Compare with the growth rate when $r = 0$ (in which case s does not matter). Briefly comment on your results.

3 Output feedback for maximum damping

Consider the discrete-time linear dynamical system

$$x(t+1) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t),$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$. In output-feedback control, we use an input that is a linear function of the output: that is,

$$u(t) = Ky(t),$$

where $K \in \mathbb{R}^{m\times p}$ is the feedback-gain matrix. The resulting state trajectory is identical to that of an autonomous system

$$x(t+1) = \bar{A}x(t).$$

(a) Give an expression for $\bar{A}$ in terms of A, B, C and K.

(b) Consider the single-input, single-output system with

$$A = \begin{bmatrix} 0.5 & 1.0 & 0.1 \\ -0.1 & 0.5 & -0.1 \\ 0.2 & 0.0 & 0.9 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \text{and} \qquad C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.$$

In this case, the feedback gain matrix K is a scalar (which we call the feedback gain). Explain how to find a feedback gain K_{opt} such that the system is maximally damped. The asymptotic decay rate of a system is defined to be the slowest asymptotic decay rate of the trajectory of the system over all possible initial conditions. A system is maximally damped if it has the fastest possible asymptotic decay rate. Give your answer for K_{opt} to two digits; you may assume that $-5 \leq K_{\text{opt}} \leq +5$.

4 Analyzing investment-allocation strategies

Each year an investor buys one-, two-, and three-year certificates of deposit (CDs), which have interest rates 5 %, 6 %, and 7 %, respectively. For simplicity, we assume that the CDs can be bought in any amount, although in practice there are minimum purchase requirements. Let $B_k(t)$ denote the amount of k-year CDs bought in year t. We assume that $B_1(0) + B_2(0) + B_3(0) = 1$: that is, a total of one unit of capital is invested at time $t = 0$. The total payout $p(t)$ to the investor in period t is the sum of six terms:

- $1.05B_1(t-1)$, which is the principle plus 5 % interest on the amount of one-year CDs bought one year ago,
- $1.06B_2(t-2)$, which is the principle plus 6 % interest on the amount of two-year CDs bought two years ago,
- $1.07B_3(t-3)$, which is the principle plus 7 % interest on the amount of three-year CDs bought two years ago,
- $0.06B_2(t-1)$, which is 6 % interest on the amount of two-year CDs bought one year ago,
- $0.07B_3(t-1)$, which is 7 % interest on the amount of three-year CDs bought one year ago,
- $0.07B_3(t-2)$, which is 7 % interest on the amount of three-year CDs bought two years ago.

The total wealth held by the investor at time t is

$$w(t) = B_1(t) + B_2(t) + B_2(t-1) + B_3(t) + B_3(t-1) + B_3(t-2).$$

Consider the following investment allocation strategies.

- *The 35-35-30 strategy.* The initial investment allocation is $B_1(0) = 0.35$, $B_2(0) = 0.35$, and $B_3(0) = 0.30$; each year the total payout is re-invested, with 35 % in one-year CDs, 35 % in two-year CDs, and 30 % in three-year CDs.

- *The 60-20-20 strategy.* The initial investment allocation is $B_1(0) = 0.60$, $B_2(0) = 0.20$, and $B_3(0) = 0.20$; each year the total payout is re-invested, with 60 % in one-year CDs, 20 % in two-year CDs, and 20 % in three-year CDs.

(a) For each of the investment strategies, describe the wealth of the investor using a linear dynamical system:

$$x(t+1) = Ax(t), \qquad w(t) = Cx(t),$$

where $x(t)$ is the state (which you must choose), and $w(t)$ is the output.

(b) *Asymptotic wealth growth rate.* For each of the investment strategies, determine the asymptotic growth rate, which is defined to be

$$G = \lim_{t \to \infty} \frac{w(t+1)}{w(t)}.$$

(c) *Asymptotic liquidity.* The total wealth at time t can be divided into three components:

- $B_1(t) + B_2(t-1) + B_3(t-2)$, which is the amount that matures in one year,
- $B_2(t) + B_3(t-1)$, which is the amount that matures in two years,
- $B_3(t)$, which is the amount that matures in three years.

The first component is the most liquid, the third component is the least liquid, and the second component is somewhere in between. The liquidity ratios are defined to be

$$L_1(t) = \frac{B_1(t) + B_2(t-1) + B_3(t-2)}{w(t)},$$
$$L_2(t) = \frac{B_2(t) + B_3(t-1)}{w(t)},$$
$$L_3(t) = \frac{B_3(t)}{w(t)}.$$

For each of the investment strategies, determine if the liquidity ratios converge to limiting values as $t \to \infty$. If the liquidity ratios do converge, give their limits.

(d) Suppose you can change the initial investment strategy for the 35-35-30 strategy: that is, you can choose different values of $B_1(0)$, $B_2(0)$, and $B_3(0)$ (subject to the constraints $B_1(0), B_2(0), B_3(0) \geq 0$, and $B_1(0) + B_2(0) + B_3(0) = 1$); in every other time period, you still use the 35-35-30 strategy. What allocation would you pick? How would it be better than using 35-35-30 initial investment strategy?

5 System identification of a linear dynamical system

Suppose we measure the sequence of inputs

$$u(1), \ldots, u(N) \in \mathbb{R}^m$$

of a discrete-time dynamical system, as well as the corresponding sequence of states

$$x(1), \ldots, x(N) \in \mathbb{R}^n.$$

We want to use this data to find a linear dynamical system that approximately describes the observed behavior of our system. In particular, we want to find matrices $A \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times m}$ that minimize the root mean squared state residual:

$$R = \left(\frac{1}{N-1} \sum_{t=1}^{N-1} \|x(t+1) - (Ax(t) + Bu(t))\|^2 \right)^{\frac{1}{2}}.$$

The RMS value of x over the same period is

$$S = \left(\frac{1}{N-1} \sum_{t=1}^{N-1} \|x(t+1)\|^2 \right)^{\frac{1}{2}},$$

and the normalized residual is $\rho = R/S$. For example, if we have $\rho = 0.05$, then our linear dynamical system models the observed behavior of the system to within about 5 %.

(a) Explain how to choose A and B in order to minimize R. State any assumptions that are needed for your method to work.

(b) Apply your method to the data in `lds_system_identification_data.m`. Report your estimates A and B, and the corresponding normalized residual.