EE263: Introduction to Linear Dynamical Systems
Midterm Exam

Name: _____

SUID: _____

Please circle the appropriate option for each of the following:

Grading option:	Letter grade	Credit/No credit
Date:	July 18-19 (5pm)	July 19-20 (10am)
	July 19-20 (5pm)	Other (please specify):

I acknowledge and accept the Honor Code.

(Signed) _____

(For EE263 staff only)

Problem	Score
1	/20
2	/20
3	/20
4	/20
5	/20
Total	/100

This is a 24-hour take-home midterm exam. If you are a local student, please submit your exam in Bytes Cafe (in the Packard building) 24 hours after you pick it up; if you are an SCPD student, please upload your solutions as a single PDF file using the drop box on CourseWork 24 hours after you receive the exam via email. (All of your work, including code, must be in one file.)

- You may use any books, notes, or computer programs (such as `MATLAB`), but you may not discuss the exam with others until Tuesday July 22, after everyone has taken the exam. The only exception is that you can ask the course staff for clarification by emailing to the staff email address `ee263-sum1314-staff@lists.stanford.edu`. Do not ask about the exam questions on Piazza! However, we've tried pretty hard to make the exam unambiguous and clear, so we're unlikely to say much.

- Since you have 24 hours, we expect your solutions to be legible, neat, and clear. Do not hand in your rough notes, and please try to simplify your solutions as much as you can. We will deduct points from solutions that are technically correct, but much more complicated than they need to be.

- Please check your email a few times during the exam, just in case we need to send out a clarification or other announcement. It's unlikely we'll need to do this, but you never know.

- Attach the official exam cover page to your exam, and assemble your solutions to the problems in order: that is, problem 1, problem 2, etc. Start each solution on a new page. We will not go hunting for your work if the problems are not arranged in order; make sure that the code for each problem appears with your discussion and solution. (In particular, do not put all the code at the end.)

- Please make a copy of your exam before handing it in. We have never lost one, but it might occur.

- When a problem involves some computation, we do not want just the final answers. We want a clear discussion and justification of exactly what you did, the `MATLAB` source code that produces the result, and the final numerical result.

- Some of the problems are described in a practical setting, such as energy consumption, population dynamics, or wireless communications. *You do not need to understand anything about the application area to solve these problems.* We've taken special care to make sure all the information and math needed to solve the problem are given in the problem description.

- Some of the problems require you to download and run a `MATLAB` file to generate problem data. These files can be found at the URL

  `http://www.stanford.edu/class/ee263/summer_midterm/FILENAME`

  where you should substitute the particular filename (given in the problem) for `FILENAME`. *There are no links on the course web page pointing to these files, so you'll have to type in the whole URL yourself.*

1 Numerical differentiation formulas

Suppose we want to compute the nth derivative $f^{(n)}(x)$ of a function $f : \mathbb{R} \to \mathbb{R}$ at a point $x \in \mathbb{R}$. A $(2d+1)$-point approximation of $f^{(n)}(x)$ has the form

$$\hat{f}_d^{(n)}(x) = \frac{1}{\delta^n} \sum_{k=-d}^{+d} c_k f(x + k\delta),$$

where $c_{-d}, \ldots, c_{+d} \in \mathbb{R}$ are coefficients that must be determined, and δ is the grid size. (We assume that the function f is known on a grid of equally spaced points, and δ is the distance between points where f is known; the points where f is known are called sample points).

(a) Suppose f has a Taylor-series expansion at x:

$$f(z) = \sum_{m=0}^{\infty} \frac{f^{(m)}(x)}{m!}(z-x)^m$$

Show that we can express $\hat{f}_d^{(n)}(x)$ as

$$\hat{f}_d^{(n)}(x) = \sum_{m=0}^{\infty} \alpha_m(n,d)\delta^{m-n} f^{(m)}(x).$$

Give an expression for $\alpha_m(n,d)$.

(b) We need to choose $2d+1$ coefficients: $c_{-d}, \ldots, c_{+d}$. This suggests that we can control $2d+1$ of the coefficients $\alpha_m(n,d)$. Suppose we choose the c_k such that

$$\alpha_m(n,d) = \begin{cases} 1 & m = n, \\ 0 & m \in \{0, \ldots, 2d\} - \{n\}. \end{cases}$$

Then, we have that

$$\hat{f}_d^{(n)}(x) = \sum_{m=0}^{\infty} \alpha_m(n,d)\delta^{m-n} f^{(m)}(x) = f^{(n)}(x) + \mathcal{O}(\delta^{2d-n+1}).$$

In other words, the error in our approximation of $f^{(n)}(x)$ decreases like δ^{2d-n+1}, where δ is the distance between sample points. Explain how to choose the c_k in order to satisfy the given conditions on the $\alpha_m(n,d)$. Report the values of c_k you get with $n = 2$, and $d = 1, 2, 3$.

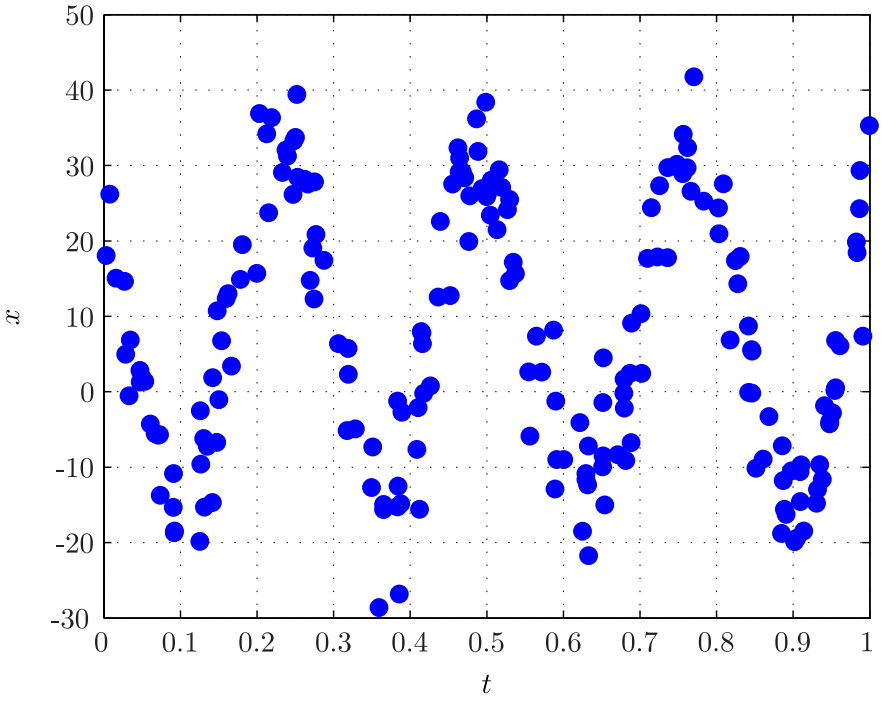
(c) Consider the function

$$f(x) = \exp(\cos(10x)).$$

Compute the second derivative of $f(x)$. (We want you to differentiate the function using the rules of calculus, so you can compare your numerical differentiation rules to the exact value of the derivative.) For $n = 2$, and $d = 1, 2, 3$, make a plot of the log approximation error $\log(|\hat{f}_d^{(n)}(1) - f^{(n)}(1)|)$ versus the negative log grid size $-\log(\delta)$, using values of δ ranging from 10^{-6} to 10^{-1}. Briefly comment on the results.

2 Fitting a sinusoid to data

Consider the following data set.



The file `fit_sinusoid_data.m` defines the data points

$$(t_1, x_1), \ldots, (t_N, x_N).$$

We want to fit a sinusoidal model to the data:

$$\hat{x}(t) = A\cos(2\pi f t + \phi) + b,$$

where $A \in \mathbb{R}$ is the amplitude, $f \in \mathbb{R}$ is the frequency, $\phi \in \mathbb{R}$ is the phase, and $b \in \mathbb{R}$ is the offset.

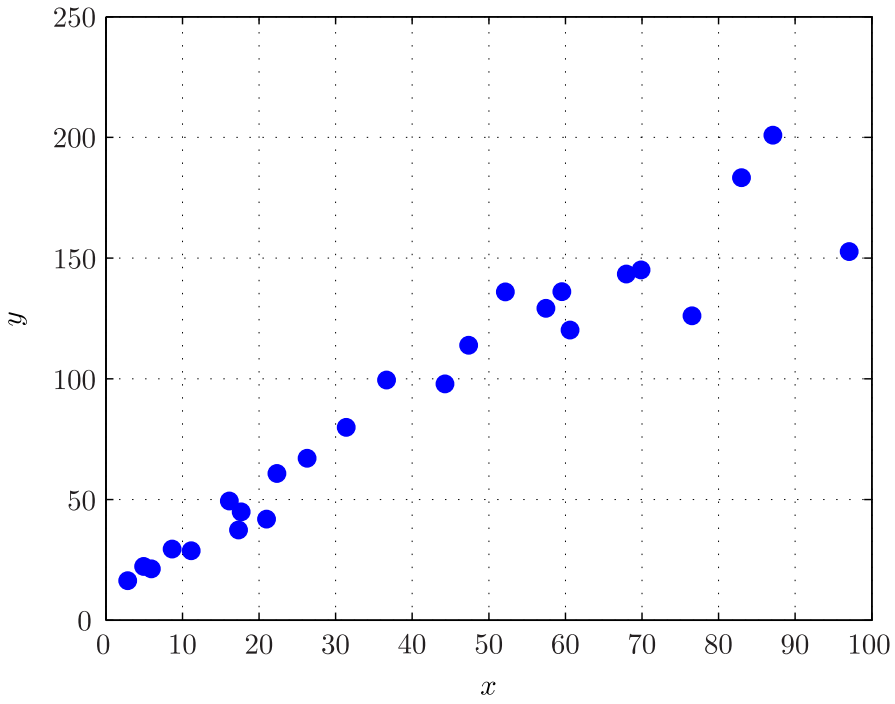
(a) Explain how to use the Gauss-Newton method to choose the parameters A, f, ϕ, and b in order to minimize the sum of squared errors:

$$\sum_{i=1}^{N} (\hat{x}(t_i) - x_i)^2.$$

(b) Apply your method to the data given in `fit_sinusoid_data.m`. Explain how you chose the initial parameter estimates. Report your final estimates of the parameters, and the sums of squared errors for your initial and final parameter estimates. On a single set of axes, plot the data, the sinusoid corresponding to your initial parameter estimates, and the sinusoid corresponding to your final parameter estimates.

3 Minimum-percent-error regression

Consider the following data set.



The file `percent_error_regression_data.m` defines the data points

$$(x_1, y_1), \ldots, (x_m, y_m).$$

We want to fit a linear model to the data:

$$\hat{y}(x) = ax + b,$$

where $a, b \in \mathbb{R}$ are parameters.

(a) Fit a line to the data using least-squares: that is, find a_{ls} and b_{ls} that minimize the sum of squared errors

$$\sum_{i=1}^{m} (\hat{y}(x_i) - y_i)^2.$$

Report your values of a_{ls} and b_{ls}. Submit a plot with the data points (x_i, y_i), and the least-squares line $y = a_{\text{ls}}x + b_{\text{ls}}$.

(b) The data points seem to follow a line closely for small values of y, but there seem to be large deviations from the linear trend for large values of y. In this case, it may be reasonable to choose parameter estimates a_{mpe} and b_{mpe} that minimize the total percent error:

$$\sum_{i=1}^{m} 100 \times \frac{|\hat{y}(x_i) - y_i|}{|y_i|}.$$

Explain how to use iteratively reweighted least squares to compute a_{mpe} and b_{mpe}. In particular, what is the weight function, and what is the update equation? Apply your method to the example data; use a_{ls} and b_{ls} as your initial values. Report your final estimates of a_{mpe} and b_{mpe}. Submit a plot with the data points (x_i, y_i), and the minimum-percent-error line $y = a_{\text{mpe}}x + b_{\text{mpe}}$.

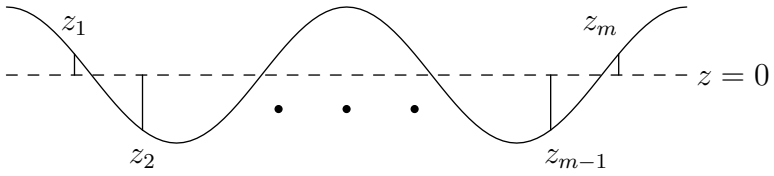
(c) The data points were generated using the model

$$y_i = (1 + \epsilon_i)(ax_i + b), \qquad i = 1, \ldots, m,$$

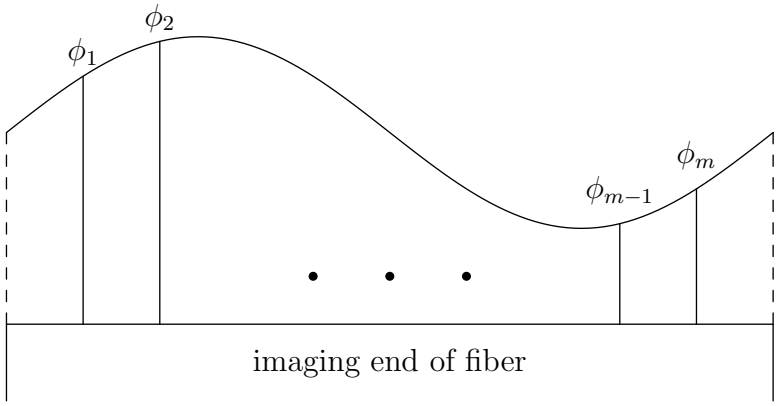
where ϵ_i is uniformly distributed on $[-0.25, +0.25]$, $a = 2$, and $b = 8$. In light of this revelation, briefly compare the least-squares and minimum-percent-error lines. In particular, give the root mean squared errors in $(a_{\text{ls}}, b_{\text{ls}})$ and $(a_{\text{mpe}}, b_{\text{mpe}})$ as approximations of (a, b).

4 Surface imaging using a multimode fiber

Suppose we want to use an optical fiber to image a surface. In particular, we want to determine the profile of the surface, which we can describe by a vector $z \in \mathbb{R}^m$, where z_i is the elevation of the ith reference point of the surface relative to some baseline value. An example of a surface is shown below.



The fiber has n different modes, which correspond to different patterns of light intensity that appear at the imaging end of the fiber. We can describe the pattern of light intensity corresponding to the ith mode by a vector $\phi_i \in \mathbb{R}^m$, which is a discretized version of the pattern of light intensity, as shown below.



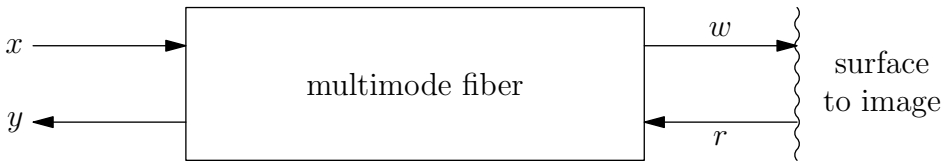
Each of the modes can be excited independently, and we let x_i denote the excitation applied to the ith mode. The pattern of light intensity at the imaging end of the fiber is a superposition of the modes:

$$w = x_1\phi_1 + \cdots + x_n\phi_n.$$

If we shine the light coming out of the imaging end of the fiber onto a surface, then some amount of light is reflected back into the fiber. We assume that the amount of light reflected back into the fiber is

$$r = w_1 z_1 + \cdots + w_m z_m.$$

We get a noisy measurement $y = r + \epsilon$ of r at the end of the fiber opposite the imaging surface. The complete experiment setup is shown below.



The file `multimode_fiber_data.m` defines the following variables.

- `Phi`, an $m \times n$ matrix whose jth column is ϕ_j
- `Y`, an $m \times k$ matrix giving measurements of y (more details are given below)
- `ztrue`, a vector of length m giving the true surface profile

(a) *Spot formation.* If we can choose the excitation vector x in such a way that $w = e_i$, then the resulting measurement y is a noisy measurement of z_i. This process is called forming a spot at location i. The total energy used to excite the fiber is $\|x\|^2$. In practice, the amount of energy required to form an exact spot may be prohibitively large (for example, applying excitations that are too large may damage the fiber). Thus, we want to find a set of excitation vectors $x^{(1)}, \ldots, x^{(m)} \in \mathbb{R}^n$ such that $x^{(i)}$ gives an approximate spot at location i, while also using as little input energy as possible. We can balance these objectives by minimizing the cost

$$J = \sum_{i=1}^{m} \|w(x^{(i)}) - e_i\|^2 + \lambda \sum_{i=1}^{m} \|x^{(i)}\|^2,$$

where $\lambda > 0$ is a tradeoff parameter. For a given value of λ, explain how to choose $x^{(1)}, \ldots, x^{(m)}$ in order to minimize J. Apply your method to the example data, and make a plot of the tradeoff curve for the costs

$$J_1 = \sum_{i=1}^{m} \|w(x^{(i)}) - e_i\|^2, \qquad \text{and} \qquad J_2 = \sum_{i=1}^{m} \|x^{(i)}\|^2.$$

For $\lambda = 10$, report the optimal value of J, and make a plot of $w(x^{(15)})$. Additionally, mark the point corresponding to $\lambda = 10$ on your tradeoff curve.

(b) *Surface imaging.* Suppose you decide to use the values of $x^{(1)}, \ldots, x^{(m)}$ corresponding to $\lambda = 10$. You take k measurements of y with each of the $x^{(i)}$. The data file defines a matrix of noisy measurements $Y \in \mathbb{R}^{m \times k}$, where Y_{ij} is jth measurement taken with excitation vector $x^{(i)}$. Explain how to choose an estimate $\hat{z} \in \mathbb{R}^m$ that minimizes the sum of squared errors:

$$E = \sum_{i=1}^{m} \sum_{j=1}^{k} (\hat{y}_i - Y_{ij})^2,$$

where we define the vector $y \in \mathbb{R}^m$ such that

$$\hat{y}_i = w_1(x^{(i)})\hat{z}_1 + \cdots + w_m(x^{(i)})\hat{z}_m.$$

Apply your method to the example data. Make a plot of the estimated surface profile $\hat{z}$.

(c) Explain how to choose $\hat{z}^{(1)}$ in order to minimize

$$E_1 = \sum_{i=1}^{m} (\hat{y}_i^{(1)} - Y_{i1})^2,$$

where

$$\hat{y}_i^{(1)} = w_1(x^{(i)})\hat{z}_1^{(1)} + \cdots + w_m(x^{(i)})\hat{z}_m^{(1)}.$$

(Intuitively, we only use the first measurement taken with each excitation vector $x^{(i)}$.) Make a plot of the estimated surface profile $\hat{z}^{(1)}$. Report the RMS errors in $\hat{z}$ and $\hat{z}^{(1)}$ as approximations of the true z:

$$E^{\text{rms}} = \sqrt{\frac{1}{m}\sum_{i=1}^{m}(z_i - \hat{z}_i)^2}, \qquad \text{and} \qquad E_1^{\text{rms}} = \sqrt{\frac{1}{m}\sum_{i=1}^{m}(z_i - \hat{z}_i^{(1)})^2}.$$

Compare $\hat{z}$ and $\hat{z}^{(1)}$ to the true surface profile. Briefly comment on your results.

5 Designing an equalizer for backwards-compatible wireless transceivers

You are given the job of designing the equalizer for a new line of wireless handheld transceivers (more commonly called walkie-talkies). The transmitter of the new line of transceivers has already been designed – if the input signal is $x \in \mathbb{R}^n$, then the transmitted signal is $y = A_{\text{new}}x \in \mathbb{R}^m$, where $A_{\text{new}} \in \mathbb{R}^{m\times n}$ is known. An equalizer is a matrix $B \in \mathbb{R}^{n\times m}$ such that $By = x$ for every $x \in \mathbb{R}^n$.

The new line of transceivers will replace an older model. Given an input signal $x \in \mathbb{R}^n$, the old line of transceivers transmit a signal $y = A_{\text{old}}x \in \mathbb{R}^m$, where $A_{\text{old}} \in \mathbb{R}^{m\times n}$ is known. In addition to providing exact equalization for the new line of transceivers, you want your equalizer to be able to at least partially equalize signals transmitted using the old line of transceivers. In other words, to the extent that it is possible, you want the new line of transceivers to be backwards compatible with the old line of transceivers.

(a) Explain how to find an equalizer B that minimizes

$$J = \|BA_{\text{old}} - I\|_F^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}(BA_{\text{old}} - I)_{ij}^2$$

among all B that exactly equalize A_{new}. Such a B is an exact equalizer for A_{new}, and an approximate equalizer for A_{old}. State any assumptions that are needed for your method to work.

(b) The file `backwards_compatible_transceiver_data.m` defines the following variables.

- `Anew`, the $m\times n$ matrix describing the transmitter used in the new line of transceivers
- `Aold`, the $m\times n$ matrix describing the transmitter used in the old line of transceivers
- `x`, a vector of length n that serves as an example input signal

Apply your method to this example data. Report the optimal value of J. The pseudoinverse $A_{\text{new}}^{\dagger}$ is another exact equalizer for A_{new}. Compare the optimal value of J, and the value of J achieved by $A_{\text{new}}^{\dagger}$.

(c) The example signal x defined in the data file is a binary signal. Form the signal $y_{\text{old}} = A_{\text{old}}x$ transmitted by the old line of transceivers, and construct an estimate of x by equalizing y_{old} using B, and then rounding the result to a binary signal. More concretely, compute the estimate $\hat{x} \in \mathbb{R}^n$, where

$$\hat{x}_i = \begin{cases} 1 & (By_{\text{old}})_i > \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Report the bit error rate of your estimate, which is defined as

$$\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}(x_i \neq \hat{x}_i),$$

where $\mathbf{1}(x_i \neq \hat{x}_i)$ is an indicator function:

$$\mathbf{1}(x_i \neq \hat{x}_i) = \begin{cases} 1 & x_i \neq \hat{x}_i, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, report the bit error rate if $A_{\text{new}}^{\dagger}$ is used as the equalizer.