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EE263 (Introduction to Linear Dynamical Systems)
Problem Session 1
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1 Gradient of the norm function

Recall that the gradient of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ at a point $x \in \mathbb{R}^n$ is defined to be the vector

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(x) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x) \end{bmatrix}.$$

The first-order Taylor expansion of f near x is given by

$$\hat{f}_1(z) = f(x) + \nabla f(x)^\mathsf{T}(z - x).$$

This function is affine: that is, a linear function plus a constant offset. If z is near x, then $\hat{f}_1(z)$ is very near $f(z)$. Find the gradient of the function $f(x) = \|x\|$.

2 Some standard time-series models

In some contexts, a discrete-time signal is called a time series. The study of time series predates the extensive study of linear state-space systems, and is used in many fields. Let u and y be two time series, which we will think of as the input and output, respectively.

(a) The relation (or time-series model)

$$y(k) = a_0u(k) + a_1u(k-1) + \cdots + a_ru(k-r)$$

is called a moving-average (MA) model. Since the output at time k is a weighted average of the previous r inputs, we can think of the output as the average of the inputs in a moving window. Express this model as a linear dynamical system with input u, output y, and state

$$x(k) = \begin{bmatrix} u(k-1) \\ \vdots \\ u(k-r) \end{bmatrix}.$$

(b) Another model is

$$y(k) = u(k) + b_1y(k-1) + \cdots + b_py(k-p).$$

This model is called an autoregressive (AR) model, since the current output is a linear combination of the current input, and some previous values of the output. Express this model as a linear dynamical system with input u, output y, and state

$$x(k) = \begin{bmatrix} y(k-1) \\ \vdots \\ y(k-p) \end{bmatrix}.$$

(c) A third widely used model is the autoregressive, moving-average (ARMA) model, which combines the MA and AR models:

$$y(k) = a_0 u(k) + \cdots + a_r u(k-r) + b_1 y(k-1) + \cdots + b_p y(k-p).$$

Express this model as a linear dynamical system with input u and output y (you can choose the state; there are many possible choices, and not all choices have the same dimension).

3 Some linear functions associated with a convolution system

Suppose that u and y are discrete-time scalar signals related via convolution:

$$y(t) = \sum_{\tau=-\infty}^{+\infty} h(t-\tau)u(\tau), \qquad t \in \mathbb{Z},$$

where $(h(t) : t \in \mathbb{Z})$ is a known discrete-time scalar signal. You may assume that the system is causal: that is, $h(t) = 0$ for all $t < 0$.

(a) Suppose $u(t) = 0$ for all $t < 0$. Define the vectors

$$\bar{u} = \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(T) \end{bmatrix} \qquad \text{and} \qquad \bar{y} = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(T) \end{bmatrix}.$$

Thus, $\bar{u}$ and $\bar{y}$ are the first $T+1$ values of the input and output signals, respectively. Find the matrix $G \in \mathbb{R}^{(T+1)\times(T+1)}$ such that $\bar{y} = G\bar{u}$. The matrix G describes the linear mapping from (a segment of) the input sequence to (a segment of) the output sequence; G is called the input/output or Toeplitz matrix of size $T+1$ associated with the convolution system.

(b) Now suppose that $u(t) = 0$ for all $t < 0$ and $t > T$. Define the vectors

$$\tilde{u} = \begin{bmatrix} u(T) \\ u(T-1) \\ \vdots \\ u(0) \end{bmatrix} \qquad \text{and} \qquad \tilde{y} = \begin{bmatrix} y(T) \\ y(T+1) \\ \vdots \\ y(2T) \end{bmatrix}.$$

Thus, $\tilde{u}$ is the input to the system, and $\tilde{y}$ is (a segment of) the future output of the system. Find the matrix $H \in \mathbb{R}^{(T+1)\times(T+1)}$ such that $\tilde{y} = H\tilde{u}$. The matrix H describes the linear mapping from the input sequence to (a segment of) the future output sequence; H is called the Hankel matrix of size $T+1$ associated with the convolution system.

4 Matrix representation of polynomial differentiation

Consider the set of all polynomials with real coefficients, and degree less than n:

$$\mathcal{P}_n = \{p(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0 \mid a_0, a_1, \ldots, a_{n-2}, a_{n-1} \in \mathbb{R}\}.$$

We can represent a polynomial $p \in \mathcal{P}_n$ by a vector $\tilde{p} = (a_0, a_1, \ldots, a_{n-2}, a_{n-1}) \in \mathbb{R}^n$. The differentiation function $\mathcal{D}_n : \mathcal{P}_n \to \mathcal{P}_n$ such that $\mathcal{D}_n(p) = \frac{dp}{dx}$ is a linear transformation. Find the matrix representation $D_n \in \mathbb{R}^{n \times n}$ of $\mathcal{D}_n$: that is, find the matrix D_n such that $\tilde{q} = D_n \tilde{p}$ whenever $q = \mathcal{D}_n(p)$.

5 Counting paths in an undirected graph

Consider an undirected graph with n nodes, and no self loops. Let $A \in \mathbb{R}^{n \times n}$ be the node-adjacency matrix, which is defined such that

$$A_{ij} = \begin{cases} 1 & \text{there is an edge between nodes } i \text{ and } j, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $A = A^\mathsf{T}$ because the graph is undirected, and $A_{ii} = 0$ since there are no self loops. Give an interpretation of $(A^p)_{ij}$ (that is the (i, j)-entry of A^p) for $p \in \mathbb{N}$.