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EE263 (Introduction to Linear Dynamical Systems)
Problem of the Day 14
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1 Regularized least squares and the SVD

Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix with singular-value decomposition

$$A = \sum_{i=1}^{n} \sigma_i u_i v_i^\mathsf{T}.$$

(We consider the case where A is square and nonsingular for simplicity; it is not too hard to extend the following results to the general case.) Recall that the regularized approximate solution of $Ax = y$ is defined to be the vector $\hat{x}(\mu) \in \mathbb{R}^n$ that minimizes

$$\|Ax - y\|^2 + \mu\|x\|^2,$$

where $\mu > 0$ is the regularization parameter. The regularized solution is a linear function of y, so it can be expressed as $\hat{x}(\mu) = B(\mu)y$ for some matrix $B(\mu) \in \mathbb{R}^{n\times n}$.

(a) Let the singular-value decomposition of B be

$$B = U\Sigma V^\mathsf{T} = \sum_{i=1}^{n} \tilde{\sigma}_i \tilde{u}_i \tilde{v}_i^\mathsf{T}.$$

Express $\tilde{\sigma}_i$, $\tilde{u}_i$, and $\tilde{v}_i$ in terms of σ_i, u_i, v_i, and μ. In particular, make sure that your answer follows the convention $\tilde{\sigma}_1 \geq \cdots \geq \tilde{\sigma}_n$.

(b) Find the norm of B in terms of μ, and the SVD of A.

(c) Find the worst-case relative inversion error:

$$\max_{y \neq 0} \left\{ \frac{\|ABy - y\|}{\|y\|} \right\}.$$

Give your answer in terms of μ, and the SVD of A.