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EE263 (Introduction to Linear Dynamical Systems)
Problem of the Day 10
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1 Eigenvalue power iteration

Suppose $A \in \mathbb{R}^{n \times n}$ is diagonalizable. Let the dyadic eigendecomposition of A be

$$A = \sum_{k=1}^{n} \lambda_k v_k w_k^\mathsf{T},$$

where $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ are the eigenvalues of A, $v_1, \ldots, v_n \in \mathbb{R}^n$ are corresponding right eigenvectors, and $w_1, \ldots, w_n \in \mathbb{R}^n$ are corresponding left eigenvectors. Additionally, assume that $v_1, \ldots, v_n$ are normalized with $\|v_1\| = 1$.

(a) Suppose λ_1 is the unique dominant eigenvalue of A: that is, $|\lambda_1| > |\lambda_k|$ for $k = 2, \ldots, n$. Define the sequence of vectors $x_1(t) \in \mathbb{R}^n$ such that

$$x_1(t+1) = \frac{\operatorname{sgn}(\lambda_1)}{\|Ax_1(t)\|} Ax_1(t),$$

where we assume that $w_1^\mathsf{T} x_1(0) \neq 0$. Show that

$$\lim_{t \to \infty} x_1(t) = \operatorname{sgn}(w_1^\mathsf{T} x_1(0)) v_1.$$

(b) Now suppose that $|\lambda_2| > |\lambda_k|$ for $k = 3, \ldots, n$. Define the sequence of vectors $x_2(t) \in \mathbb{R}^n$ such that

$$x_2(t+1) = \frac{\operatorname{sgn}(\lambda_2)}{\|Ax_2(t)\|} Ax_2(t),$$

where we assume that $w_1^\mathsf{T} x_2(0) = 0$, and $w_2^\mathsf{T} x_2(0) \neq 0$. Show that

$$\lim_{t \to \infty} x_2(t) = \operatorname{sgn}(w_2^\mathsf{T} x_2(0)) v_2.$$

(c) Use the observations above to design an algorithm for computing the eigenvalues and eigenvectors of a matrix. You may assume that the eigenvalues are all distinct. Implement your algorithm in `MATLAB`, and use it to compute the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 0.5 & -1.0 & 2.5 \\ 0.0 & 1.0 & 0.0 \\ 2.5 & -1.0 & 0.5 \end{bmatrix}.$$

Compare the eigendecomposition you find using your algorithm with the eigendecomposition returned by `MATLAB`'s `eig` function.