Alex Lemon
EE263 (Introduction to Linear Dynamical Systems)
Problem Session 2
June 26, 2013

1 The orthogonal complement of a subspace

Given a subspace $\mathcal{V}$ of $\mathbb{R}^n$, we define $\mathcal{V}^\perp$ to be the set of all vectors that are orthogonal to every vector in $\mathcal{V}$: that is,

$$\mathcal{V}^\perp = \{x \in \mathbb{R}^n : \langle x \,|\, y \rangle = 0 \text{ for all } y \in \mathcal{V}\}.$$

We call $\mathcal{V}^\perp$ the orthogonal complement of $\mathcal{V}$.

(a) Verify that $\mathcal{V}^\perp$ is a subspace of $\mathbb{R}^n$.

(b) Suppose $\mathcal{V} = \text{span}(v_1, \ldots, v_r)$. Define the matrix $V = \begin{bmatrix} v_1 & \cdots & v_r \end{bmatrix} \in \mathbb{R}^{n \times r}$. Express $\mathcal{V}$ and $\mathcal{V}^\perp$ in terms of the range or nullspace of an appropriate matrix.

(c) Show that every $x \in \mathbb{R}^n$ can be written as $x = v + v^\perp$, where $v \in \mathcal{V}$ and $v^\perp \in \mathcal{V}^\perp$. Is this representation unique?

(d) Show that $\dim(\mathcal{V}) + \dim(\mathcal{V}^\perp) = n$.

(e) Show that if $\mathcal{V} \subseteq \mathcal{U}$, then $\mathcal{U}^\perp \subseteq \mathcal{V}^\perp$.

2 Right inverses

Consider the matrix

$$A = \begin{bmatrix} -1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

This matrix is fat and full rank, so it has at least one right inverse. In fact, there are many right inverses of A, which opens the possibility that we can seek right inverses that satisfy additional side constraints. For each of the cases below, either find a specific matrix $B \in \mathbb{R}^{5 \times 3}$ that satisfies $AB = I$ and the given property, or explain why there is no such B. If there is a right inverse B with the required property, briefly explain how you can find such a B. In addition, attach a `MATLAB` script checking that $AB = I$. If there is no right inverse with the required property, explain why.

(a) The second row of B is zero.

(b) The nullspace of B has dimension 1.

(c) The third column of B is zero.

(d) The second and third rows of B are the same.

(e) B is upper triangular: that is, $b_{ij} = 0$ for $i > j$.

(f) B is lower triangular: that is, $b_{ij} = 0$ for $i < j$.

3 Memory of a linear, time-invariant system

Suppose an input signal $(u_t : t \in \mathbb{Z})$, and an output signal $(y_t : t \in \mathbb{Z})$ are related by a convolution operator:

$$y_t = \sum_{\tau=1}^{M} h_\tau u_{t-\tau},$$

where $h = (h_1, \ldots, h_M)$ are the impulse-response coefficients of the convolution system. (Convolution systems are also called linear, time-invariant systems.) If $h_M \neq 0$, then M is called the memory of the system. You are given the input and output signals for $t = 1, \ldots, T$:

$$u_1, \ldots, u_T \qquad \text{and} \qquad y_1, \ldots, y_T.$$

However, you do not know u_t or y_t for $y < 1$ or $t > T$, and you do not know the impulse response, h.

(a) Explain how to find the smallest value of M, and a corresponding impulse response $(h_t : t = 1, \ldots, M)$ that is consistent with the given data. You may assume that $T > 2M$.

(b) Apply your method to the data in `lti_memory_data.m`. Report the value of M that you find.

Hint. The function `toeplitz` may be useful.

4 Projection matrices

A matrix $P \in \mathbb{R}^{n\times n}$ is called a projection matrix if $P = P^\mathsf{T}$, and $P^2 = P$. (These properties are sometimes called symmetry and idempotency, respectively.)

(a) Show that if P is a projection matrix, then $I - P$ is also a projection matrix.

(b) Suppose $U \in \mathbb{R}^{n\times k}$ has orthonormal columns. Show that UU^T is a projection matrix. (The converse is also true: every projection matrix can be written as UU^T for some matrix U with orthonormal columns; you do not need to prove this.)

(c) Suppose $A \in \mathbb{R}^{n\times k}$ is skinny, and full rank. Show that $A(A^\mathsf{T}A)^{-1}A^\mathsf{T}$ is a projection matrix.

(d) Given $S \subseteq \mathbb{R}^n$, and $x \in \mathbb{R}^n$, the point $\hat{x} \in S$ that is closest to x is called the projection of x onto S. Show that if P is a projection matrix, then $\hat{x} = Px$ is the projection of x onto range(P). (This is the origin of the term "projection matrix.")