Alex Lemon
EE263 (Introduction to Linear Dynamical Systems)
Problem Session 4
July 10, 2013

1 Fitting a Gaussian function to data

A Gaussian function has the form

$$f(t) = a \exp\left(- \left(\frac{t - \mu}{\sigma} \right)^2 \right),$$

where $t \in \mathbb{R}$ is the independent variable, $a \in \mathbb{R}$ is called the amplitude, $\mu \in \mathbb{R}$ is called the center, and σ is called the spread or width. We can assume without loss of generality that $\sigma > 0$. Thus, the vector of parameters is $p = (a, \mu, \sigma)$. We are given a set of data pairs

$$(t_1, y_1), \ldots, (t_N, y_N),$$

and our goal is to fit a Gaussian function to the data. We measure the quality of a fit using the root mean squared error:

$$E = \left(\frac{1}{N} \sum_{i=1}^{N} (f(t_i) - y_i)^2 \right)^{\frac{1}{2}}.$$

(a) Explain how to use the Gauss-Newton method to compute a parameter vector p that minimizes E.

(b) Fit a Gaussian function to the data in `fit_gaussian_data.m`. Explain how you chose your initial guess for the parameters. Plot the RMS error as a function of the iteration number. Plot the data together with the fitted Gaussian function. Repeat for a different, but still reasonable, initial guess for the parameters. Repeat for an unreasonable initial guess for the parameters. Comment on your results.

2 Curve smoothing

We are given a function $F : [0, 1] \to \mathbb{R}$, and we want to find a function $G : [0, 1] \to \mathbb{R}$ that is a smoothed version of F. We will evaluate our smoothed version G of F using the following two criteria.

- *The mean squared deviation from* F:

$$D = \int_0^1 (F(t) - G(t))^2 \, dt$$

- *The mean squared curvature*:

$$C = \int_0^1 G''(t)^2 \, dt.$$

The first criterion penalizes G for being far away from F, while the second criterion penalizes G for not being smooth. If we only cared about D, we would take $G = F$; if we only cared about C, we would take G to be the best affine approximation of F (this gives $C = 0$). However, in general we want D and C to both be small: that is, we have a multiobjective problem. We can approximate F and G by vectors $f, g \in \mathbb{R}^n$ such that

$$f_i = F(i/n) \qquad \text{and} \qquad g_i = G(i/n)$$

for $i = 1, \ldots, n$. You can assume that n is large enough that f and g are good representations of F and G. The discretized versions of our evaluation criteria are as follows.

- *The mean squared deviation from f*:

$$d = \frac{1}{n} \sum_{i=1}^{n} (f_i - g_i)^2$$

- *The mean squared curvature*:

$$c = \frac{1}{n-2} \sum_{i=2}^{n-1} \left(\frac{g_{i+1} - 2g_i + g_{i-1}}{1/n^2} \right)^2$$

In our definition of c, note that

$$\frac{g_{i+1} - 2g_i + g_{i-1}}{1/n^2}$$

is a discrete approximation of $G''(i/n)$.

(a) Explain how to choose g in order to minimize $d + \mu c$ for a given $\mu \geq 0$. State any assumptions that are needed for your method to work. In particular, explain how to find g in the extreme cases $\mu = 0$ and $\mu \to \infty$.

(b) The file `curve_smoothing_data.m` defines the following variables.

- `n`, the number of samples in our discretizations of F and G
- `f`, the vector representing the function G

Plot the optimal tradeoff curve between d and c. Be sure to identify critical points such as the intersection of the curve with an axis. Plot f and g on the same set of axes for $\mu = 0$; repeat for $\mu = \infty$, and three additional intermediate values of μ.

3 Minimum-energy roundtrip

Consider the linear dynamical system

$$x(t+1) = Ax(t) + Bu(t), \qquad x(0) = 0,$$

with input $u(t) \in \mathbb{R}$, and state $x(t) \in \mathbb{R}^n$. We need to complete a roundtrip journey from the origin to a given destination state $x_d \in \mathbb{R}^n$, and back to the origin. We must complete the journey by time T, so that $x(T) = 0$. Let $t_d \in \{1, \ldots, T-1\}$ be the time at which the system reaches the destination state, so that $x(t_d) = x_d$. Note that t_d is not given: we must choose it. The total input energy is

$$E = \sum_{t=0}^{T-1} u(t)^2.$$

(a) Explain how to choose t_d and $u(0), \ldots, u(T-1)$ in order to minimize the total input energy subject to the roundtrip constraints: $x(t_d) = x_d$ and $x(T) = 0$. State any assumptions that are needed for your method to work. You may assume that $n \leq t_d \leq T - n$.

(b) Carry out your method on the instance of the problem with

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad T = 30 \qquad \text{and} \qquad x_d = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

Give the optimal values of t_d and E. Plot the optimal input sequence $u(t)$, and the state sequences $x_j(t)$ for $j = 1, \ldots, n$.