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EE263 (Introduction to Linear Dynamical Systems)
Problem of the Day 6
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1 Fitting a quadratic form to data

A quadratic form is a function $f : \mathbb{R}^n \to \mathbb{R}$ of the form

$$f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j = x^\mathsf{T} A x.$$

We can assume without loss of generality that A is symmetric because

$$x^\mathsf{T} A x = x^\mathsf{T} \left(\tfrac{1}{2}(A + A^\mathsf{T}) \right) x,$$

where $\frac{1}{2}(A + A^\mathsf{T})$ is a symmetric matrix (called the symmetric part of A). This observation follows from the fact that

$$x^\mathsf{T} A x = (x^\mathsf{T} A x)^\mathsf{T} = x^\mathsf{T} A^\mathsf{T} x,$$

where the first step follows from the fact that any scalar is equal to its own transpose, and the second step is an application of the identity that the transpose of a product is equal to the product of the transposes in the reverse order. Suppose you are given data points $(x_1, y_1), \ldots, (x_N, y_N) \in \mathbb{R}^2$, where y_i is a noisy measurement of $f(x_i)$.

(a) Explain how to find a symmetric matrix $A \in \mathbb{R}^{n \times n}$ that minimizes the mean squared fitting error:

$$J = \sum_{i=1}^{N} (x_i^\mathsf{T} A x_i - y_i)^2.$$

(b) Apply your method to the data given in `fit_quadratic_form_data.m`. Report your matrix A, and the corresponding relative fitting error:

$$e_{\text{rel}} = \frac{\sqrt{\sum_{i=1}^{N} (x_i^\mathsf{T} A x_i - y_i)^2}}{\sqrt{\sum_{i=1}^{N} y_i^2}}.$$